Uniqueness of minimal coverings of maximal partial clones

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Abstract. A partial function f on an k-element set E_k is a partial Sheffer function if every partial function on E_k is definable in terms of f. Since this holds if and only if f belongs to no maximal partial clone on E_k, a characterization of partial Sheffer functions reduces to finding families of minimal coverings of maximal partial clones on E_k. We show that for each $k \geq 2$ there exists a unique minimal covering.

1. Introduction

In many-valued logic the set of truth values is finite and without loss of generality we can assume it to be $E_k := \{0, 1, \ldots, k-1\}$, $k \in \mathbb{N}$, $k \geq 2$.

The set $P_k := \{f^{(n)} \mid f^{(n)} \colon E_k^n \to E_k, n \geq 1\}$ is the set of all total functions on E_k. Let $D \subseteq E_k^n$, $n \geq 1$ and $f^{(n)} \colon D \to E_k$. Then $f^{(n)}$ is called an n-ary partial function on E_k with domain D. We also write $\operatorname{dom}(f) = D$. If the arity of the function is known we omit the upper index and write f instead of $f^{(n)}$. Denote by $\widetilde{P}_k^{(n)}$ the set of all n-ary partial functions on E_k and set

$$\widetilde{P}_k := \bigcup_{n \geq 1} \widetilde{P}_k^{(n)}.$$

Let $C_\emptyset := \left\{ f \in \widetilde{P}_k \;\middle|\; \operatorname{dom}(f) = \emptyset \right\}$.

For $i \in \{1, \ldots, n\}$ the n-ary function $e_i^{(n)}$ defined by setting $e_i^{(n)}(x_1, \ldots, x_n) := x_i$ for all $x_1, \ldots, x_n \in E_k$ is called the n-ary *projection* onto the i-th coordinate. Let $J_k := \left\{ e_i^{(n)} \;\middle|\; n \in \mathbb{N}, 1 \leq i \leq n \right\}$ be the set of all projections.

For $f \in \widetilde{P}_k^{(n)}$ and $g_1, \ldots, g_n \in \widetilde{P}_k^{(m)}$ let $f(g_1, \ldots, g_n) \in \widetilde{P}_k^{(m)}$ be the *composition* as given in [2], i.e.,

$$x \in \operatorname{dom}(f(g_1, \ldots, g_n)) \iff \left(x \in \bigcap_{i=1}^{n} \operatorname{dom}(g_i) \right) \wedge (g_1(x), \ldots, g_n(x)) \in \operatorname{dom}(f)$$

and $f(g_1, \ldots, g_n)(x) := f(g_1(x), \ldots, g_n(x))$ for all $x \in \operatorname{dom}(f(g_1, \ldots, g_n))$. A *partial clone* (*clone*) on E_k is a composition closed subset of $\widetilde{P}_k$ (P_k) containing the set of projections J_k.

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The set of all partial clones on E_k (clones on E_k), ordered by inclusion, forms an algebraic lattice $\mathbb{L}\widetilde{P}_k$ ($\mathbb{L}P_k$), whose smallest element is the set of all projections and greatest element is $\widetilde{P}_k$ (P_k), respectively. A *maximal partial clone* (a *maximal clone*) on E_k is a co-atom of $\widetilde{P}_k$ and P_k, respectively. Thus a partial clone (clone) M is a maximal partial clone (maximal clone) if the inclusions $M \subset C \subset \widetilde{P}_k$ ($M \subset C \subset P_k$) hold for no partial clone (hold for no clone) C on E_k.

For $F \subseteq \widetilde{P}_k$ ($F \subseteq P_k$), we denote by $[F]_{\mathrm{P}}$ ($[F]$) the partial clone (clone) on E_k generated by F, i.e., the intersection of all partial clones (clones) containing the set F on E_k. Clearly $[F]_{\mathrm{P}}$ ($[F]$) is the least partial clone (clone) on E_k containing F.

A set F of partial functions (functions) on E_k is *complete* if $[F]_{\mathrm{P}} = \widetilde{P}_k$ ($[F] = P_k$), respectively. It is well known that a set $F \subseteq \widetilde{P}_k$ ($F \subseteq P_k$) is complete if and only if F is contained in no maximal partial clone (maximal clone) on E_k (see, e.g., [6] for the partial case and e.g., [7], Theorem 1.5.4.1, for the total case). Therefore maximal clones fully described in [9, 10] (see also [11]) play a fundamental role for completeness.

Similarly, maximal partial clones play a very important role for the completeness problem of finite partial algebras. The description of all maximal partial clones on a finite set can be found in the literature. We refer the reader to the papers of Haddad and Rosenberg [3, 5] for the description of all maximal partial clones.

Sheffer [17] described two binary functions $f \in P_2$ such that $[\{f\}] = P_2$, i.e., such that every function on E_2 can be expressed in terms of f only. A function $f \in P_k$ is a *Sheffer function* if every function on E_k can be obtained by composition from f and the projections, i.e., if $[f] := [\{f\}] = P_k$.

Next Webb [18] showed that the function f defined by

$$f(x, y) := \min(x, y) + 1 \pmod{k}$$

is a Sheffer function for P_k. Sheffer functions have been well studied and characterized by Rousseau [12] and Schofield [13]. We refer the reader to [11] for a detailed list of references on the subject.

Partial Sheffer functions are defined similarly. A partial function f on E_k is a *partial Sheffer function* if every partial function on E_k can be obtained by composition from f and the projections, i.e., if $[f]_{\mathrm{P}} = \widetilde{P}_k$. However due to the difficulty of the problem, very little is known about partial Sheffer functions for $\widetilde{P}_k$. Already the family of all maximal partial clones on E_k is far more complex than the family of all maximal clones on E_k. This is already shown in the following table where $|\mathscr{M}_k|$ and $|p\mathscr{M}_k|$ denote the number of maximal clones (see [7] p. 185) and maximal partial clones (see [15]), respectively.

k	$\lvert\mathscr{M}_k\rvert$	$\lvert p\mathscr{M}_k\rvert$
2	5	8
3	18	58
4	82	1 102
5	643	325 722
6	15 182	5 242 621 816
7	7 848 984	?

Results on partial Sheffer functions can be found in the papers by Haddad and Rosenberg [4], Romov [8], and Haddad and Lau [2]. Many examples of partial Sheffer functions are known, see e.g. [1] and [4].

The *completeness problem for partial Sheffer functions* is the question if for a given partial function $f \in \widetilde{P}_k$ the identity $[f]_{\mathrm{P}} = \widetilde{P}_k$ holds. That means, criteria are investigated to decide if a partial function is a partial Sheffer function. The problem has been solved for $k = 2$ by Haddad and Rosenberg [4], for $k = 3$ by Haddad and Lau [2], and for $k = 4$ by the author in [14] (see also [16]). A specific notion used there is a *minimal covering* of the maximal partial clones, which for $k \in \{2, 3, 4\}$ has been shown to be unique and has been determined in the papers mentioned above. The aim of this paper is to show that for all $k \geq 2$ there is a unique minimal covering.

2. Definitions and the Theorem of Haddad and Rosenberg

Relations are useful to describe the clones in $\widetilde{P}_k$. We often write the elements of relations as columns and a relation can then be given as a matrix. For example, the ternary relation $\varrho = \{(0, 1, 2), (1, 2, 0), (3, 4, 5), (2, 3, 1)\}$ can also be written as

$$\varrho = \begin{pmatrix} 0 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 2 & 0 & 5 & 1 \end{pmatrix}.$$

Denote by $E_k^{a\times b}$ be the set of all $(a \times b)$–matrices over E_k. Let a matrix be given by $C = (c_{ij})_{h,n} \in E_k^{h\times n}$. Then denote by $c_{i*} = (c_{i1}, \ldots, c_{in})$ the i-th row of the matrix where $i \in \{1, \ldots, h\}$), and denote by $c_{*j} = (c_{1j}, \ldots, c_{hj})^{\mathrm{T}}$ the j-th column of the matrix where $j \in \{1, \ldots, n\}$.

Let $\mathcal{R}_k^{(h)}$ be the set of all h-ary relations on E_k and $\mathcal{R}_k := \bigcup_{h\geq 1} \mathcal{R}_k^{(h)}$. For a relation $\varrho \in \mathcal{R}_k$ we write $\varrho^{(h)}$ to indicate that $\varrho \in \mathcal{R}_k^{(h)}$, i.e., that ϱ is an h-ary relation.

An n-ary function $f^{(n)} \in \widetilde{P}_k$ *preserves* an h-ary relation $\varrho^{(h)} \in \mathcal{R}_k$ iff for all $c_{*1}, c_{*2}, \ldots, c_{*n} \in \varrho$ with $c_{1*}, \ldots, c_{h*} \in \mathrm{dom}(f)$ holds

$$f(c_{*1}, \ldots, c_{*n}) := \begin{pmatrix} f(c_{1*}) \\ f(c_{2*}) \\ \vdots \\ f(c_{h*}) \end{pmatrix} := \begin{pmatrix} f(c_{11}, c_{12}, \ldots, c_{1n}) \\ f(c_{21}, c_{22}, \ldots, c_{2n}) \\ \vdots \\ f(c_{h1}, c_{h2}, \ldots, c_{hn}) \end{pmatrix} \in \varrho.$$

Denote by $\mathrm{pPOL}_k\, \varrho$ the set of all functions $f \in \widetilde{P}_k$ which preserve the relation $\varrho \in \mathcal{R}_k$. For example, for $h = 1$ and $\varrho = \{0\}$ the set $\mathrm{pPOL}_k\{0\}$ is the set of all functions $f \in \widetilde{P}_k$ for which $f(0, \ldots, 0) = 0$ or $(0, \ldots, 0) \notin \mathrm{dom}\, f$.

For each $m \in \mathbb{N}$ set $\eta_m := (0, 1, \ldots, m - 1)^{\mathrm{T}}$.

Denote by $\omega(v)$ the set of distinct entries of $v = (v_1, \ldots, v_h) \in E_k^h$, that means, $\omega(v) = \omega((v_1, \ldots, v_h)) := \{v_1, \ldots, v_h\}$. Additionally for some relation $\varrho \subseteq E_k^h$ we set $\omega(\varrho) = \bigcup_{v\in\varrho} \omega(v)$. For example, for $v = (0, 0, 1) \in E_k^3$ we get $\omega(v) = \{0, 1\}$.

Definition 2.1. Set for all h with $1 \leq h \leq k$

$$\varrho_1 := \left\{(a,a,b,b),(a,b,a,b) \mid a,b \in E_k\right\},$$
$$\varrho_2 := \left\{(a,a,b,b),(a,b,a,b),(a,b,b,a) \mid a,b \in E_k\right\},$$
$$\iota_k^h := \left\{x \in E_k^h \;\middle|\; |\omega(x)| \leq h-1\right\}.$$

Definition 2.2. For an arbitrary equivalence relation ε on E_h define

$$\delta_{k,\varepsilon}^{(h)} := \left\{(a_0,\dots,a_{h-1}) \in E_k^h \;\middle|\; (i,j) \in \varepsilon \Longrightarrow a_i = a_j\right\}.$$

If h or k is understood from the context we just write δ_ε or $\delta_\varepsilon^{(h)}$ or $\delta_{k,\varepsilon}$. If $\varepsilon_1,\dots,\varepsilon_r$ are the non-singular equivalence classes of the relation ε then we write $\delta_{k;\varepsilon_1,\dots,\varepsilon_r}^{(h)}$ or $\delta_{\varepsilon_1,\dots,\varepsilon_r}$ instead of $\delta_{k,\varepsilon}^{(h)}$. For example, $\delta_{k;E_h}^{(h)} = \left\{(x,x,\dots,x) \in E_k^h \;\middle|\; x \in E_k\right\}$. These relations are called *diagonal* relations. Especially E_k^h for any h is a diagonal relation.

Definition 2.3. For $\varrho^{(h)} \subseteq E_k^h$ we set $\sigma(\varrho) := \varrho \setminus \iota_k^h$ and $\delta(\varrho) := \varrho \cap \iota_k^h = \varrho \setminus \sigma(\varrho)$. If $\delta(\varrho) = \delta_\gamma$ for some equivalence relation γ on E_h then we write $\varepsilon(\varrho) := \gamma$.

Definition 2.4. A relation $\varrho^{(h)} \subseteq E_k^h$ is

- *areflexive*, if $h \geq 2$ and $\delta(\varrho) = \emptyset$, i.e., $\varrho = \sigma(\varrho)$ meaning that for each $(x_1,\dots,x_h) \in \varrho$ we have that $x_i \neq x_j$ for all $1 \leq i < j \leq h$.
- *quasi-diagonal*, if $\sigma(\varrho)$ is a non-empty areflexive relation, and $\delta(\varrho) = \delta_\varepsilon$ where $\varepsilon \neq \iota_h^2$ is an equivalence relation on E_h.

Definition 2.5. For $\varrho^{(h)} \subseteq E_k^h$ set $\sigma := \sigma(\varrho)$, $\delta := \delta(\varrho)$, and denote by S_h the set of all permutations on E_h.

For $r = (r_0,\dots,r_{h-1}) \in \varrho$ and $\pi \in S_h$ we write

$$r^{[\pi]} := (r_{\pi(0)}, r_{\pi(1)}, \dots, r_{\pi(n-1)}), \text{ and } \varrho^{[\pi]} := \left\{r^{[\pi]} \;\middle|\; r \in \varrho\right\}.$$

Let $\Gamma_\sigma := \left\{\pi \in S_h \;\middle|\; \sigma \cap \sigma^{[\pi]} \neq \emptyset\right\}$.

The *model* of ϱ is the h-ary relation $M(\varrho) := \left\{\eta_h^{[\pi]} \;\middle|\; \pi \in \Gamma_\sigma\right\} \cup (\delta \cap E_h^h)$ on E_h.

The relation ϱ is *coherent*, if the following conditions hold:

(1) $\varrho \neq E_k^h$, $\varrho \neq \emptyset$,
(2) (a) ϱ is a unary relation, i.e., $h = 1$, or
  (b) ϱ is areflexive with $2 \leq h \leq k$, or
  (c) ϱ is quasi-diagonal with $2 \leq h \leq k$, or
  (d) $\delta = \iota_k^h$ with $3 \leq h \leq k$, or
  (e) $h = 4$ and $\delta = \varrho_i$ with $i \in \{1,2\}$ (see Definition 2.1),
(3) $r^{[\pi]} \in \sigma$ for all $r \in \sigma$ and all $\pi \in \Gamma_\sigma$,
(4) for every σ' with $\emptyset \neq \sigma' \subseteq \sigma$ there is a relational homomorphism $\varphi\colon E_k \to E_h$ from σ' to $M(\varrho)$, such that $\varphi(r) = \eta_h$ for some $r \in \sigma'$, i.e., there is some $r = (r_0,\dots,r_{h-1}) \in \sigma'$ with $(\varphi(r_0),\dots,\varphi(r_{h-1})) = (0,\dots,h-1)$,
(5) (a) if $\delta = \iota_k^h$ and $h \geq 3$ then $\Gamma_\sigma = S_h$,
  (b) if $\delta = \varrho_1$ then $\Gamma_\sigma = \langle(0231),(12)\rangle$ (Γ_σ is the permutation group which is generated by the cycles (0231) and (12)),
  (c) if $\delta = \varrho_2$ then $\Gamma_\sigma = S_4$.

We remark that all non-empty non-diagonal totally reflexive, totally symmetric relations are coherent.

Denote by $\widetilde{\mathcal{R}}_k^{\max}$ the set of all coherent relations. Due to [15] (Chapter: Different Relations – Different Clones) we can assume that $\mathrm{pPOL}_k\,\varrho \neq \mathrm{pPOL}_k\,\chi$ for all $\varrho^{(h)}, \chi^{(h)} \in \widetilde{\mathcal{R}}_k^{\max}$ with $\varrho \neq \chi$. Let

$$p\mathscr{M}_k := \{P_k \cup C_\emptyset\} \cup \left\{\mathrm{pPOL}_k\,\varrho \;\middle|\; \varrho \in \widetilde{\mathcal{R}}_k^{\max}\right\}.$$

Theorem 2.6 (of Haddad and Rosenberg; [3, 5])**.** *Let $k \geq 2$. For each $A \subset \widetilde{P}_k$ with $A = [A]_{\mathrm{P}}$ there is a maximal partial clone M_A with $A \subseteq M_A$. A clone M is a maximal partial clone of $\widetilde{P}_k$ if and only if $M \in p\mathscr{M}_k$, i.e., in other words $p\mathscr{M}_k$ is the set of all maximal partial clones of $\widetilde{P}_k$.*

Theorem 2.7 (Completeness criterion for $\widetilde{P}_k$; [5])**.** *Let $C \subseteq \widetilde{P}_k$. Then $[C]_{\mathrm{P}} = \widetilde{P}_k$ if and only if $C \not\subseteq M$ for all $M \in p\mathscr{M}_k$.*

Definition 2.8. The set of coherent relations $\widetilde{\mathcal{R}}_k^{\max}$ can be divided into the following sets:

$$\begin{aligned}
\mathcal{U} &:= \{\chi^{(\mu)} \in \widetilde{\mathcal{R}}_k^{\max} \mid \mu = 1\},\\
\mathcal{A} &:= \{\chi^{(\mu)} \in \widetilde{\mathcal{R}}_k^{\max} \mid \mu \geq 2 \wedge \chi \text{ is areflexive}\},\\
\mathcal{Q} &:= \{\chi^{(\mu)} \in \widetilde{\mathcal{R}}_k^{\max} \mid \mu \geq 2 \wedge \chi \text{ is quasi-diagonal}\},\\
\mathcal{S} &:= \{\chi^{(\mu)} \in \widetilde{\mathcal{R}}_k^{\max} \mid \mu \geq 3 \wedge \delta(\chi) = \iota_k^{\mu}\},\\
\mathcal{L} &:= \{\chi^{(\mu)} \in \widetilde{\mathcal{R}}_k^{\max} \mid \mu = 4 \wedge \delta(\chi) \in \{\varrho_1, \varrho_2\}\}.
\end{aligned}$$

Definition 2.9. Let $\varrho^{(h)} \in \mathcal{R}_k$ and $A = \{a_0, \ldots, a_{l-1}\} \subseteq E_h$ with $a_i < a_j$ for all $i < j$. Then set

$$\begin{aligned}
\mathrm{pr}_A\,\varrho &:= \mathrm{pr}_{a_0,\ldots,a_{l-1}}\,\varrho\\
&:= \{(x_{a_0}, \ldots, x_{a_{l-1}}) \mid \exists x_0, \ldots, x_{h-1} \in E_k : (x_0, \ldots, x_{h-1}) \in \varrho\}.
\end{aligned}$$

Definition 2.10. For $\varrho^{(h)} \in \mathcal{Q}$ denote by $\varrho^\star$ the union of the non-singleton classes of the equivalence relation $\varepsilon(\varrho)$. We define

$$\begin{aligned}
\mathrm{pp}\,\varrho &:= \mathrm{pr}_{\varrho^\star}\,\varrho,\\
\|\varrho\| &:= |\varrho^\star|,\\
\mathcal{Q}_0 &:= \{\chi^{(\mu)} \in \mathcal{Q} \mid \varepsilon(\chi) \text{ has no singular equivalence class}\}\\
&\quad\big(= \{\chi^{(\mu)} \in \mathcal{Q} \mid \mathrm{pp}\,\chi = \chi\} = \{\chi^{(\mu)} \in \mathcal{Q} \mid \|\chi\| = \mu\}\big),\\
\mathcal{Q}_1 &:= \mathcal{Q} \setminus \mathcal{Q}_0.
\end{aligned}$$

If $\varrho \in \mathcal{Q}_1$ then define

$$\mathcal{Q}_\varrho := \left\{\chi \in \mathcal{Q}_1 \;\middle|\; \begin{array}{l}(\|\chi\| < \|\varrho\|) \vee\\ (\|\chi\| = \|\varrho\| \wedge \mathrm{pp}\,\chi \not\subseteq \mathrm{pp}\,\varrho)\end{array}\right\}. \tag{2.1}$$

Because $\mathrm{pPOL}_k\,\varrho = \mathrm{pPOL}_k\,\varrho^{[\pi]}$ for all $\pi \in S_h$ we use the convention $\mathrm{pp}\,\varrho = \mathrm{pr}_{E_{\|\varrho\|}}\,\varrho$ for all $\varrho \in \mathcal{Q}$.

The relations in $\mathcal{Q}_1$ are exactly the coherent quasi-diagonal relations ϱ where $\varepsilon(\varrho)$ has at least one singular class.

Example 2.11. Let $k = 10$ and

$$\varrho^{(5)} := \begin{pmatrix} 0 & 5 \\ 1 & 6 \\ 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{pmatrix} \cup \delta^{(5)}_{\{0,1\},\{2,3\}}.$$

Then $\varrho \in \mathcal{Q}$, $\varepsilon(\varrho)$ has the blocks $\{0,1\}, \{2,3\}, \{4\}$,

$$\varrho^\star = \{0,1,2,3\},$$
$$\|\varrho\| = 4, \text{ and}$$
$$\operatorname{pp} \varrho = \begin{pmatrix} 0 & 5 \\ 1 & 6 \\ 2 & 7 \\ 3 & 8 \end{pmatrix} \cup \delta^{(4)}_{\{0,1\},\{2,3\}}.$$

Then $\varrho \in \mathcal{Q}_1$, since $\varepsilon(\varrho)$ has a singleton block $\{4\}$, and $\operatorname{pp} \varrho = \operatorname{pr}_{E_4} \varrho \in \mathcal{Q}_0$.

3. Minimal covering

We want to determine which maximal partial clones in the criterion in Theorem 2.7 are needed to characterize partial Sheffer functions. According to Theorem 2.7 a function $f \in \widetilde{P}_k$ is a partial Sheffer function if and only if $f \in \widetilde{P}_k \setminus (\bigcup p\mathscr{M}_k)$. It turns out that the union $\bigcup p\mathscr{M}_k$ of maximal partial clones is also $\bigcup \mathscr{X}$ for a proper subset $\mathscr{X}$ of $p\mathscr{M}_k$. This leads to the following definition.

Definition 3.1. A set $\mathscr{X} \subseteq p\mathscr{M}_k$ is a *minimal covering* of $p\mathscr{M}_k$, if for every $f \in \widetilde{P}_k$ holds

$$[f]_\mathrm{P} = \widetilde{P}_k \iff \forall A \in \mathscr{X} : f \notin A$$

and for each $A \in \mathscr{X}$ there is some $f \in \widetilde{P}_k$ with

$$[f]_\mathrm{P} \neq \widetilde{P}_k \wedge (\forall B \in \mathscr{X} \setminus \{A\} : f \notin B)\,.$$

Lemma 3.2. *Let C be a maximal partial clone and $f \in C$ with $f \notin B$ for all $B \in p\mathscr{M}_k \setminus \{C\}$. Then C is in every minimal covering of $p\mathscr{M}_k$.*

Proof. Let $f \in C \in p\mathscr{M}_k$ with $f \notin B$ for all $B \in p\mathscr{M}_k \setminus \{C\}$. Assume there is a minimal covering $\mathscr{X}$ of $p\mathscr{M}_k$ with $C \notin \mathscr{X}$. Then $[f]_\mathrm{P} \subseteq C \subset \widetilde{P}_k$ and $f \notin A$ for each $A \in \mathscr{X} \subseteq p\mathscr{M}_k \setminus \{C\}$, in contradiction to the first condition of a minimal covering. □

Lemma 3.3. *Let $C \in p\mathscr{M}_k$ and $\mathscr{C} \subseteq p\mathscr{M}_k \setminus \{C\}$ be such that every $C' \in \mathscr{C}$ is contained in every minimal covering of $p\mathscr{M}_k$ and for all $f \in C$ there is some $C' \in \mathscr{C}$ with $f \in C'$. Then C is in no minimal covering of $p\mathscr{M}_k$.*

Proof. Assume C is in some minimal covering $\mathscr{X}$ of $p\mathscr{M}_k$. Then there is some $f \in \widetilde{P}_k$ with $[f]_{\mathrm{P}} \neq \widetilde{P}_k$ and $f \notin B$ for all $B \in \mathscr{X} \setminus \{C\}$. From $\mathscr{C} \subseteq \mathscr{X}$ follows $f \notin C$. Thus $f \notin A$ for all $A \in \mathscr{X}$, and $[f]_{\mathrm{P}} \neq \widetilde{P}_k$ contradicting $\mathscr{X}$ minimal covering of $p\mathscr{M}_k$. Thus C is in no minimal covering. □

4. A Product of Functions

Definition 4.1. Let $D' \in E_k^{a\times b}$ be an (a,b)-matrix on E_k, i.e.,

$$D' = \begin{pmatrix} d_{11} & \dots & d_{1b} \\ \vdots & \ddots & \vdots \\ d_{a1} & \dots & d_{ab} \end{pmatrix}$$

with $d_{ij} \in E_k$ for all i, j.

If a function $f^{(n)} \in \widetilde{P}_k$ is defined by

$$f(D') := v = (v_1, \dots, v_a)^{\mathrm{T}}$$

then

$$\begin{aligned} n &:= b, \\ \operatorname{dom} f := D &:= \{(d_{i1}, \dots, d_{ib}) \mid i \in \{1, \dots, a\}\}, \\ f(d_{i1}, \dots, d_{ib}) &:= v_i \end{aligned}$$

for all $i \in \{1, \dots, a\}$. If the domain $D := \operatorname{dom} f$ is given then D' is a matrix whose rows are the entries of D in lexicographical order.

Let $\chi^{(h)} \in \mathcal{R}_k$ and $f^{(n)} \in \widetilde{P}_k$ be defined by $f(\chi) = v$ then assume χ be given as a matrix as explained before, i.e., $n = |\chi|$ and $v \in E_k^h$.

Definition 4.2. Let $f^{(n)} \in \widetilde{P}_k$ with $D = \operatorname{dom} f$ and $g^{(m)} \in \widetilde{P}_k$ with $E = \operatorname{dom} g$. Then $D' \in E_k^{|D|\times n}$ and $E' \in E_k^{|E|\times m}$.

Define $F^{(N)} := (f \otimes g) \in \widetilde{P}_k^{(n\cdot m)}$ by

$$F(D' \otimes E') := F \left(\begin{array}{c|c|c} D'_{*1} & \dots & D'_{*n} \\ \hline E' & \dots & E' \end{array} \right) := \left(\frac{f(D')}{g(E')} \right). \tag{4.1}$$

We assume E' has no constant rows so F is well-defined. Then

$$\begin{aligned} \operatorname{dom} F = \{(&\underbrace{a_1, \dots, a_1}_{m \text{ times}}, \dots, \underbrace{a_i, \dots, a_i}_{m \text{ times}}, \dots, \underbrace{a_n, \dots, a_n}_{m \text{ times}}) \mid (a_1, \dots, a_n) \in D\} \\ &\cup \{(b_1, b_2, \dots, b_m, b_1, b_2, \dots, b_m, \dots, b_1, b_2, \dots, b_m) \mid (b_1, \dots, b_m) \in E\}. \end{aligned}$$

Let $c = (c_1, \dots, c_N) \in \operatorname{dom} F$. Then we say it is from the E-part or g-part of F if $c = \mathrm{pr}_i(E', E', \dots, E')$ for some i. Otherwise we say it is from the D-part or f-part of F.

Likewise we inductively set $f \otimes g_1 \otimes \dots \otimes g_{l-1} \otimes g_l := (f \otimes g_1 \otimes \dots \otimes g_{l-1}) \otimes g_l$ with $g_i \in \widetilde{P}_k$ for all $i \in \{1, \dots, l\}$.

Example 4.3. Let $f, g \in \widetilde{P}_k$ be given by

$$f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} := \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ g\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 5 \end{pmatrix} := \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

where

$$D' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ D = \operatorname{dom} f = \{(0,0),(0,1)\},$$
$$E = \operatorname{dom} g = \{(0,2,3),(2,4,5)\}.$$

Then

$$(f \otimes g)\left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 2 & 3 & 0 & 2 & 3 \\ 2 & 4 & 5 & 2 & 4 & 5 \end{array}\right) = \left(\begin{array}{c} 1 \\ 2 \\ \hline 4 \\ 0 \end{array}\right)$$

and

$$\operatorname{dom}(f \otimes g) = \{(0,0,0,0,0,0),(0,0,0,1,1,1),$$
$$(0,2,3,0,2,3),(2,4,5,2,4,5)\}.$$

5. Criteria

For the remainder of this paper we will assume $k \geq 3$ as the case $k = 2$ is already solved.

Lemma 5.1 (Lemma 4 [2])**.** *The maximal partial clone $P_k \cup C_\emptyset$ belongs to every minimal covering of $\mathrm{p}\mathscr{M}_k$.*

Lemma 5.2 (Lemmas 5, 7 [2])**.** *Let $\varrho \in \mathcal{U}$, i.e., $\emptyset \subset \varrho \subset E_k$. Then* $\mathrm{pPOL}_k\,\varrho$ *belongs to every minimal covering of $\mathrm{p}\mathscr{M}_k$.*

Lemma 5.3. *Let $\varrho^{(h)} \in \widetilde{\mathcal{R}}_k^{\max}$ with $h \geq 2$ and $f^{(n)} \in \widetilde{P}_k$. Let $c_{*1}, c_{*2}, \ldots, c_{*n} \in \varrho$ with $c_{1*}, \ldots, c_{h*} \in \operatorname{dom}(f)$ and $c_{i'*} = c_{i''*}$ for some $i', i'' \in \{1, \ldots, h\}$ with $i' < i''$. Then $d := f(c_{*1}, c_{*2}, \ldots, c_{*n}) \in \varrho$.*

Proof. Because two rows are equal we have $c_{*i} \in \delta(\varrho) \subseteq \iota_k^h$ for all $i \in \{1, 2, \ldots, n\}$.

Because ϱ is coherent there are the following cases for $\delta := \delta(\varrho)$:

$\delta = \emptyset$**:** Then ϱ is areflexive and $c_{*1} \notin \varrho$ contradicting the assumption.

$\delta = \delta_\varepsilon$ *for some equivalence relation* $\varepsilon \neq \iota_h^2$**:** Then $c_{*1}, c_{*2}, \ldots, c_{*n} \in \delta_\varepsilon$ and thus $d \in \delta_\varepsilon \subseteq \varrho$.

$\delta = \iota_k^h$**:** Then $d_{i'} = f(c_{i'*}) = f(c_{i''*}) = d_{i''}$, i.e., $d \in \iota_k^h \subseteq \varrho$.

$\delta = \varrho_1$**:** Then

$$\delta \;=\; \{(a,a,b,b) \mid a,b \in E_k, a \neq b\} \,\cup\, \{(a,b,a,b) \mid a,b \in E_k, a \neq b\} \,\cup\, \{(a,a,a,a) \mid a \in E_k\}$$

and there are the following subcases:

$i' = 1$ *and* $i'' = 2$**:** Then

$$c_{*j} \in \delta \setminus \{(a,b,a,b) \mid a,b \in E_k, a \neq b\} = \delta_{\{0,1\},\{2,3\}}$$

for all $j \in \{1,2,\ldots,n\}$ and thus $d \in \delta_{\{0,1\},\{2,3\}} \subset \varrho_1 \subseteq \varrho$.
The case $i' = 3$ and $i'' = 4$ is analogous.

$i' = 1$ *and* $i'' = 3$**:** Then

$$c_{*j} \in \delta \setminus \{(a,a,b,b) \mid a,b \in E_k, a \neq b\} = \delta_{\{0,2\},\{1,3\}}$$

for all $j \in \{1,2,\ldots,n\}$ and thus $d \in \delta_{\{0,2\},\{1,3\}} \subset \varrho_1 \subseteq \varrho$.
The case $i' = 2$ and $i'' = 4$ is analogous.

$i' = 1$ *and* $i'' = 4$**:** Then $c_{*j} \in \{(a,a,a,a) \mid a \in E_k\} = \delta_{\{0,1,2,3\}}$ for all $j \in \{1,2,\ldots,n\}$ and thus $d \in \delta_{\{0,1,2,3\}} \subset \varrho_1 \subseteq \varrho$.
The case $i' = 2$ and $i'' = 3$ is analogous.

$\delta = \varrho_2$**:** is done analogously. □

Lemma 5.4. *Let* $\varrho^{(h)} \in \mathcal{S}$ *(see Definition 2.8) with either*

$$h \geq 4, \text{ or}$$

$$h = 3 \text{ and } \exists x \in \sigma(E_k^2)\, \forall a \in E_k \setminus \omega(x)\, \exists y \in \sigma(\varrho) : \omega(x) \cup \{a\} = \omega(y). \tag{5.1}$$

Then

$$\forall f \in \mathrm{pPOL}_k\, \varrho\, \exists \gamma \in \mathcal{U} \cup \{\chi\} : f \in \mathrm{pPOL}_k\, \gamma \tag{5.2}$$

with $\chi := \left\{x \in E_k^{h-1} \;\middle|\; \{x\} \times E_k \subseteq \varrho\right\}$ *and* $\mathrm{pPOL}_k\, \chi \in p\mathscr{M}_k$.

Proof. The definition of χ implies that χ is totally symmetric and totally reflexive. We have to show that χ is non-diagonal. For $h = 3$ we have $\chi \neq \iota_k^2$ because of (5.1). Assume $h \geq 3$ and $\chi = E_k^{h-1}$ to the contrary. Since $\varrho \neq E_k^h$, there is an $x := (x_1, x_2, \ldots, x_h) \in E_k^h \setminus \varrho$ and hence $(x_1, \ldots, x_{h-1}) \notin \chi$. Thus χ is a non-diagonal totally symmetric totally reflexive relation and thus $\mathrm{pPOL}_k\, \chi \in p\mathscr{M}_k$.

Let $f^{(n)} \in \mathrm{pPOL}_k\, \varrho$ be arbitrary. Assume to the contrary, that $f \notin \mathrm{pPOL}_k\, \gamma$ for all $\gamma \in \mathcal{U} \cup \{\chi\}$. Then there are $c_{*1}, \ldots, c_{*n} \in \chi$ with $c := f(c_{*1}, \ldots, c_{*n}) \in E_k^{h-1} \setminus \chi$. This means,

$$\exists q \in E_k \setminus \omega(c)\, \forall y \in \sigma(\varrho) : \omega(c) \cup \{q\} \neq \omega(y).$$

Because $f \notin \mathrm{pPOL}_k(E_k \setminus \{q\})$ there are $q_1, \ldots, q_n \in E_k \setminus \{q\}$ with $f(q_1, \ldots, q_n) = q$. Thus follows

$$f \begin{pmatrix} c_{*1} & \ldots & c_{*n} \\ q_1 & \ldots & q_n \end{pmatrix} = \begin{pmatrix} c \\ q \end{pmatrix} =: t$$

with $|\omega(t)| = h$, and therefore $t \notin \iota_k^h$. Because of $\omega(t) \neq \omega(y)$ for every $y \in \sigma(\varrho)$ by construction, $t \notin \varrho$ holds. But $c_{*1}, \ldots, c_{*n}$ are chosen with $\begin{pmatrix} c_{*i} \\ q_i \end{pmatrix} \in \varrho$ for all $i \in \{1, \ldots, n\}$ contradicting $f \in \mathrm{pPOL}_k\, \varrho$. Thus (5.2) holds. □

Let the set $\mathcal{S}'$ consist of all relations in $\mathcal{S}$ not fulfilling the conditions of Lemma 5.4, i.e., $\mathcal{S}' := \{\chi^{(\mu)} \mid \chi \in \mathcal{S}, \mu = 3 \text{ and (5.1) is not fulfilled by } \chi\}$.

Lemma 5.5. *Let* $\varrho^{(h)} \in \mathcal{Q}_0$ *(see Definition 2.10) with* $h = 2$ *and*

$$\exists x \in E_k \, \exists \pi \in S_2 : \{x\} \times E_k \subseteq \varrho^{[\pi]}.$$

Then $\mathrm{pPOL}_k\, \varrho$ *belongs to no minimal covering of* $p\mathscr{M}_k$.

Proof. Let $f^{(n)} \in \mathrm{pPOL}_k\, \varrho$ be arbitrary. Assume to the contrary $f \notin \mathrm{pPOL}_k\, \theta$ for all $\theta \in \mathcal{U}$.

Let $A \subset E_k$ be a maximal set with $A \times E_k \subseteq \varrho^{[\pi]}$. Let $y \in E_k$ be arbitrary. Because $f \notin \mathrm{pPOL}_k\, A$ and $f \notin \mathrm{pPOL}_k\, (E_k \setminus \{y\})$ there are rows $c_A \in A^n$ and $c_y \in (E_k \setminus \{y\})^n$ with $f(c_A) =: a \in E_k \setminus A$ and $f(c_y) = y$. Thus $(a, y) \in \varrho^{[\pi]}$ and because y is arbitrary we get $(A \cup \{a\}) \times E_k \subseteq \varrho^{[\pi]}$ contradicting the maximality of A.

Thus the assumption is wrong and $f \in \mathrm{pPOL}_k\, \theta$ for some $\theta \in \mathcal{U}$. This implies $\mathrm{pPOL}_k\, \varrho$ is in no minimal covering, because $\mathrm{pPOL}_k\, \theta$ is in every minimal covering of $p\mathscr{M}_k$ by Lemma 5.2. □

Let the set $\mathcal{Q}_0'$ consist of all relations in $\mathcal{Q}_0$ not fulfilling the conditions of Lemma 5.5.

If ϱ is symmetric, then Lemma 5.5 follows from Theorem 15, (b) in [2].

6. Sorting the minimal coverings

Definition 6.1. Let $\varrho, \chi \in \widetilde{\mathcal{R}}_k^{\max}$ with $\varrho \neq \chi$, i.e., $\mathrm{pPOL}_k\, \varrho \neq \mathrm{pPOL}_k\, \chi$ by definition of $\widetilde{\mathcal{R}}_k^{\max}$. We write $\varrho \ll \chi$ iff

$$\forall f \in \mathrm{pPOL}_k\, \varrho \, \exists g \in \mathrm{pPOL}_k\, \varrho$$
$$\Big((g \notin \mathrm{pPOL}_k\, \chi) \wedge \Big(\forall \psi \in \widetilde{\mathcal{R}}_k^{\max} \, (f \notin \mathrm{pPOL}_k\, \psi \Longrightarrow g \notin \mathrm{pPOL}_k\, \psi) \Big) \Big).$$

Lemma 6.2. *Let* $X = \mathrm{pPOL}_k\, \varrho \in p\mathscr{M}_k$, $f \in X$, *and* $\mathscr{Y}, \mathscr{Z} \subseteq p\mathscr{M}_k$ *with* $f \notin Y$ *for all* $Y \in \mathscr{Y}$ *and* $\mathscr{Z} = \{\mathrm{pPOL}_k\, \psi \mid \psi \in \widetilde{\mathcal{R}}_k^{\max} \wedge \varrho \ll \psi\} \neq \emptyset$.

Then there is some $F \in X$ *with* $F \notin Y$ *for all* $Y \in \mathscr{Y} \cup \mathscr{Z}$.

Proof. Let $l := |\mathscr{Z}|$ and $\mathscr{Z} =: \{\mathrm{pPOL}_k\, \psi_1, \ldots, \mathrm{pPOL}_k\, \psi_l\}$. If $l = 1$ then the statement of this Lemma follows from Definition 6.1. Now let $l \geq 2$. Assume there is some $f_i \in X$ with $i \in \{1, \ldots, l-1\}$, $f_i \notin Y$ for all $Y \in \mathscr{Y}$ and $f_i \notin \mathrm{pPOL}_k\, \chi_j$ for all $j \leq i$. Since $i + 1 \leq l$ and $\varrho \ll \chi_{i+1}$, there is some $f_{i+1} \in X$ with $f_{i+1} \notin \mathrm{pPOL}_k\, \chi_{i+1}$ and $f_{i+1} \notin Y$ for all $Y \in \mathscr{Y} \cup \{\mathrm{pPOL}_k\, \chi_j \mid 1 \leq j \leq i\}$. Thus, by induction on l, there is some $F := f_l \in X$ with $F \notin Y$ for all $Y \in \mathscr{Y} \cup \mathscr{Z}$. □

Remark 6.3. With the help of $\ll$ we can define a directed graph $\mathcal{G} = (p\mathscr{M}_k, E)$ without loops such that $(X, Y) \notin E$ for all $X, Y \in p\mathscr{M}_k$ with $X = \mathrm{pPOL}_k\, \varrho$, $Y = \mathrm{pPOL}_k\, \psi$ and $\varrho \ll \psi$.

If $X \in p\mathscr{M}_k$ is a sink in $\mathcal{G}$, then X is in every minimal covering of $p\mathscr{M}_k$. Assume this is false. Then there is a minimal covering $\mathscr{Y}$ of $p\mathscr{M}_k$ with $X \notin \mathscr{Y}$ and

$$\forall f \in X \, \exists Y \in \mathscr{Y} : f \in Y.$$

Since X is a sink, i.e., $(X,Y) \notin E$, we have $\varrho \ll \psi$ for all $Y = \mathrm{pPOL}_k\,\psi \in \mathscr{Y}$ and thus by Lemma 6.2 there is some $F \in X$ with $F \notin Y$ for all $Y \in \mathscr{Y}$ contradicting $\mathscr{Y}$ is a covering of $p\mathscr{M}_k$. Thus X is in every minimal covering of $p\mathscr{M}_k$.

If $X \in p\mathscr{M}_k$ is not a sink in the graph $\mathcal{G}$ then X is covered by its successors $U(X) := \{Y \in p\mathscr{M}_k \mid (X,Y) \in E\}$, i.e.,

$$X \subseteq \bigcup_{Y \in U(X)} Y.$$

Assume this is false. Then there is some $f \in X$ with $f \notin X'$ for all $X' \in U(X)$. By Lemma 6.2 there is some $F \in X$ with $F \notin X'$ for all $X' \in U(X)$ and $F \notin Z$ for all $Z \in p\mathscr{M}_k$ with $(X,Z) \notin E$. Thus $F \notin Y$ for all $Y \in \mathscr{X} \setminus \{X\}$. But then $U(X) = \emptyset$ because of the existence of F, i.e., X is a sink. Thus X is covered by $U(X)$.

Then we show in following sections that $\mathcal{G}$ is acyclic. This implies if X is not a sink then X is covered by sinks since $\mathcal{G}$ is transitive and finite, i.e., X is in no minimal covering. Thus there is only one minimal covering.

Definition 6.4. Sometimes we write $\chi \subset \varrho$ to mean $\chi \subset \varrho^{[\pi]}$ for some $\pi \in S_h$. Because $\mathrm{pPOL}_k\,\varrho = \mathrm{pPOL}_k\,\varrho^{[\pi]}$ we can assume $\pi = \mathrm{id}$ in most cases where id is the identity permutation in S_h.

Similarly if we write $\chi \not\subseteq \varrho$ then $\chi \not\subseteq \varrho^{[\pi]}$ for all $\pi \in S_h$.

Lemma 6.5. *Let $\varrho^{(h)} \in \mathcal{A} \cup \mathcal{Q}_0' \cup \mathcal{L}$ and $\chi^{(\mu)} \in (\mathcal{Q} \cup \mathcal{L} \cup \mathcal{S}) \setminus \{\varrho\}$. Then $\varrho \ll \chi$.*

Proof. Let $\sigma := \sigma(\varrho)$ and $\delta := \delta(\varrho)$. Let $f \in \mathrm{pPOL}_k\,\varrho$ be arbitrary. If $f \notin \mathrm{pPOL}_k\,\chi$ then $g := f$ fulfills the conditions of $\varrho \ll \chi$. Thus assume $f \in \mathrm{pPOL}_k\,\chi$.

There are two cases:

$\mu \leq h$ *or* $\chi \in \mathcal{S}$**:**

Let $g_0(\chi) := v$ (see Definition 4.1) for some $v \in \varrho^{[\pi]} \setminus \chi$ if $\chi \subset \varrho^{[\pi]}$ for some $\pi \in S_h$ (w.l.o.g. $\pi = \mathrm{id}$) and $v \in E_k^{\mu} \setminus \chi$ otherwise. Then $g_0 \notin \mathrm{pPOL}_k\,\chi$.

We have to show $g_0 \in \mathrm{pPOL}_k\,\varrho$. Assume $g_0^{(n)} \notin \mathrm{pPOL}_k\,\varrho$. Then there are some rows $c_{1*}, \ldots, c_{h*}$ with $c_{*1}, \ldots, c_{*n} \in \varrho$ and $g_0(c_{*1}, \ldots, c_{*n}) =: d \notin \varrho$. Because of Lemma 5.3 all rows have to be different. Thus if $\mu = h$ then $\{c_{*1}, \ldots, c_{*n}\} \subseteq \chi^{[\pi']}$ for some $\pi' \in S_h$.

There are some cases:

$\mu < h$**:** Because g_0 is only defined on μ different rows Lemma 5.3 applies.

$\mu = h$ *and* $\chi \subset \varrho$**:** Then $\pi' \in \Gamma_{\sigma(\varrho)}$ because $\chi^{[\pi']} \subset \varrho$ and $\chi \subset \varrho$. Thus we have $d = v^{[\pi']} \in \varrho$ because $v \in \varrho$.

$\mu = h$ *and* $\chi \not\subseteq \varrho^{[\pi]}$ *for all* $\pi \in S_h$**:** Thus there is some $j \in \{1, 2, \ldots, n\}$ with $c_{*j} \notin \varrho$ contradicting the assumption.

$\chi \in \mathcal{S}$ *and* $\mu > h$**:** Then $E_k^h = \mathrm{pr}_A\,\iota_k^{\mu} = \{c_{*1}, \ldots, c_{*n}\} \subseteq \varrho$ contradicting that ϱ is coherent.

Thus $g_0 \in \mathrm{pPOL}_k\,\varrho$.

Let $G_0 := f \otimes g_0$ and $L = 0$. By construction $G_0 \notin \mathrm{pPOL}_k\,\chi$.

$\mu > h$ *and* $\chi \notin \mathcal{S}$**:**

Let $\sigma_0 := \sigma$ and define the relations $\sigma_1, \sigma_2, \ldots, \sigma_l, \sigma_{l+1}$ recursively until $\sigma_{l+1} = \emptyset$ and $\emptyset \notin \{\sigma_0, \sigma_1, \ldots, \sigma_l\}$ hold.

Let $\emptyset \subset \sigma_i \subseteq \sigma$ be given. Because ϱ is coherent, there is a relational homomorphism $\varphi_i \colon E_k \to E_h$ from σ_i to $M(\varrho)$ and an $s_i \in \sigma_i$ with $\varphi_i(s_i) = \eta_h$. Let $\sigma_{i+1} := \left\{ s \in \sigma_i \;\middle|\; \varphi_i(s) \in \delta \cap E_h^h \right\}$. From $\varphi_i(s_i) = \eta_h \notin \delta \cap E_h^h$ follows $|\sigma_{i+1}| < |\sigma_i|$. Because $|\sigma|$ is finite there is an $l \in \mathbb{N}$ with $\sigma_{l+1} = \emptyset$.

Define $\varphi_\star \colon E_h \to E_k$ by $\varphi_\star(\eta_h) := s_0$. Then define for $i \in \{0, 1, \dots, l\}$ the function $q_i \colon E_k \to E_k$ with $q_i(x) := \varphi_\star(\varphi_i(x))$. Then q_i is a relational homomorphism from σ_i to ϱ. For $v := (v_0, v_1, \dots, v_{m-1}) \in E_k^m$ and $m \in \{1, 2, \dots, k\}$ let

$$Q_{-1}(v) := \delta^{(m)}_{E_m},$$
$$Q_i(v) := Q_{i-1}(v) \cup \{(x_{v_0}, x_{v_1}, \dots, x_{v_{m-1}}) \in E_k^m \mid \varphi_i(a) = \varphi_i(b) \Longrightarrow x_a = x_b\},$$
$$Q_{l+j}(v) := Q_l(v) \text{ for all } j \geq 1.$$

Because of $|\varphi_i(E_k)| = h$ we have $|\omega(x)| \leq h$ for all $x \in Q_i(\eta_k)$.

Let

$$\forall i \in \{0, 1, 2, \dots, l\} : g_i(\{\eta_k\} \cup Q_{i-1}(\eta_k)) := q_i(\eta_k),$$
$$\forall j \in \{1, 2, \dots, |E_k^k|\} : g_{l+j}(\{\eta_k\} \cup Q_l(\eta_k)) := w_j,$$

where $\{w_1, w_2, \dots, w_{|E_k^k|}\} = E_k^k$. Let $L := l + |E_k^k|$.

We now show $g_i \in \mathrm{pPOL}_k\, \varrho$ for $i \in \{0, 1, \dots, L\}$. Assume $g_i^{(n)} \notin \mathrm{pPOL}_k\, \varrho$. Then there are rows $c_{1*}, \dots, c_{h*}$ with $c_{*1}, \dots, c_{*n} \in \varrho$ and $g_i(c_{*1}, \dots, c_{*n}) =: d \in E_k^h \setminus \varrho$. By construction of g_i we can w.l.o.g. assume that $c' := c_{*1} = \mathrm{pr}_{p_1,\dots,p_h} \eta_k$ with pairwise different coordinates $p_1, \dots, p_h$. Thus $c' \in \sigma(\varrho)$. There are two cases:

$c' \in \sigma_i$: We have $i \leq l$ since $\sigma_{l+1} = \emptyset$. Then $d = g_i(c', c_{*2}, \dots, c_{*n}) = q_i(c') \in \varrho$ because q_i is a relational homomorphism from σ_i to ϱ. This is in contradiction to $d \in E_k^h \setminus \varrho$.

$c' \in \varrho \setminus \sigma_i$: Then there is some $j < i$ such that $c' \in \sigma_j$ and $c' \notin \sigma_{j+1}$ hold. Then $\varphi_j(c') \in \sigma(E_k^h)$. Thus

$$E_k^h = Q_j(c') \subseteq Q_{i-1}(c') = \mathrm{pr}_{p_1,\dots,p_h} Q_{i-1}(\eta_k) = \{c_{*2}, \dots, c_{*n}\} \subseteq \varrho,$$

i.e., $\varrho = E_k^h$ in contradiction to ϱ coherent.

Thus no such c' can exist and therefore $g_i \in \mathrm{pPOL}_k\, \varrho$ for all $i \in \{0, 1, \dots, L\}$.

Let $G_i := f \otimes g_0 \otimes g_1 \otimes \cdots \otimes g_i$ for $i \in \{0, 1, \dots, L\}$.

We show $G_L \notin \mathrm{pPOL}_k\, \chi$. If $g_i \notin \mathrm{pPOL}_k\, \chi$ for some $i \in \{0, 1, \dots, L\}$, then $G_L \notin \mathrm{pPOL}_k\, \chi$. Otherwise $Q_l(v) \subseteq \chi$ for some $v \in \chi$ by construction of $Q_l(v)$ and $g_i \in \mathrm{pPOL}_k\, \chi$ for $i \in \{0, 1, \dots, L\}$. Then $g_{l+j}(\{v\} \cup \delta^{(\mu)}_{E_\mu} \cup Q_l(v)) \in \chi$ for all $j \in \{1, 2, \dots, |E_k^k|\}$ and thus $E_k^\mu \subseteq \chi$ in contradiction to χ coherent.

Let $G_{-1} = f$ and $g = G_L$. Now we show that $g \in \mathrm{pPOL}_k\, \varrho$ by induction over $i \in \{-1, 0, 1, \dots, L\}$.

The basis $G_{-1} = f \in \mathrm{pPOL}_k\, \varrho$ is given by choice of f.

The induction goes from $i - 1$ to i for $i \in \{0, 1, \dots, L\}$. Let $G_{i-1} \in \mathrm{pPOL}_k\, \varrho$. We want to show $G_i = G_{i-1} \otimes g_i \in \mathrm{pPOL}_k\, \varrho$. Let $D := \operatorname{dom} G_{i-1}$, $E := \operatorname{dom} g_i$ and D', E' the associated matrices (see Definition 4.1).

Assume $G_i \notin \mathrm{pPOL}_k\,\varrho$. Then there are some rows $c_{1*}, \dots, c_{h*}$ from $D' \otimes E'$ with

$$F\begin{pmatrix} c_{1*} \\ \vdots \\ c_{h*} \end{pmatrix} =: \begin{pmatrix} s_1 \\ \vdots \\ s_h \end{pmatrix} =: s \notin \varrho \tag{6.1}$$

and

$$\forall j \in \{1, 2, \dots, N\} : c_{*j} \in \varrho. \tag{6.2}$$

By Lemma 5.3 all the rows c_{i*} are pairwise different.

Because $G_{i-1} \in \mathrm{pPOL}_k\,\varrho$ and $g_i \in \mathrm{pPOL}_k\,\varrho$ some rows have to be from the D-part and some from the E-part of $D' \otimes E'$.

Assume w.l.o.g. c_{1*} is from the D-part and c_{h*} is from the E-part. From $\delta(\chi) \neq \emptyset$ and χ coherent follows $\delta^{(\mu)}_{E_\mu} \subseteq \delta(\chi) \subseteq \chi$.

There are three cases:

$\varrho \in \mathcal{A}$: Because $E_k \subseteq c_{h*}$ there is some column j with $c_{1j} = c_{hj}$ in contradiction to ϱ areflexive.

$\varrho \in \mathcal{Q}_0'$: If $h = 2$ then $\{x\} \times E_k \subseteq \varrho$ in contradiction to $\varrho \in \mathcal{Q}_0'$, i.e., ϱ does not fulfill the conditions of Lemma 5.5.

Let $h \geq 3$ and $i \in \{2, \dots, h-1\}$ be arbitrary. Because $c_{1*} \neq c_{i*}$ there is column j with $(c_{1j}, c_{ij}, c_{hj})^{\mathrm{T}} = (x, y, y)^{\mathrm{T}}$ and $x \neq y$. Because i is arbitrary and $\delta(\varrho) = \delta_\varepsilon$ for some equivalence relation ε we get that $(0, a) \notin \varepsilon = \varepsilon(\varrho)$ for all $a \neq 0$, i.e., $\varepsilon(\varrho)$ has a singular equivalence class contradicting $\varrho \in \mathcal{Q}_0$.

$\varrho \in \mathcal{L}$: If c_{1*}, c_{2*}, c_{3*} (which are pairwise different) are from D and is c_{4*} from E, then there is there is a column $c_{*j} = (x, y, z, w)^{\mathrm{T}} \notin \varrho$ with $|\{x, y, z\}| \geq 2$ and $|\{x, y, z, w\}| = 3$ in contradiction to (6.2). Otherwise there is some column $c_{*j} = (x, y, y, y)^{\mathrm{T}} \notin \varrho$ with $x \neq y$ contradicting (6.2).

Thus $G_i \in \mathrm{pPOL}_k\,\varrho$ and by induction $g = G_L \in \mathrm{pPOL}_k\,\varrho$. Because $g \notin \mathrm{pPOL}_k\,\chi$ we get $\varrho \ll \chi$. □

Lemma 6.6. *Let $\varrho^{(h)} \in \mathcal{Q}_1 \cup \mathcal{S}'$ and $\chi^{(\mu)} \in \mathcal{S}$. Then $\varrho \ll \chi$.*

Proof. Let $f \in \mathrm{pPOL}_k\,\varrho$ be arbitrary. If $f \notin \mathrm{pPOL}_k\,\chi$ then $g := f$ fulfills the conditions of $\ll$. Thus assume $f \in \mathrm{pPOL}_k\,\chi$.

Let $g_\chi(\chi) := v$ (see Definition 4.1) for some $v \in \varrho \setminus \chi$ if $\chi \subseteq \varrho$ and $v \in E_k^\mu \setminus \chi$ otherwise. Then $g_\chi \in \mathrm{pPOL}_k\,\varrho$ and let $g := f \otimes g_\chi$. We get $g \notin \mathrm{pPOL}_k\,\chi$ because $g_\chi \notin \mathrm{pPOL}_k\,\chi$. Let $D := \operatorname{dom} f$ and $E := \operatorname{dom} g_\chi$ as in Definition 4.2, $n = |D|$, $m = |E|$ and $N = |D| \cdot |E|$.

It suffices to show $F^{(N)} := (f \otimes g_\chi) \in \mathrm{pPOL}_k\,\varrho$. Assume this is false. Then there are some rows $c_{1*}, \dots, c_{h*}$ from $D \otimes E$ with $s_i := F(c_{i*})$, $s := (s_1, \dots, s_h)^{\mathrm{T}} \notin \varrho$ and $c_{*j} \in \varrho$ for all $j \in \{1, 2, \dots, N\}$. By Lemma 5.3 all the rows c_{i*} are pairwise different.

If all rows c_{i*} are from the D-part of $D \otimes E$, then $f(c_{*1m}, c_{*2m}, \dots, c_{*nm}) = s \notin \varrho$ in contradiction to $f \in \mathrm{pPOL}_k\,\varrho$. Thus one row is from the E-part.

Assume there are rows from both parts of $D \otimes E$. Then w.l.o.g. ϱ is given such that c_{1*} is from the D-part and c_{h*} is from the E-part. Let $c_{i_1*}, \dots, c_{i_q*}$ with $q < h$ the rows of the E-part. Then $\delta^{(q)}_{E_q} \subseteq (c_{ij})_{i=i_1,\dots,i_q, j=1,\dots,N}$.

There are two cases:

$\varrho \in \mathcal{Q}_1$ **:** Let $c_{i_1*}, c_{i_2*}, c_{h*}$ be three pairwise different rows. Then there are columns j_1 and j_2 with $(c_{i_1j_1}, c_{i_2j_1}, c_{hj_1})^{\mathrm{T}} = (x, y, x)^{\mathrm{T}}$, $(c_{i_1j_2}, c_{i_2j_2}, c_{hj_2})^{\mathrm{T}} = (x, y, y)^{\mathrm{T}}$ and $x \neq y$. Thus $(i_1-1, i_2-1), (i_1-1, h-1), (i_2-1, h-1) \notin \varepsilon(\varrho)$. Because $i_1 \neq i_2 \neq h$ are arbitrary it follows that $\delta(\varrho) = \delta^{(h)}_{\iota^2_h} = E^h_k$ in contradiction to ϱ coherent.

$\varrho \in \mathcal{S}'$ **:** If only c_{1*} is from D, then $\{x\} \times E^2_k \subseteq \varrho$ for some $x \in c_{1*}$, and thus $(x, y)^{\mathrm{T}} \times E_k \subseteq \varrho$ with $x \neq y$. If only c_{h*} is from E then $(x, y)^{\mathrm{T}} \times E_k \subseteq \varrho$ with $x \neq y$, $x \in c_{1*}$ and $y \in c_{2*}$. Thus Lemma 5.4 applies contradicting $\varrho \in \mathcal{S}'$. □

Lemma 6.7. *Let $\varrho^{(h)} \in \mathcal{Q}_1$ and $\chi^{(\mu)} \in \mathcal{Q}_\varrho$. Then $\varrho \ll \chi$.*

Proof. Let $f \in \mathrm{pPOL}_k\, \varrho$ be arbitrary. If $f \notin \mathrm{pPOL}_k\, \chi$ then $g := f$ fulfills the conditions of $\ll$. Thus assume $f \in \mathrm{pPOL}_k\, \chi$.

Let $g_\chi(\chi) := v$ (see Definition 4.1) for some $v \in E^\mu_k \setminus \chi$ and let $g := f \otimes g_\chi$. We get $g \notin \mathrm{pPOL}_k\, \chi$ because $g_\chi \notin \mathrm{pPOL}_k\, \chi$. It suffices to show

$$F^{(N)} := (f \otimes g_\chi) \in \mathrm{pPOL}_k\, \varrho. \tag{6.3}$$

Let $D := \operatorname{dom} f$ and $E := \operatorname{dom} g$ as in Definition 4.2, $n = |D|$, $m = |E|$ and $N = |D| \cdot |E|$.

Assume (6.3) is false. Then there are $c_{1*}, \ldots, c_{h*}$ from $D \otimes E$ with $s_i := F(c_{i*})$, $s := (s_1, \ldots, s_h)^{\mathrm{T}} \notin \varrho$ and $c_{*j} \in \varrho$ for all $j \in \{1, 2, \ldots, N\}$. By Lemma 5.3 all the rows c_{i*} are pairwise different.

If all rows c_{i*} are from the D-part of $D \otimes E$, then $f(c_{*1m}, c_{*2m}, c_{*nm}) = s \notin \varrho$ in contradiction to $f \in \mathrm{pPOL}_k\, \varrho$. Thus at least one row is from the E-part.

Let $l := \|\varrho\|$. Let w.l.o.g. $\mathrm{pp}\, \varrho$ be the first l rows of ϱ. Assume there is some row c_{i_1*} with $1 \leq i_1 \leq l$ such that c_{i_1*} is not from the part of E representing $\mathrm{pp}\, \chi$. Let $i_2 \neq i_1$ with $1 \leq i_2 \leq l$ be arbitrary. Then there are columns j_1 and j_2 with $(c_{i_1j_1}, c_{i_2j_1}, c_{hj_1})^{\mathrm{T}} = (x, y, x)^{\mathrm{T}}$, $(c_{i_1j_2}, c_{i_2j_2}, c_{hj_2})^{\mathrm{T}} = (x, y, y)^{\mathrm{T}}$ and $x \neq y$. Thus $(i_2 - 1, i_1 - 1) \notin \varepsilon(\varrho)$, i.e., there is a singleton class in $\varepsilon(\mathrm{pp}\, \varrho)$ in contradiction to $\mathrm{pp}\, \varrho \in \mathcal{Q}_0$.

So we need $\mathrm{pp}\, \chi \subseteq \mathrm{pp}\, \varrho$ in contradiction to the definition of $\mathcal{Q}_\varrho$. □

Definition 6.8. Let $f \in \widetilde{P}^{(1)}_k$ be a unary function. Then we define recursively $f^0 := e^{(1)}_1$ and $f^n := f(f^{n-1})$ for all $n \geq 1$.

For the proof of Theorem 6.13 some lemmas are needed using the following condition on $\varrho \in \mathcal{A}$.

$$\begin{aligned} &\exists \varphi \in \mathrm{Pol}^{(1)}_k\, \varrho\, \forall l \in \{1, 2, \ldots, h-1\}\, \forall D \subseteq \sigma(E^l_k)\, \forall v \in \sigma(E^{h-l}_k) \\ &\forall \pi \in S_h\, \exists m \geq 0 : D \times \{\varphi^m(v)\} \not\subseteq \varrho^{[\pi]}. \end{aligned} \tag{6.4}$$

Proposition 6.9. *Let $\varrho^{(h)} \in \mathcal{A}$ and ϱ fulfills* (6.4)*. Then there is some $\varphi' \in \mathrm{Pol}^{(1)}_k\, \varrho$ which suffices the conditions in* (6.4) *and $\varphi' \notin \mathrm{pPOL}_k\{x\}$ for all $x \in E_k$.*

Proof. There is some $\varphi \in \mathrm{Pol}^{(1)}_k\, \varrho$ which fulfills (6.4). Let

$$\varphi'(x) = \begin{cases} y & \text{for some } y \in E_k \setminus \{x\}, \text{ if } x \in E_k \setminus \omega(\varrho), \\ \varphi(x) & \text{otherwise.} \end{cases}$$

Let $x \in E_k \setminus \omega(\varrho)$. If $x \in \omega(D) \cup \omega(v)$ then $D \times \{(\varphi')^0(v)\} = D \times \{v\} \nsubseteq \varrho^{[\pi]}$ for all π. Thus φ' fulfills the conditions of (6.4) because it coincides with φ on $\omega(\varrho)$.

Let $x \in \omega(\varrho)$. Then there is some $D \subseteq \sigma(E_k^{h-1})$ with $D \times \{x\} \subseteq \varrho^{[\pi]}$ for some $\pi \in S_h$. But there is some $m \geq 0$ with $D \times \{(\varphi')^m(x)\} \nsubseteq \varrho^{[\pi]}$ because of (6.4). Thus $\varphi'(x) \neq x$. For $x \notin \omega(\varrho)$ follows $\varphi'(x) \neq x$ by definition of φ'. □

Lemma 6.10. *Let $\varrho^{(h)} \in \mathcal{A}$ and ϱ fulfills* (6.4). *Then* $\mathrm{pPOL}_k\,\varrho$ *is in every minimal covering of* $\mathrm{p}\mathscr{M}_k$.

Proof. Because of Lemma 6.5 we just have to find a function $f \in \mathrm{pPOL}_k\,\varrho$ with

$$\forall \chi \in (\mathcal{U} \cup \mathcal{A}) \setminus \{\varrho\} : f \notin \mathrm{pPOL}_k\,\chi.$$

Then there is some function $g \in \mathrm{pPOL}_k\,\varrho$ with

$$\forall \chi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \{\varrho\} : g \notin \mathrm{pPOL}_k\,\chi$$

and by construction also $g \notin P_k \cup C_\emptyset$. Thus $\mathrm{pPOL}_k\,\varrho$ is in every minimal covering of $p\mathscr{M}_k$ by Lemma 3.2.

We will now construct the function f mentioned above.

We can assume $\pi = \mathrm{id} \in S_h$ in (6.4) because $\mathrm{pPOL}_k\,\varrho = \mathrm{pPOL}_k\,\varrho^{[\pi]}$. Because of Proposition 6.9 we can assume $\varphi \notin \mathrm{pPOL}_k\{x\}$ for all $x \in E_k$.

Let $f_0 := \varphi$ and define $f_j := f_{j-1} \otimes f_{\chi_j}$ recursively with

$$X := \{\chi_1, \ldots, \chi_N\} := \{\chi \in \mathcal{U} \cup \mathcal{A} \mid \varphi \in \mathrm{pPOL}_k\,\chi\}.$$

Let $\chi^{(\mu)} = \chi_j \in X$. There are two cases:

$\mu \leq h$**:** Let $f_\chi(\chi) := z$ (see Definition 4.1) with $z \in \varrho \setminus \chi$ if $\chi \subset \varrho$ and $z \in E_k^\mu \setminus \chi$ otherwise. Then $f_\chi \notin \mathrm{pPOL}_k\,\chi$ and by construction $f_\chi \in \mathrm{pPOL}_k\,\varrho$.

$\mu > h$**:** Because ϱ is coherent there is a relational homomorphism $\theta_0 \colon E_k \to E_k$ from ϱ to $M(\varrho)$. Let $\theta_1 \colon E_h \to E_k$ with $\theta_1(\eta_h) = v_1$ for some $v_1 \in \varrho$. Then $\theta \in \mathrm{pPOL}_k\,\varrho$ for $\theta \colon E_k \to E_k$ with $\theta(x) = \theta_1(\theta_0(x))$.

Let $v \in \chi$ be arbitrary and let $f_\chi(\chi) := \theta(v)$ (see Definition 4.1). Because $|\omega(\theta(v))| \leq h < \mu$ we have $\theta \notin \mathrm{pPOL}_k\,\chi$, and thus $f_\chi \notin \mathrm{pPOL}_k\,\chi$.

By construction θ is a relational homomorphism from ϱ to ϱ. Thus $f_\chi \in \mathrm{pPOL}_k\,\varrho$.

Because $f_j = f_{j-1} \otimes f_{\chi_j}$ and $f_{\chi_j} \notin \mathrm{pPOL}_k\,\chi_j$ we get $f_j \notin \mathrm{pPOL}_k\,\chi_j$.

Assume $f_j^{(n)} \notin \mathrm{pPOL}_k\,\varrho$. Then there are rows $c_{1*}, \ldots, c_{h*}$ with $c_{*1}, \ldots, c_{*n} \in \varrho$ and $f(c_{*1}, \ldots, c_{*n}) = d \in E_k^h \setminus \varrho$. Then the rows c_{i*} are pairwise different and some rows belong to the f_{j-1} part of f_j and some to the f_χ part. The rows can w.l.o.g. be sorted in a way such that the first l rows for some $l \in \{1, 2, \ldots, h-1\}$ are from the f_{j-1} part of $f_j = f_{j-1} \otimes f_{\chi_j}$.

Let $D := \mathrm{pr}_{0,\ldots,l-1}\{c_{*1}, \ldots, c_{*n}\}$ and $W := \mathrm{pr}_{l,\ldots,h-1}\{c_{*1}, \ldots, c_{*n}\}$. Because the rows $c_{l*}, c_{l+1*}, \ldots, c_{h-1*}$ are from the f_χ part and f_χ is only defined on χ we get $W = \mathrm{pr}_{p_l, p_{l+1}, \ldots, p_{h-1}}\,\chi$ for pairwise different p_i. Let $v \in W \subseteq E_k$ be arbitrary. Then there is some $v' \in \chi$ with $v = \mathrm{pr}_{p_l, p_{l+1}, \ldots, p_{h-1}}\,v'$. Thus $\{\varphi^m(v) \mid m \geq 0\} \subseteq W$ because $\varphi \in \mathrm{pPOL}_k\,\chi$, i.e., $\{\varphi^m(v') \mid m \geq 0\} \subseteq \chi$.

But then $D \times \{\varphi^m(v) \mid m \geq 0\} \subseteq \varrho$ in contradiction to (6.4). □

Let $\Gamma' \subseteq S_h$ and $l \leq h$. Then we define $\Gamma'_{|E_l} \subseteq S_l$ by

$$\Gamma'_{|E_l} := \{\pi \in S_l \mid \exists \pi' \in \Gamma' : (\forall x \in E_l : \pi(x) = \pi'(x)) \wedge (\forall x \in E_h \setminus E_l : \pi'(x) = x)\}.$$

Lemma 6.11. *Let $\varrho^{(h)} \in \mathcal{A}$, $\bar{\chi}^{(l)} \subseteq E_k^l$, $V \subseteq E_k^{h-l}$, $l \in \{1, \ldots, h-1\}$ and $\chi \times V \subseteq \varrho$. Let $\Gamma' := \{\pi \in \Gamma_\sigma \mid \forall x \in E_h \setminus E_l : \pi(x) = x\}$ and $\chi' := \{c^{[\pi]} \mid c \in \bar{\chi}, \pi \in \Gamma'_{|E_l}\}$.*

Then χ' is coherent and $\chi' \times V \subseteq \varrho$.

Proof. From the definition of Γ' and ϱ coherent follows $\chi' \times V \subseteq \varrho$.

We now show that χ' is coherent.

- If $l \geq 2$ then $\chi' \neq E_k^l$ because $\chi' \subseteq \sigma(E_k^l) \subset E_k^l$.

  Let $l = 1$ and $\chi' = E_k^l = E_k$.

  If $h = 2$ then $V \neq E_k$ because otherwise $E_k^2 \subseteq \varrho$ contradicting ϱ coherent. Let V' with $V \subseteq V' \subset E_k$ be maximal with respect to inclusion such that $\chi' \times V' \subseteq \varrho$. Because $f \notin \mathrm{pPOL}_k V'$ there are $b_1, \ldots, b_n \in V'$ and $y \in E_k \setminus V'$ with $f(b_1, \ldots, b_n) = y$. Then there is some $x \in E_k$ such that $(x, y)^{\mathrm{T}} \notin \varrho$ and there are $a_1, \ldots, a_n \in E_k \setminus \{x\}$ with $f(a_1, \ldots, a_n) = x$ because $f \notin \mathrm{pPOL}_k(E_k \setminus \{x\})$ and $E_k \setminus (E_k \setminus \{x\}) = \{x\}$. Thus

  $$f \begin{pmatrix} a_1 & \ldots & a_n \\ b_1 & \ldots & b_n \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \notin \varrho$$

  but $(a_i, b_i)^{\mathrm{T}} \in \varrho$ for all $i \in \{1, \ldots, n\}$ contradicting $f \in \mathrm{pPOL}_k \varrho$.

  If $h \geq 3$ then $\begin{pmatrix} x \\ w \end{pmatrix} \notin \varrho$ for all $w \in V \subseteq \sigma(E_k^{h-l})$ and $x \in \omega(w)$.

  Thus $\chi' \neq E_k^l$.
- $\chi' \subseteq \sigma(E_k^l)$, i.e., χ' is areflexive and $1 \leq l < k$,
- $r^{[\pi]} \in \chi'$ for all $r \in \chi'$ and $\pi \in \Gamma_{\chi'}$ because $\pi \in \Gamma'$ for any $\pi \in \Gamma_{\chi'}$.
- $M(\chi') = \{\eta_l^{[\pi]} \mid \pi \in \Gamma_{\chi'} = \Gamma'_{|E_l}\}$. Let ψ with $\emptyset \subset \psi \subseteq \chi'$ and $w \in V$ be arbitrary. Because ϱ is coherent there exists a relational homomorphism $\lambda\colon E_k \to E_h$ from $\psi \times \{w\}$ to $M(\varrho)$ with

  $$\lambda \begin{pmatrix} c \\ w \end{pmatrix} = \eta_h,$$

  i.e., $\lambda(c) = \eta_l$, for some $c \in \psi$. For any $c' \in \psi$ we have

  $$\lambda \begin{pmatrix} c' \\ w \end{pmatrix} \in M(\varrho) = \{\eta_h^{[\pi]} \mid \pi \in \Gamma_\sigma\}$$

  and because $\lambda(w) = (l, \ldots, h-1)^{\mathrm{T}}$ we get

  $$\lambda \begin{pmatrix} c' \\ w \end{pmatrix} \in \{\eta_h^{[\pi]} \mid \pi \in \Gamma'\}$$

  and thus $\lambda(c') \in M(\chi')$.

  Let $\lambda'\colon E_k \to E_l$ be defined by

  $$\lambda'(x) := \begin{cases} \lambda(x) & \text{if } x \in \omega(\chi'), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\lambda'\colon E_k \to E_l$ is a relational homomorphism from χ' to $M(\chi')$ such that $\lambda'(c) = \eta_l$ for some $c \in \chi'$.

Thus χ' is coherent, i.e., $\chi' \in \mathcal{U} \cup \mathcal{A}$. $\square$

Lemma 6.12. *Let $\varrho^{(h)} \in \mathcal{A}$ and ϱ does not fulfill* (6.4). *Then* $\mathrm{pPOL}_k\,\varrho$ *is in no minimal covering of $\mathrm{p}\mathscr{M}_k$.*

Proof. Let $f^{(n)} \in \mathrm{pPOL}_k\,\varrho$ be arbitrary with

$$\forall \chi^{(\mu)} \in \mathcal{U} \cup \mathcal{A} : (\mu < h \Longrightarrow f \notin \mathrm{pPOL}_k\,\chi)\,. \tag{6.5}$$

Let $\varphi(x) := f(x, \ldots, x)$. Then $\varphi \in \mathrm{pPOL}_k^{(1)}\,\varrho$ and by (6.5) we have $\varphi(x) \in E_k \setminus \{x\}$ for all $x \in E_k$, specifically $\varphi \in \mathrm{Pol}_k^{(1)}\,\varrho$. Because (6.4) is false, there exist l, $D = \{D_{*1}, \ldots, D_{*|D|}\} \subseteq \sigma(E_k^l)$, $v \in \sigma(E_k^{h-l})$ and $\pi \in S_h$ with $0 < l < h$ and

$$\forall m \geq 0 : D \times \{\varphi^m(v)\} \subseteq \varrho^{[\pi]}.$$

Because $\mathrm{pPOL}_k\,\varrho = \mathrm{pPOL}_k\,\varrho^{[\pi]}$ we assume w.l.o.g. $\pi = \mathrm{id}$, i.e.,

$$\forall m \geq 0 : D \times \{\varphi^m(v)\} \subseteq \varrho. \tag{6.6}$$

We show $\varphi^m(v) \in \sigma(E_k^{h-l})$ for all $m \geq 0$. Assume otherwise. Then

$$\begin{pmatrix} D_{*1} \\ \varphi^m(v) \end{pmatrix} \in \iota_k^h$$

for some m, but this contradicts (6.6) because $\varrho \subseteq \sigma(E_k^h)$.

Because $\sigma(E_k^{h-l})$ is finite, there are $0 \leq m_1 < m_2$ such that $\varphi^{m_1}(v) = \varphi^{m_2}(v)$. Let $V := \{\varphi^{m_1+m}(v) \mid m \geq 0\}$. Then for any $w \in V$ there is some $w' \in V$ with $\varphi(w') = w$.

Let $\chi \in \left\{\psi^{(\mu)} \in \mathcal{U} \cup \mathcal{A} \;\middle|\; \mu = l\right\}$ with $\chi \times V \subseteq \varrho$. Then there are rows $c_{1*}, \ldots, c_{l*}$ with $c_{*1}, \ldots, c_{*n} \in \chi$ and $f(c_{*1}, \ldots, c_{*n}) =: d \in E_k^l \setminus \chi$.

Let $w' \in V$ arbitrary and $w = \varphi(w') \in V$. Then $\chi \times \{w'\} \subseteq \varrho$ and

$$f\begin{pmatrix} c_{*1} & \ldots & c_{*n} \\ w' & \ldots & w' \end{pmatrix} = \begin{pmatrix} d \\ w \end{pmatrix} \in E_k^h,$$

i.e.,

$$\begin{pmatrix} d \\ w \end{pmatrix} \in \varrho$$

because $f \in \mathrm{pPOL}_k\,\varrho$. Thus $(\chi \cup \{d\}) \times V \subseteq \varrho$. This also implies $\chi \cup \{d\} \subseteq \sigma(E_k^l)$ as shown before.

Let $\Gamma' := \{\pi \in \Gamma_\sigma \mid \forall x \in E_h \setminus E_l : \pi(x) = x\}$ and $\chi' := \{c^{[\pi]} \mid c \in \chi \cup \{d\}, \pi \in \Gamma'_{|E_l}\}$.

By Lemma 6.11 with $\bar{\chi} = \chi \cup \{d\}$ we get χ' coherent, i.e., $\chi' \in \mathcal{U} \cup \mathcal{A}$, and $\chi \subset \chi'$ with $\chi' \times V \subseteq \varrho$.

Now let $\chi_0 := \{D_{*1}\}$ then χ_0 is coherent and $\chi_0 \times V \subseteq \varrho$. By the argument above there is an infinite chain $\chi_0 \subset \chi_1 \subset \chi_2 \subset \ldots$ with $\chi_i \in \mathcal{U} \cup \mathcal{A}$ and $\chi_i \times V \subseteq \varrho$ for all $i \in \mathbb{N}$. But this contradicts $|\mathcal{U} \cup \mathcal{A}| < \infty$ and thus the assumption (6.5) is wrong.

Thus for any $f \in \mathrm{pPOL}_k\,\varrho$ there is some $\chi^{(\mu)} \in \mathcal{U} \cup \mathcal{A}$ with $\mu < h$ and $f \in \mathrm{pPOL}_k\,\chi$. By induction there is some $\psi^{(\mu')} \in \mathcal{U} \cup \mathcal{A}$ with $\mu' \leq \mu$, $f \in \mathrm{pPOL}_k\,\psi$ and $\mathrm{pPOL}_k\,\psi$ is in every minimal covering of $p\mathscr{M}_k$.

Thus $\mathrm{pPOL}_k\,\varrho$ is in no minimal covering of $p\mathscr{M}_k$. □

Theorem 6.13. *Let $\varrho^{(h)} \in \mathcal{A}$. Then $\mathrm{pPOL}_k\,\varrho$ is in any minimal covering of $p\mathscr{M}_k$ if and only if ϱ fulfills* (6.4)*.*

Proof. If ϱ fulfills (6.4) then $\mathrm{pPOL}_k\,\varrho$ is in every minimal covering by Lemma 6.10. If ϱ does not fulfill (6.4) then $\mathrm{pPOL}_k\,\varrho$ is no minimal covering by Lemma 6.12. □

7. Uniqueness of minimal coverings

Lemma 7.1. *Let $\mathscr{X}, \mathscr{Y}$ be two different minimal coverings of $p\mathscr{M}_k$. Then $\mathrm{pPOL}_k\,\varrho \in \mathscr{X}$ if and only if $\mathrm{pPOL}_k\,\varrho \in \mathscr{Y}$ for all $\varrho \in \mathcal{U} \cup \mathcal{A}$.*

Proof. By Lemma 5.2 we have $\mathrm{pPOL}_k\,\varrho \in \mathscr{X}$ and $\mathrm{pPOL}_k\,\varrho \in \mathscr{Y}$ for all $\varrho \in \mathcal{U}$. By Theorem 6.13

$$\forall \varrho \in \mathcal{A}\,(\mathrm{pPOL}_k\,\varrho \in \mathscr{X} \iff \mathrm{pPOL}_k\,\varrho \in \mathscr{Y}).$$

□

Lemma 7.2. *Let $\mathscr{X}, \mathscr{Y}$ be two different minimal coverings of $p\mathscr{M}_k$. Then $\mathrm{pPOL}_k\,\varrho \in \mathscr{X}$ if and only if $\mathrm{pPOL}_k\,\varrho \in \mathscr{Y}$ for all $\varrho \in \mathcal{Q}_0 \cup \mathcal{L}$.*

Proof. Assume this is false. Then there exists some $\varrho \in \mathcal{Q}_0 \cup \mathcal{L}$ such that $X := \mathrm{pPOL}_k\,\varrho \in \mathscr{X} \setminus \mathscr{Y}$. Because X is in some minimal covering of $p\mathscr{M}_k$ we obtain $\varrho \in \mathcal{Q}_0' \cup \mathcal{L}$. By Lemma 6.5 we have

$$\mathscr{Z} := \{\mathrm{pPOL}_k\,\psi \mid \psi \in \widetilde{\mathcal{R}}_k^{\max} \wedge \varrho \ll \psi\} \supseteq \{\mathrm{pPOL}_k\,\psi \mid \psi \in (\mathcal{Q} \cup \mathcal{L} \cup \mathcal{S}) \setminus \{\varrho\}\}.$$

Since $\mathscr{X}$ is a minimal covering there exists some $f \in X$ with $f \notin X'$ for all $X' \in \mathscr{X} \setminus \{X\}$. By Lemma 6.2 there is some $F \in X$ with $F \notin Y$ for all $X' \in \mathscr{X} \cup \mathscr{Z}$ and $X' \neq X$. Since $\mathscr{Y}$ is a covering there is some $Y \in \mathscr{Y}$ with $F \in Y$. But then $Y \in \mathscr{Y} \setminus (\mathscr{X} \cup \mathscr{Z}) = (\mathscr{Y} \setminus \mathscr{Z}) \setminus \mathscr{X} = (\mathscr{Y} \cap \{\mathrm{pPOL}_k\,\chi \mid \chi \in \mathcal{U} \cup \mathcal{A}\}) \setminus \mathscr{X} = \emptyset$ by Lemma 7.1. This is a contradiction. □

Lemma 7.3. *Let $\mathscr{X} \subseteq p\mathscr{M}_k$ be a minimal covering of $p\mathscr{M}_k$. Then $\mathrm{pPOL}_k\,\varrho \notin \mathscr{X}$ for all $\varrho \in \mathcal{S} \setminus \mathcal{S}'$.*

Proof. Assume $X := \mathrm{pPOL}_k\,\varrho \in \mathscr{X}$ for some $\varrho \in \mathcal{S} \setminus \mathcal{S}'$. Then there is some $f \in X$ with $f \notin Y$ for all $Y \in \mathscr{X} \setminus \{X\}$. Applying Lemma 5.4 recursively on X implies $f \in \mathrm{pPOL}_k\,\chi$ for some $\chi \in \mathcal{U} \cup \mathcal{Q} \cup \mathcal{S}'$. By Lemmas 6.5 and 6.6 there is some $g \in \mathrm{pPOL}_k\,\chi$ with $g \notin Y$ for all $Y \in \mathscr{X}$ in contradiction to $\mathscr{X}$ minimal covering. □

Lemma 7.4. *Let $\mathscr{X}, \mathscr{Y}$ be two different minimal coverings of $p\mathscr{M}_k$. Then $\mathrm{pPOL}_k\,\varrho \in \mathscr{X}$ if and only if $\mathrm{pPOL}_k\,\varrho \in \mathscr{Y}$ for all $\varrho \in \mathcal{S}$.*

Proof. Assume this is false. Then there is some $\varrho \in \mathcal{S}$ with $X := \mathrm{pPOL}_k\, \varrho \in \mathscr{X} \setminus \mathscr{Y}$. In particular is $\varrho \in \mathcal{S}'$ by Lemma 7.3. Then there is some $f \in X$ with $f \notin X'$ for all $X' \in \mathscr{X} \setminus \{X\}$.

Then $f \notin Y$ for all $Y \in \mathscr{Y}$ with $Y = \mathrm{pPOL}_k\, \chi$ and $\chi \in \mathcal{U} \cup \mathcal{A} \cup \mathcal{Q}_0 \cup \mathcal{L}$ by Lemmas 7.1 and 7.2. Thus there is some $Z \in \mathscr{Y}$ with $Z = \mathrm{pPOL}_k\, \psi$ and $\psi \in \mathcal{Q}_1 \cup \mathcal{S}'$ and $f \in Z$. By Lemma 6.6 there is some $g \in Z$ with $g \notin X$ and $g \notin X'$ for all $X' \in \mathscr{X} \setminus \{X\}$, i.e., $g \notin X'$ for all $X' \in \mathscr{X}$. This contradicts $\mathscr{X}$ minimal covering because $g \in Z \in \mathscr{Y}$. □

Theorem 7.5. *Let $\mathscr{X}, \mathscr{Y}$ be two different minimal coverings of $\mathrm{p}\mathscr{M}_k$. Then $\mathscr{X} \setminus \mathscr{Y} \subseteq \{\mathrm{pPOL}_k\, \psi \mid \psi \in \mathcal{Q}_1\}$.*

Proof. The theorem follows from Lemmas 7.1, 7.2 and 7.4, and Lemma 5.2 for the partial clone $P_k \cup C_\emptyset$. □

Lemma 7.6. *Let $\mathscr{X}, \mathscr{Y}$ be different minimal coverings of $\mathrm{p}\mathscr{M}_k$. Furthermore let $X := \mathrm{pPOL}_k\, \varrho \in \mathscr{X} \setminus \mathscr{Y}$ for some $\varrho \in \mathcal{Q}_1$. Then there is some $\chi \in \mathcal{Q}_1$ with $Y := \mathrm{pPOL}_k\, \chi \in \mathscr{Y} \setminus \mathscr{X}$ and $\mathrm{pp}\, \chi = \mathrm{pp}\, \varrho$.*

Proof. By $\mathscr{X} \neq \mathscr{Y}$ and Theorem 7.5 we have $\emptyset \subset \mathscr{X} \setminus \mathscr{Y} \subseteq \{\mathrm{pPOL}_k\, \psi \mid \psi \in \mathcal{Q}_1\}$. Let $X := \mathrm{pPOL}_k\, \varrho \in \mathscr{X} \setminus \mathscr{Y}$ be arbitrary with $\varrho \in \mathcal{Q}_1$. Then there is some $f \in X$ with $f \notin X'$ for all $X' \in \mathscr{X} \setminus \{X\}$. Then $f \in Y$ with $Y := \mathrm{pPOL}_k\, \chi \in \mathscr{Y} \setminus \mathscr{X}$ for some $\chi \in \mathcal{Q}_1$.

Assume $\mathrm{pp}\, \chi \neq \mathrm{pp}\, \varrho$. Then $\chi \in \mathcal{Q}_\varrho$ or $\varrho \in \mathcal{Q}_\chi$. If $\chi \in \mathcal{Q}_\varrho$ then there is some $g \in X$ with $g \notin X'$ for all $X' \in \mathscr{X} \setminus \{X\}$ and $g \notin \mathrm{pPOL}_k\, \chi$ by Lemma 6.7. Thus $\varrho \in \mathcal{Q}_\chi$ has to be true. But then there is some $G \in Y = \mathrm{pPOL}_k\, \chi$ with $G \notin X'$ for all $X' \in \mathscr{X}$ again by Lemma 6.7 contradicting $\mathscr{X}$ minimal covering. □

Definition 7.7. Let $\varrho^{(h)} \in \mathcal{Q}_1$. We call ϱ *irreducible* iff

$$\forall \emptyset \subset A \subset E_h\, \forall v \in \sigma(E_k^{h-|A|})\, \forall \pi \in S_h : (\mathrm{pr}_A\, \varrho) \times \{v\} \not\subseteq \varrho^{[\pi]}.$$

Otherwise we call it *reducible.*

Example 7.8. Let $k = 4$ and $h = 3$. Let

$$\varrho = \begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 2 & 3 \end{pmatrix} \cup \delta^{(3)}_{\{0,1\}}.$$

We show that ϱ is irreducible. There are three cases:

$|A| = 1$: Then $\mathrm{pr}_A\, \varrho = E_4$ and $v = (v_1, v_2) \in \sigma(E_4^2)$. Assume $(\mathrm{pr}_A\, \varrho) \times \{v\} \subseteq \varrho^{[\pi]}$. Then $(v_1, v_1, v_2), (v_2, v_1, v_2) \in \varrho^{[\pi]}$. Thus $\delta^{(3)}_{\{0,1\}} \cup \delta_{\{0,2\}} \subseteq \varrho$ because ϱ coherent. But this contradicts $\varrho \in \mathcal{Q}_1$.

$A = \{0, 1\}$: If $\pi \neq \mathrm{id}$ then $\delta^{(3)}_X \subseteq \varrho$ with $X \subset E_3$, $|X| = 2$ and $X \neq \{0, 1\}$ in contradiction to $\varrho \in \mathcal{Q}_1$. Thus $\pi = \mathrm{id}$.

Because for all $x \in E_k$

$$\begin{pmatrix} 0 & 0 \\ 1 & 2 \\ x & x \end{pmatrix} \not\subseteq \varrho \quad \text{and} \quad \mathrm{pr}_A\, \varrho = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \cup \delta^{(2)}_{\{0,1\}},$$

we get $(\mathrm{pr}_A\,\varrho) \times \{v\} \not\subseteq \varrho$.

$|A| = 2$ *and* $A \neq \{0,1\}$**:** Then $\mathrm{pr}_A\,\varrho = E_4^2$. Let $v = (x)$. Assume that the inclusion $(\mathrm{pr}_A\,\varrho) \times \{v\} \subseteq \varrho^{[\pi]}$ holds. Then $(x,y,x),(y,x,x),(y,y,x) \in \varrho^{[\pi]}$ and thus we have $\iota_4^3 \subseteq \varrho^{[\pi]}$ because ϱ is coherent. But this contradicts $\varrho \in \mathcal{Q}_1$.

Thus ϱ is irreducible.

Now let

$$\varrho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 2 \end{pmatrix} \cup \delta^{(3)}_{\{0,1\}}.$$

Then ϱ is reducible because

$$(\mathrm{pr}_A\,\varrho) \times \{v\} = \begin{pmatrix} 0 & 1 & 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 1 & 2 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix} \subseteq \varrho = \varrho^{[\pi]}$$

holds with $A = \{0,1\}$, $v = (2)$ and $\pi = \mathrm{id}$.

Lemma 7.9. *Let $\varrho^{(h)} \in \mathcal{Q}_1$ be reducible. Then for every $f \in \mathrm{pPOL}_k\,\varrho$ there is some*

$$\chi \in \mathcal{X}_\varrho := \{\{a\} \mid a \in E_k\} \cup \{\psi^{(\mu)} \in \mathcal{Q} \mid \mathrm{pp}\,\psi = \mathrm{pp}\,\varrho \wedge \mu < h\}$$

with $f \in \mathrm{pPOL}_k\,\chi$.

Proof. Let $\sigma := \sigma(\varrho)$. Assume there is some $f^{(n)} \in \mathrm{pPOL}_k\,\varrho$ such that $f \notin \mathrm{pPOL}_k\,\chi$ for all $\chi \in \mathcal{X}_\varrho$. Then $f(x,\dots,x) \in E_k \setminus \{x\}$ for each $x \in E_k$.

Because ϱ is reducible there are some A with $\emptyset \subset A \subset E_h$, and $\pi \in S_h$ and $v \in \sigma(E_k^{h-|A|})$ such that

$$(\mathrm{pr}_A\,\varrho) \times \{v\} \subseteq \varrho^{[\pi]}.$$

Because $\mathrm{pPOL}_k\,\varrho = \mathrm{pPOL}_k\,\varrho^{[\pi]}$ we assume w.l.o.g. $\pi = \mathrm{id}$.

We show that $\mathrm{pp}\,\mathrm{pr}_A\,\varrho = \mathrm{pp}\,\varrho$. If $|A| = 1$ then $\mathrm{pr}_A\,\varrho = E_k$ and thus $E_k \times \{v\} \subseteq \varrho$. This implies $\delta^{(h)}_{\{0,i\}} \subseteq \varrho$ for all $i \in E_h \setminus \{0\}$ contradicting $\varrho \in \mathcal{Q}$. Let $|A| \geq 2$. We know $s_0 = (0,\dots,0), s_1 = (1,\dots,1) \in \delta(\mathrm{pr}_A\,\varrho)$. Then $\{s_0, s_1\} \times \{v\} \subseteq \delta(\varrho)$, i.e., for all $i \in E_h$ and $j \in E_h \setminus (A \cup \{i\})$ we get $(i,j) \notin \varepsilon(\varrho)$. Thus all non-singular classes of $\varepsilon(\varrho)$ are covered by A, i.e., the projection pr_A preserves them, and this implies $\mathrm{pp}\,\mathrm{pr}_A\,\varrho = \mathrm{pp}\,\varrho$.

We show that $(\mathrm{pr}_A\,\sigma) \times \{v\} \subseteq \sigma$. Assume the contrary. Then there exists some $s \in \mathrm{pr}_A\,\sigma$ with $\{s\} \times \{v\} \subseteq \delta(\varrho)$. But this contradicts $\mathrm{pr}_A\,\delta(\varrho) \cap \mathrm{pr}_A\,\sigma = \emptyset$ because $\mathrm{pp}\,\mathrm{pr}_A\,\varrho = \mathrm{pp}\,\varrho$. So $s \notin \mathrm{pr}_A\,\delta(\varrho)$ in contradiction to the assumption. We proved $(\mathrm{pr}_A\,\sigma) \times \{v\} \subseteq \sigma$, and thus $\omega(\mathrm{pr}_A\,\sigma) \cap \omega(v) = \emptyset$.

Now we show that $\gamma := \mathrm{pr}_A\,\varrho \in \mathcal{Q}$, i.e., that it is coherent. Let $\theta \in \Gamma_{\sigma(\gamma)}$ and $w \in \gamma$ arbitrarily. There is some $\hat{w} \in \gamma$ with $\hat{w}^{[\theta]} \in \gamma$. Then $\{\hat{w}, \hat{w}^{[\theta]}\} \times \{v\} \subseteq \varrho$, i.e., $\theta \in \Gamma_\sigma$ and thus $\{w, w^{[\theta]}\} \times \{v\} \subseteq \varrho$. This implies $w^{[\theta]} \in \gamma$.

$M(\gamma) = \mathrm{pr}_A\,M(\varrho)$ because $\mathrm{pp}\,\mathrm{pr}_A\,\varrho = \mathrm{pp}\,\varrho$.

Let $\gamma' \subseteq \sigma(\gamma)$. Then $\gamma' \times \{v\} \subseteq \sigma$. Thus there is a relational homomorphism $\varphi\colon E_k \to E_h$ from $\gamma' \times \{v\}$ to $M(\varrho)$ and some $w \in \gamma'$ with $\varphi\begin{pmatrix} w \\ v \end{pmatrix} = \eta_h$. Let

$\hat{\varphi}\colon E_k \to E_{|A|}$ be given by

$$\hat{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } \varphi(x) \in E_{|A|}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\hat{\varphi}$ is a relational homomorphism from γ' to $M(\gamma)$ with $\hat{\varphi}(w) = \eta_{|A|}$. Thus γ is a coherent relation and $\gamma \in \mathcal{X}_\varrho$ because $\mathrm{pp}\,\gamma = \mathrm{pp}\,\varrho$ and $|A| < h$. Since $f \notin \mathrm{pPOL}_k\,\chi$ for all $\chi \in \mathcal{X}_\varrho$ there are rows $c_{1*}, \ldots, c_{|A|*}$ with $c_{*1}, \ldots, c_{*n} \in \gamma$ and $f(c_{*1} \ldots c_{*n}) = d \in E_k^{|A|} \setminus \gamma$.

Then

$$f\begin{pmatrix} c_{*1} & \ldots & c_{*n} \\ v & \ldots & v \end{pmatrix} \in E_k^h \setminus \varrho,$$

i.e., $f \notin \mathrm{pPOL}_k\,\varrho$ contradicting the assumption. □

Proposition 7.10. *Let $\varrho^{(h)}, \chi^{(\mu)} \in \mathcal{Q}$ with $\mu \geq 3$, $f, g \in \mathrm{pPOL}_k\,\chi$ with $g(\varrho) \in E_k^h$, and g is not defined anywhere else, and $F^{(n)} := f \otimes g \notin \mathrm{pPOL}_k\,\chi$.*

Then there are rows $c_{1*}, \ldots, c_{\mu*}$

1. *with $c_{*1}, \ldots, c_{*n} \in \chi$ and $F(c_{*1} \ldots c_{*n}) = d \in E_k^\mu \setminus \chi$, and*
2. *there is some j with $c_{*j} \in \sigma(E_k^\mu)$, and*
3. *the rows $c_{1*}, \ldots, c_{\|\chi\|*}$ belong to the g-part of F, and*
4. *if $\mathrm{pp}\,\varrho = \mathrm{pp}\,\chi$, then the rows $c_{1*}, \ldots, c_{\|\chi\|*}$ belong to the first $\|\chi\|$ rows of the g-part of F.*

Proof. Statement (1) follows directly from $F \notin \mathrm{pPOL}_k\,\chi$. Choose some rows $c_{1*}, \ldots, c_{\mu*}$ such that (1) holds.

(2)**:** Assume (2) is false. Then $\{c_{*1}, \ldots, c_{*n}\} \subseteq \delta(\chi) = \delta_{\varepsilon(\chi)}$ contradicting all rows c_{i*} are pairwise different by Lemma 5.3. Thus for any two rows there is a column in which they differ.

(3)**:** Because $\varrho \in \mathcal{Q}$ we have $\delta_{E_h}^{(h)} \subseteq \varrho$. Because $g \in \mathrm{pPOL}_k\,\chi$ there is at least one row from the f-part of F and because $f \in \mathrm{pPOL}_k\,\chi$ there is at least one row from the g-part of F. Let c_{i_f*} be an arbitrary row from the f-part and c_{i_g*} be an arbitrary row from the g-part. Because $\mu \geq 3$ there is a third row $c_{i'*}$ different from c_{i_f*} and c_{i_g*}. Let $c_{i'*}$ be arbitrary with this condition.

There are two cases to consider:

*The row $c_{i'*}$ is from the f-part***:** Then there is some column j in which the rows c_{i_f*} and $c_{i'*}$, i.e., $c_{i_fj} = x$, $c_{i'j} = y$ and $x \neq y$. By construction and $\varrho \in \mathcal{Q}$, i.e., $\delta_{E_h}^{(h)} \subset \varrho$, we can choose j more specifically such that

$$\begin{pmatrix} c_{i_fj} \\ c_{i'j} \\ c_{i_gj} \end{pmatrix} = \begin{pmatrix} x \\ y \\ y \end{pmatrix}.$$

*The row $c_{i'*}$ is from the g-part***:** Then there is some j with

$$\begin{pmatrix} c_{i_fj} \\ c_{i'j} \\ c_{i_gj} \end{pmatrix} = \begin{pmatrix} x \\ y \\ y \end{pmatrix}$$

and $x \neq y$ by construction and $\varrho \in \mathcal{Q}$, i.e., $\delta_{E_h}^{(h)} \subset \varrho$.
Thus $(i_f, i_g), (i_f, i') \notin \varepsilon(\chi)$. Because i_f, i_g and i' are chosen arbitrarily any row $c_{i_f *}$ from the f-part belongs to a singular class of $\varepsilon(\chi)$. Because the first $\|\chi\|$ rows of χ belong to non-singular classes of $\varepsilon(\chi)$ the first $\|\chi\|$ rows $c_{1*}, \ldots, c_{\|\chi\|*}$ belong to the g-part of F. Thus (3) is true.

(4): Let $\operatorname{pp}\varrho = \operatorname{pp}\chi$. Assume one of the rows $c_{1*}, \ldots, c_{\|\chi\|*}$ does not belong to the first $\|\chi\|$ rows of the g-part of F, w.l.o.g. let this be the row c_{1*}. As shown before c_{1*} belongs to the g-part of F. Because $\operatorname{pp}\varrho = \operatorname{pp}\chi$ the row c_{1*} belongs to a singular class of $\varepsilon(\varrho)$. Now let $c_{i_1 *}, c_{i_2 *}$ be two arbitrarily chosen different rows. Then there are three different cases:

$c_{i_1 *}$ and $c_{i_2 *}$ are both from the f-part: Then they differ at some point and by construction we get columns c_{*j}, $c_{*j'}$ with

$$\begin{pmatrix} c_{1j} & c_{1j'} \\ c_{i_1 j} & c_{i_1 j'} \\ c_{i_2 j} & c_{i_2 j'} \end{pmatrix} = \begin{pmatrix} x & y \\ x & x \\ y & y \end{pmatrix}$$

and $x \neq y$.

$c_{i_1 *}$ is from the f-part and $c_{i_2 *}$ from the g-part: Then by construction and because c_{1*} belongs to a singular class of $\varepsilon(\varrho)$ there is some column c_{*j} with

$$\begin{pmatrix} c_{1j} \\ c_{i_1 j} \\ c_{i_2 j} \end{pmatrix} = \begin{pmatrix} x \\ y \\ y \end{pmatrix}$$

and $x \neq y$.

$c_{i_1 *}$ and $c_{i_2 *}$ are both from the g-part: Then because c_{1*} belongs to a singular class of $\varepsilon(\varrho)$ there is some column c_{*j} with

$$\begin{pmatrix} c_{1j} \\ c_{i_1 j} \\ c_{i_2 j} \end{pmatrix} = \begin{pmatrix} x \\ y \\ y \end{pmatrix}$$

and $x \neq y$.

Thus for all cases $(1, i_1), (1, i_2) \notin \varepsilon(\chi)$. Because i_1 and i_2 are chosen arbitrarily the row c_{1*} belongs to a singular class of $\varepsilon(\chi)$ in contradiction to the convention that the first $\|\chi\|$ rows of χ belong to the non-singular classes of $\varepsilon(\chi)$, see Definition 2.10. Thus (4) is true.

□

Definition 7.11. Let $\varrho \in \mathcal{Q}_1$. Define $\mathcal{T}_\varrho \subseteq \widetilde{\mathcal{R}}_k^{\max}$ by

$$\mathcal{T}_\varrho := \{\psi \in \mathcal{Q}_1 \mid \operatorname{pp}\psi = \operatorname{pp}\varrho\}.$$

Lemma 7.12. *Let $\varrho \in \mathcal{Q}_1$, $\mathcal{T} \subseteq \mathcal{T}_\varrho$, $|\mathcal{T}| \geq 2$ and $f \in \widetilde{P}_k$ with*

$$\left(\forall \psi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T} : f \notin \operatorname{pPOL}_k \psi\right) \wedge \left(\exists \chi \in \mathcal{T} : f \in \operatorname{pPOL}_k \chi\right).$$

Then

(1) *there are* $\chi_0 \in \mathcal{T}$ *and* $F \in \mathrm{pPOL}_k\, \chi_0$ *with*

$$\forall \psi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \{\chi_0\} : F \notin \mathrm{pPOL}_k\, \psi,$$

or

(2) *there are* $F \in \widetilde{P}_k$ *and* $\mathcal{T}' \subset \mathcal{T}$, $\mathcal{T}' \neq \emptyset$ *with*

$$\left(\forall \psi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}' : F \notin \mathrm{pPOL}_k\, \psi\right) \wedge \left(\exists \chi \in \mathcal{T}' : F \in \mathrm{pPOL}_k\, \chi\right).$$

Proof. Assume (1) is false. By this assumption there exists some $\psi_1 \in \mathcal{T}$ with $f \in \mathrm{pPOL}_k\, \psi_1$. Then there is some $\psi_2 \in \widetilde{\mathcal{R}}_k^{\max} \setminus \{\psi_1\}$ with $f \in \mathrm{pPOL}_k\, \psi_2$ because (1) is false. Because $f \notin \mathrm{pPOL}_k\, \psi$ for all $\psi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}$ we get $\psi_2 \in \mathcal{T}$. Thus there are $\psi_1^{(\mu_1)}, \psi_2^{(\mu_2)} \in \mathcal{T}$ with $\psi_1 \neq \psi_2$ and $f \in (\mathrm{pPOL}_k\, \psi_1) \cap (\mathrm{pPOL}_k\, \psi_2)$. We can choose ψ_1 such that μ_1 is minimal. This implies

$$\forall \chi \in \{\{a\} \mid a \in E_k\} \cup \{\psi^{(\mu)} \in \mathcal{Q} \mid \mathrm{pp}\, \psi = \mathrm{pp}\, \varrho \wedge \mu < \mu_1\} : f \notin \mathrm{pPOL}_k\, \chi.$$

Thus ψ_1 is irreducible because $f \in \mathrm{pPOL}_k\, \psi_1$ and Lemma 7.9.

Furthermore $\mu_1 \leq \mu_2$. If $\mu_2 = \mu_1$ then ψ_2 is also irreducible by the same argument.

We construct a function $F^{(n)} := f \otimes g$ such that $F \in \mathrm{pPOL}_k\, \psi_1$ and $F \notin \mathrm{pPOL}_k\, \psi_2$ holds (or the other way round).

For any set E let $\mathcal{P}(E) := \{A \subseteq E \mid A \neq \emptyset \wedge A \neq E\}$.

There are the following cases:

$\exists A \in \mathcal{P}(E_{\mu_1})\ \exists v \in \sigma(E_k^{\mu_2 - |A|})\ \exists \pi \in S_{\mu_2} : (\mathrm{pr}_A\, \psi_1) \times \{v\} \subseteq \psi_2^{[\pi]}$:

Without loss of generality $\pi = \mathrm{id}$.

Assume to the contrary that $\mathrm{pp}(\mathrm{pr}_A\, \psi_1) \neq \mathrm{pp}\, \psi_1$ holds. Then the inequality $\|(\mathrm{pr}_A\, \psi_1) \times \{v\}\| = \|\mathrm{pr}_A\, \psi_1\| < \|\psi_1\| = \|\psi_2\|$ holds in contradiction to the fact $\delta(\mathrm{pr}_A\, \psi_1) \times \{v\} \subseteq \delta\left((\mathrm{pr}_A\, \varrho) \times \{v\}\right) \subseteq \delta(\psi_2)$. Thus $\mathrm{pp}(\mathrm{pr}_A\, \psi_1) = \mathrm{pp}\, \psi_1 = \mathrm{pp}\, \psi_2$.

Let $g(\mathrm{pr}_A\, \psi_1) := d$ (see Definition 4.1) for some $d \in E_k^{|A|}$ with the property $g(\mathrm{pp}(\mathrm{pr}_A\, \psi_1)) \in E_k^{\|\psi_2\|} \setminus \mathrm{pp}\, \psi_2$. Then $F \notin \mathrm{pPOL}_k\, \psi_2$ because

$$\underbrace{F\begin{pmatrix} \mathrm{pr}_A\, \psi_1 \\ v \end{pmatrix}}_{\subseteq \psi_2} \in (E_k^{\|\psi_2\|} \setminus \mathrm{pp}\, \psi_2) \times E_k^{\mu_2 - \|\psi_2\|} \subseteq E_k^{\mu_2} \setminus \psi_2.$$

We have $g \in \mathrm{pPOL}_k\, \psi_1$ because g is defined on less than μ_1 rows. Assume $F \notin \mathrm{pPOL}_k\, \psi_1$. Then there are rows $c_{1*}, \dots, c_{\mu_1 *}$ with $c_{*1}, \dots, c_{*n} \in \psi_1$ and the first $\|\psi_1\|$ rows belong to the g-part of F, and a column $c_{*j} \in \sigma(E_k^{\mu_1})$ by Proposition 7.10. Let w.l.o.g. the rows $c_{1*}, \dots, c_{l*}$ belong to the g-part of F and $c_{l+1*}, \dots, c_{\mu_1 *}$ to the f-part of F with $\|\chi\| \leq l < \mu_1$. Then let

$$v := \begin{pmatrix} c_{l+1j} \\ \dots \\ c_{\mu_1 j} \end{pmatrix}$$

and

$$C' := \mathrm{pr}_{1,2,\dots,l}\{c_{*1}, \dots, c_{*n}\}.$$

Then $C' = \operatorname{pr}_{A'} \psi_1$ for some $A' \in \mathcal{P}(E_{\mu_1})$ with $A' \subseteq A$ by construction of g. By construction of F we get

$$(\operatorname{pr}_{A'} \psi_1 \times \{v\}) = C' \times \{v\} \subseteq \{c_{*1}, \ldots, c_{*n}\} \subseteq \psi_1$$

contradicting ψ_1 irreducible.

Thus there is some $\mathcal{T}'$ with $\{\psi_1\} \subseteq \mathcal{T}' \subseteq \mathcal{T} \setminus \{\psi_2\}$ and

$$\left(\forall \psi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}' : F \notin \operatorname{pPOL}_k \psi\right) \wedge (\exists \chi \in \mathcal{T}' : F \in \operatorname{pPOL}_k \chi).$$

$\mu_1 = \mu_2 \wedge \left(\exists A \in \mathcal{P}(E_{\mu_2})\ \exists v \in \sigma(E_k^{\mu_1 - |A|})\ \exists \pi \in S_{\mu_1} : (\operatorname{pr}_A \psi_2) \times \{v\} \subseteq \psi_1^{[\pi]}\right)$**:**
This is a restriction of the previous case with the roles of ψ_1 and ψ_2 switched. Thus there is some $\mathcal{T}'$ with $\{\psi_2\} \subseteq \mathcal{T}' \subseteq \mathcal{T} \setminus \{\psi_1\}$ and

$$\left(\forall \psi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}' : F \notin \operatorname{pPOL}_k \psi\right) \wedge (\exists \chi \in \mathcal{T}' : F \in \operatorname{pPOL}_k \chi).$$

$\mu_1 < \mu_2 \wedge \left(\exists v \in \sigma(E_k^{\mu_2 - \mu_1})\ \exists \pi \in S_{\mu_2} : \psi_1 \times \{v\} \subseteq \psi_2^{[\pi]}\right)$**:**
Without loss of generality $\pi = \mathrm{id}$.

Because ψ_1 is coherent there is some relational homomorphism $\varphi\colon E_k \to E_{\mu_1}$ from $\sigma(\psi_1)$ to $M(\psi_1)$ and some $s \in \sigma(\psi_1)$ with $\varphi(s) = \eta_{\mu_1}$. Define $\varphi^\star\colon E_{\mu_1} \to E_k$ by $\varphi^\star(\eta_{\mu_1}) = s$.

Let

$$g\left((\sigma(\psi_1) \times \{v\}) \cup \delta(\psi_2)\right) := d := \varphi^\star\left(\varphi\left(\begin{array}{c} s \\ v \end{array}\right)\right)$$

(see Definition 4.1). Then $g \in \operatorname{pPOL}_k \psi_1$ by construction.

Assume $g \in \operatorname{pPOL}_k \psi_2$. Then $d \in \delta(\psi_2)$ because $|\omega(d)| = |\omega(s)| = \mu_1 < \mu_2$. But $|\omega(\operatorname{pr}_{E_{\|\psi_2\|}} d)| = \|\psi_2\|$ in contradiction to the assumption that the first $\|\psi_2\|$ rows belong to the non-singular classes of $\varepsilon(\psi_2)$. Thus $g \notin \operatorname{pPOL}_k \psi_2$ and this implies $F \notin \operatorname{pPOL}_k \psi_2$.

Because $\sigma(\psi_1) \times \{v\} \subseteq \psi_2$ and the first $\|\psi_1\|$ rows belong to the non-singular classes of $\varepsilon(\psi_2)$ we get $\sigma(\psi_1) \times \{v\} \subseteq \sigma(\psi_2)$ and thus $\omega(v) \cap \omega(\sigma(\psi_1)) = \emptyset$. Assume $F \notin \operatorname{pPOL}_k \psi_1$. Then there are rows $c_{1*}, \ldots, c_{\mu_1 *}$ with $c_{*1}, \ldots, c_{*n} \in \psi_1$ and $F(c_{*1}, \ldots, c_{*n}) \in E_k^{\mu_1} \setminus \psi_1$. By Proposition 7.10 the rows $c_{1*}, \ldots, c_{\|\psi_1\|*}$ are the first rows in the definition of g. Thus the other rows can not belong to the last $(\mu_2 - \mu_1)$ rows in the definition of g because $\omega(v) \cap \omega(\sigma(\psi_1)) = \emptyset$. Thus this part of the definition of g can be ignored here, and thus $F \in \operatorname{pPOL}_k \psi_1$ because ψ_1 is irreducible.

Thus there is some $\mathcal{T}'$ with $\{\psi_1\} \subseteq \mathcal{T}' \subseteq \mathcal{T} \setminus \{\psi_2\}$ and

$$\left(\forall \psi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}' : F \notin \operatorname{pPOL}_k \psi\right) \wedge (\exists \chi \in \mathcal{T}' : F \in \operatorname{pPOL}_k \chi).$$

$\mu_1 = \mu_2 \wedge \left(\exists \pi \in S_{\mu_2} : \psi_1 \subset \psi_2^{[\pi]}\right)$**:**
Without loss of generality $\pi = \mathrm{id}$.

Let $g(\psi_2) := d$ (see Definition 4.1) for some $d \in E_k^{\mu_2} \setminus \psi_2$. Because $\operatorname{pr}_A \psi_1 \subseteq \operatorname{pr}_A \psi_2$ for all $A \in \mathcal{P}(E_{\mu_1})$, ψ_1 irreducible and $g \in \operatorname{pPOL}_k \psi_1$ we get $F \in \operatorname{pPOL}_k \psi_1$. Furthermore $g \notin \operatorname{pPOL}_k \psi_2$ implies $F \notin \operatorname{pPOL}_k \psi_2$.

Thus there is some $\mathcal{T}'$ with $\{\psi_1\} \subseteq \mathcal{T}' \subseteq \mathcal{T} \setminus \{\psi_2\}$ and

$$\left(\forall \psi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}' : F \notin \mathrm{pPOL}_k\, \psi\right) \wedge (\exists \chi \in \mathcal{T}' : F \in \mathrm{pPOL}_k\, \chi)\,.$$

$\mu_1 = \mu_2 \wedge \left(\exists \pi \in S_{\mu_1} : \psi_2 \subset \psi_1^{[\pi]}\right)$**:**
Analogous to the previous case because ψ_2 is irreducible in this case. Thus there is some $\mathcal{T}'$ with $\{\psi_2\} \subseteq \mathcal{T}' \subseteq \mathcal{T} \setminus \{\psi_1\}$ and

$$\left(\forall \psi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}' : F \notin \mathrm{pPOL}_k\, \psi\right) \wedge (\exists \chi \in \mathcal{T}' : F \in \mathrm{pPOL}_k\, \chi)\,.$$

Otherwise **:**
Then we have

$$\forall A \in \mathcal{P}(E_{\mu_1})\ \forall v \in \sigma(E_k^{\mu_2-|A|})\, \forall \pi \in S_{\mu_2} : (\mathrm{pr}_A\, \psi_1) \times \{v\} \not\subseteq \psi_2^{[\pi]},$$
$$\mu_1 < \mu_2 \vee \left(\forall A \in \mathcal{P}(E_{\mu_2})\ \forall v \in \sigma(E_k^{\mu_1-|A|})\, \forall \pi \in S_{\mu_1} : (\mathrm{pr}_A\, \psi_2) \times \{v\} \not\subseteq \psi_1^{[\pi]}\right),$$
$$\mu_1 = \mu_2 \vee \left(\forall v \in \sigma(E_k^{\mu_2-\mu_1})\, \forall \pi \in S_{\mu_2} : \psi_1 \times \{v\} \not\subseteq \psi_2^{[\pi]}\right),$$
$$\mu_1 < \mu_2 \vee \left(\forall \pi \in S_{\mu_2} : \psi_1 \not\subseteq \psi_2^{[\pi]}\right),$$
$$\mu_1 < \mu_2 \vee \left(\forall \pi \in S_{\mu_1} : \psi_2 \not\subseteq \psi_1^{[\pi]}\right).$$

Let $g(\psi_1) := d$ (see Definition 4.1) for some $d \in E_k^{\mu_1} \setminus \psi_1$. Because $\psi_2 \not\subseteq \psi_1^{[\pi]}$ for all $\pi \in S_{\mu_1}$, and $\mu_1 \leq \mu_2$ we have $g \in \mathrm{pPOL}_k\, \psi_2$. Assume $F^{(n)} = f \otimes g \notin \mathrm{pPOL}_k\, \psi_2$. Then there are $c_{1*}, \dots, c_{\mu_2 *}$ with $c_{*1}, \dots, c_{*n} \in \psi_2$ and $F(c_{*1}, \dots, c_{*n}) \notin \psi_2$ and the rows $c_{1*}, \dots, c_{\|\psi_2\| *}$ belong to the g-part of F by Proposition 7.10, i.e., one of the following cases apply

- there is some $A \subset E_{\mu_1}$ and $v \in \sigma(E_k^{\mu_2-|A|})$ with $(\mathrm{pr}_A\, \psi_1) \times \{v\} \subseteq \psi_2$ contradicting the first assumption, or
- $\mu_1 < \mu_2$ and there is some $v \in \sigma(E_k^{\mu_2-|A|})$ with $\psi_1 \times \{v\} \subseteq \psi_2$ contradicting the third assumption.

Thus $F \in \mathrm{pPOL}_k\, \psi_2$. Furthermore $F \notin \mathrm{pPOL}_k\, \psi_1$ because $g \notin \mathrm{pPOL}_k\, \psi_1$.

Thus there is some $\mathcal{T}'$ with $\{\psi_2\} \subseteq \mathcal{T}' \subseteq \mathcal{T} \setminus \{\psi_1\}$ and

$$\left(\forall \psi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}' : F \notin \mathrm{pPOL}_k\, \psi\right) \wedge (\exists \chi \in \mathcal{T}' : F \in \mathrm{pPOL}_k\, \chi)\,.$$

Thus in every case there is some $\mathcal{T}' \subset \mathcal{T}$ with

$$\left(\forall \psi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}' : F \notin \mathrm{pPOL}_k\, \psi\right) \wedge (\exists \chi \in \mathcal{T}' : F \in \mathrm{pPOL}_k\, \chi)\,,$$

i.e., (2) is true. □

Theorem 7.13. *For every $k \geq 2$ there is exactly one minimal covering of $\mathrm{p}\mathscr{M}_k$.*

Proof. For $k = 2$ one finds this statement in [4]. Thus we can assume $k \geq 3$. Assume the statement is false. Then there are pairwise different minimal coverings $\mathscr{X}_1, \dots, \mathscr{X}_l$ with $l \geq 2$. Choose $\varrho \in \widetilde{\mathcal{R}}_k^{\max}$ with $\mathrm{pPOL}_k\, \varrho \in \mathscr{X}_1 \setminus \mathscr{X}_2$ arbitrarily. Then $\varrho \in \mathcal{Q}_1$ because of Theorem 7.5. Let

$$\mathcal{T} := \{\psi \in \mathcal{Q}_1 \mid \mathrm{pp}\, \psi = \mathrm{pp}\, \varrho \wedge (\exists a, b \in \{1, \dots, l\} : \mathrm{pPOL}_k\, \psi \in \mathscr{X}_a \setminus \mathscr{X}_b)\} \subseteq \mathcal{T}_\varrho.$$

Then $\varrho \in \mathcal{T}$ and $|\mathcal{T}| \geq 2$ by Lemma 7.6. Additionally there is some $f \in \widetilde{P}_k \setminus (P_k \cup C_\emptyset)$ with

$$(\forall \psi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T} : f \notin \mathrm{pPOL}_k\, \psi) \wedge (\exists \chi \in \mathcal{T} : f \in \mathrm{pPOL}_k\, \chi). \tag{7.1}$$

Otherwise $\mathrm{pPOL}_k\, \varrho$ would be in no minimal covering contradicting the assumption.

Now we can assume that $\widehat{\mathcal{T}} \subseteq \mathcal{T}$ has minimal size $|\widehat{\mathcal{T}}| \geq 2$ and fulfills (7.1) (with $\widehat{\mathcal{T}}$ instead of $\mathcal{T}$).

By Lemma 7.12 there are two cases:

- There are $\chi_0 \in \widehat{\mathcal{T}}$ and $F \in \mathrm{pPOL}_k\, \chi_0$ with
$$\forall \psi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \{\chi_0\} : F \notin \mathrm{pPOL}_k\, \psi.$$
Then $\mathrm{pPOL}_k\, \chi_0$ is in every minimal covering of $p\mathscr{M}_k$ by Lemma 3.2 in contradiction to the definition of $\mathcal{T}$ and the assumption.
- There are $F \in \widetilde{P}_k$, $\mathcal{T}'$ with $\emptyset \subset \mathcal{T}' \subset \widehat{\mathcal{T}}$ and
$$(\forall \psi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}' : F \notin \mathrm{pPOL}_k\, \psi) \wedge (\exists \chi \in \mathcal{T}' : F \in \mathrm{pPOL}_k\, \chi).$$
Because $\widehat{\mathcal{T}}$ is minimal under the condition $|\widehat{\mathcal{T}}| \geq 2$ we conclude $|\mathcal{T}'| = 1$. Then $\mathcal{T}' = \{\chi_0\}$, $F \in \mathrm{pPOL}_k\, \chi_0$ and
$$\forall \psi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \{\chi_0\} : F \notin \mathrm{pPOL}_k\, \psi.$$
Thus $\mathrm{pPOL}_k\, \chi_0$ is in every minimal covering of $p\mathscr{M}_k$ by Lemma 3.2, in contradiction to the definition of $\mathcal{T}$ and the assumption.

Thus there are no two different minimal coverings of $p\mathscr{M}_k$. □

Let $p\mathscr{C}_k$ be the unique minimal covering of $p\mathscr{M}_k$. Using the uniqueness of minimal coverings we can improve the statements of Lemmas 3.3 and 3.2.

Lemma 7.14. *Let $C \in p\mathscr{M}_k$ and $\mathscr{C} \subseteq p\mathscr{M}_k \setminus \{C\}$ such that for all $f \in C$ there is some $C' \in \mathscr{C}$ with $f \in C'$. Then $C \notin p\mathscr{C}_k$.*

Proof. Assume C is in the minimal covering $p\mathscr{C}_k$ of $p\mathscr{M}_k$. Let $\mathscr{Y} := (p\mathscr{C}_k \setminus \{C\}) \cup \mathscr{C}$. Then $\mathscr{Y}$ is a covering of $p\mathscr{M}_k$ because for all $f \in X \in p\mathscr{M}_k$ there is

- some $Y \in p\mathscr{C}_k \setminus \{C\}$ with $f \in Y$, or
- $f \in C$ and then there is some $Y \in \mathscr{C}$ with $f \in Y$.

Then there is some minimal covering $\tilde{\mathscr{Y}} \subseteq \mathscr{Y}$ of $p\mathscr{M}_k$. But $\mathscr{Y} \cap p\mathscr{C}_k \subset p\mathscr{C}_k$ and thus $\tilde{\mathscr{Y}} \neq p\mathscr{C}_k$ contradicting Theorem 7.13. □

Lemma 7.15. *Let $C \in p\mathscr{M}_k$. Then*
$$C \in p\mathscr{C}_k \iff (\exists f \in C\, \forall B \in p\mathscr{M}_k \setminus \{C\} : f \notin B).$$

Proof. We split the proof into two directions:

⇐: Follows from Lemma 3.2 and Theorem 7.13.

⇒: Let $C \in p\mathscr{C}_k$. Assume,
$$\forall f \in C\, \exists B \in p\mathscr{M}_k \setminus \{C\} : f \in B.$$
By Lemma 7.14 with $\mathscr{C} = p\mathscr{M}_k \setminus \{C\}$ follows $C \notin p\mathscr{C}_k$ in contradiction to the assumption. □

Lemma 7.16. *Let $\varrho^{(h)} \in \mathcal{Q}_1$ be reducible. Then* $\mathrm{pPOL}_k\, \varrho$ *is not in the minimal covering $p\mathscr{C}_k$ of $p\mathscr{M}_k$.*

Proof. This follows directly from Lemma 7.9 with the help of Lemma 7.14. □

8. Conclusion

The minimal coverings for $k = 2, 3, 4$ have been given and shown to be unique in [4], [2] and [14] respectively. In following table the sizes of these minimal coverings $p\mathscr{C}_k$ are given with respect to the number of all maximal partial clones $|p\mathscr{M}_k|$.

k	$\lvert p\mathscr{M}_k\rvert$	$\lvert p\mathscr{C}_k\rvert$
2	8	4
3	58	26
4	1 102	449

We have now shown that the minimal coverings of $p\mathscr{M}_k$ are unique for each $k \geq 2$. Many elements of the minimal coverings have been determined (see e.g. [2, 16]) and for some maximal partial clones we have shown in this paper that they are not in a minimal covering (see Lemmas 5.5, 7.3 and 7.16). Furthermore for maximal partial clones $\mathrm{pPOL}_k\, \varrho$ with $\varrho \in \mathcal{A}$ we have a criterion which only needs to check the functions from $\mathrm{Pol}_k^{(1)}\, \varrho$ to see if $\mathrm{pPOL}_k\, \varrho$ belongs to $p\mathscr{C}_k$ (see Theorem 6.13). Still many elements of the minimal coverings have to be determined, and it seems to be a very hard problem, especially for the relations $\mathcal{Q}_1$.

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