Galois Theory for Partial Clones and Some Relational Clones

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***Abstract*—A Galois connection between partial clones and a new variant of relation algebras is established. We introduce a new elementary operation on relations which captures the difference between total and partial clones and allows us to adapt the proof of the Galois connection from the total case to the partial case. This Galois connection is able to capture all partial clones and is not restricted to strong partial clones as in previous work.**

The study of partial clones yielded many interesting results. Since partial clones are infinite objects it is often more complicated to deal with these compared to finite objects. Thus it is reasonable to look for a characterization of the functions in a clone by other means, e.g., example relations preserved by all functions in the clone.

We establish an antitone Galois connection to relation algebras. This helps especially with big clones, e.g., the maximal clones, since the corresponding relation algebras are small and can often be described by only one relation.

For the clones of total functions (total clones) this kind of Galois connection was established by Bodnarchuk, Kaluzhnin, Kotov and Romov in 1969 [1]. An refurbished proof can be found in [2] or Lau's book [3]. We use the latter one as main guidance and refer the reader to it at some places in this paper.

Galois connections for partial clones have been developed for example by Romov [4] and Rosenberg [5]. They proved very fruitful in the study of partial clones. Yet the theory developed therein can only handle strong partial clones, where a strong partial clone is a partial clones which contains every subfunction to each of its functions.

This paper is devoted to the task of describing all partial clones and we give a characterization similar to the one for total clones with the help of elementary operations on relations. All the elementary operations for the total relation algebras are used and one new elementary operation $\hat{\kappa}$ on relations is introduced which suffices to capture the differences between total and partial clones.

The approach taken here uses a one-point extension. This way fictitious variables get lost and we overcome this difficulty by restricting the set of relations allowed and adding an additional operation on relations.

I. DEFINITIONS

Let $E_k := \{0, 1, \ldots, k-1\}$ and $\widetilde{E}_k := E_k \cup \{\infty\}$ for any $k \in \mathbb{N}$ with $k \geq 2$ where ∞ is just some symbol which is not in E_k.

Then $f^{(n)} : E_k^n \to \widetilde{E}_k$ is called an n-ary partial function. If the arity n of the function $f^{(n)}$ is known we just write f instead of $f^{(n)}$. Let $\widetilde{P}_k^{(n)}$ be the set of all n-ary partial functions, i.e.,

$$\widetilde{P}_k^{(n)} := \left\{ f^{(n)} \;\middle|\; f^{(n)} : E_k^n \to \widetilde{E}_k \right\}$$

and

$$\widetilde{P}_k := \bigcup_{n \geq 1} \widetilde{P}_k^{(n)}.$$

The set of all total functions $P_k \subset \widetilde{P}_k$ is defined by

$$P_k := \{ f^{(n)} \in \widetilde{P}_k \mid \forall x \in E_k^n : f^{(n)}(x) \in E_k \}.$$

The i-th n-ary *projection* $e_i^{(n)} : E_k^n \to \widetilde{E}_k$ is defined by $e_i^{(n)}(x_1, \ldots, x_n) = x_i$ for all $x_1, \ldots, x_n \in E_k$.

Example 1. Let $k = 2$ and $f_1^{(2)}, f_2^{(2)} \in \widetilde{P}_k$ defined by

$$f_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} := \begin{pmatrix} f_1(0,0) \\ f_1(0,1) \\ f_1(1,0) \\ f_1(1,1) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \infty \\ 0 \end{pmatrix}$$

and

$$f_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} := \begin{pmatrix} f_2(0,0) \\ f_2(0,1) \\ f_2(1,0) \\ f_2(1,1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then $f_1 \in \widetilde{P}_k \setminus P_k$ and $f_2 \in P_k$.

Sometimes we want to consider the functions of $\widetilde{P}_k$ as functions from $\widetilde{E}_k^n$ to $\widetilde{E}_k$ which can be done in a natural way.

Definition 2. Let $f^{(n)} \in \widetilde{P}_k$. Then we define the n-ary function $(f^\infty)^{(n)} : \widetilde{E}_k^n \to \widetilde{E}_k$ by

$$(f^\infty)(x_1,\ldots,x_n) := \begin{cases} f(x_1,\ldots,x_n) & \text{if } (x_1,\ldots,x_n) \in E_k^n, \\ \infty & \text{otherwise.} \end{cases}$$

Since it is clear from the context when we use the extended function (f^∞) we also write f for it. Moreover we can assume

f to be (f^{∞}) since the restriction of (f^{∞}) to tuples from E_k^n is the function f.
Beware that the projection $e_1^{(2)}$ also depends on the second variable when interpreted as $\left((e_1^{(2)})^{\infty}\right)$.

We consider the following *elementary operations* (or *superposition operations* or *Mal'tsev operations*; see [6] for the original definition for the total case) ζ, τ, Δ, $\star$ on $\widetilde{P}_k$, with which we can form more complex operations.

For arbitrary $f^{(n)}, g^{(m)} \in \widetilde{P}_k$ one can define these operations by $\zeta f \in \widetilde{P}_k^{(n)}, \tau f \in \widetilde{P}_k^{(n)}, \Delta f \in \widetilde{P}_k^{(\max(1,n-1))}, f \star g \in \widetilde{P}_k^{(m+n-1)}$ and

$$\begin{aligned}
&(\zeta f)(x_1, x_2, \dots, x_n) := f(x_2, x_3, \dots, x_n, x_1),\\
&(\tau f)(x_1, x_2, \dots, x_n) := f(x_2, x_1, x_3, \dots, x_n),\\
&(\Delta f)(x_1, x_2, \dots, x_{n-1}) := f(x_1, x_1, x_2, \dots, x_{n-1})\\
&\ \text{for } n \geq 2,\\
&\zeta f := \tau f := \Delta f := f \text{ for } n = 1,\\
&(f \star g)(x_1, \dots, x_{m+n-1}) :=\\
&\quad (f^{\infty})\,(g(x_1, \dots, x_m), x_{m+1}, \dots, x_{m+n-1})
\end{aligned}$$

The algebra
$$(\widetilde{P}_k; \zeta, \tau, \Delta, \star, e_1^{(2)})$$
is called the *full partial clone* on E_k. Every subalgebra A of $\widetilde{P}_k$ is a *partial clone* on E_k.

Let $A \subseteq \widetilde{P}_k$. Then $[A]_{\mathrm{P}}$ denotes the universe of the smallest partial clone, which contains A. If $[A]_{\mathrm{P}} = A \subseteq \widetilde{P}_k$ we say that A is *closed*. If A is a partial clone and $A \subseteq P_k$ then A is called a *total clone*.

II. RELATIONS

Relations are useful to describe the partial clones of $\widetilde{P}_k$. We often write the elements of relations as columns and a relation can then be given as a matrix. For example the relation $\varrho = \{(0,1,2), (1,2,0), (3,4,5), (2,3,1)\}$ can also be written as $\varrho = \left(\begin{smallmatrix} 0 & 1 & 3 & 2\\ 1 & 2 & 4 & 3\\ 2 & 0 & 5 & 1 \end{smallmatrix}\right)$.

Denote by $\widehat{\mathcal{R}}_k^{(h)}$ the set of all h-ary relations on $\widetilde{E}_k$, i.e., $\widehat{\mathcal{R}}_k^{(h)} := \left\{\varrho \;\middle|\; \varrho \subseteq \widetilde{E}_k^h\right\}$ for $h \geq 1$, and let
$$\widehat{\mathcal{R}}_k := \bigcup_{h \geq 1} \widehat{\mathcal{R}}_k^{(h)}.$$

For a relation $\varrho \in \widehat{\mathcal{R}}_k$ we write $\varrho^{(h)}$ to indicate that $\varrho \in \widehat{\mathcal{R}}_k^{(h)}$, i.e., that ϱ is an h-ary relation.

For a set of relations $X \subseteq \widehat{\mathcal{R}}_k$ we denote by $X^{(h)} := X \cap \widehat{\mathcal{R}}_k^{(h)}$ the h-ary relations in X.

Definition 3. Let $f^{(n)} \in \widetilde{P}_k$ and $r_1, \dots, r_n \in \widetilde{E}_k^h$ with $h \geq 1$ and $r_i = (r_{1i}, \dots, r_{hi})$ for all $i \in \{1, \dots, n\}$. Then
$$f(r_1, \dots, r_n) := \begin{pmatrix} f(r_{11}, \dots, r_{1n})\\ f(r_{21}, \dots, r_{2n})\\ \cdots\\ f(r_{h1}, \dots, r_{hn}) \end{pmatrix}$$

Remember that we write f but use (f^{∞}) in this context since the elements r_{ij} belong to $\widetilde{E}_k$, i.e., $r_{ij} = \infty$ is possible for some i and j.

A. Preserving of Relations

We say a function $f^{(n)} \in \widetilde{P}_k$ *preserves the relation* $\varrho \in \widehat{\mathcal{R}}_k^{(h)}$ (or ϱ *is an invariant of* f or f *is a polymorphism of* ϱ), if
$$\forall r_1, \dots, r_n \in \varrho : f(r_1, \dots, r_n) \in \varrho.$$

By $\operatorname{pPol}_k \varrho$ or, briefly $\operatorname{pPol} \varrho$, we denote the set of all polymorphisms of ϱ. For $Q \subseteq \widehat{\mathcal{R}}_k$ we put
$$\operatorname{pPol}_k Q := \bigcap_{\varrho \in Q} \operatorname{pPol}_k \varrho.$$

The set of all relations $\varrho \in \widehat{\mathcal{R}}_k$ that are preserved by the function $f \in \widetilde{P}_k$ is denoted by $\operatorname{pInv}_k f$. The set of all invariants of $A \subseteq \widetilde{P}_k$ is denoted by
$$\operatorname{pInv}_k A := \bigcap_{f \in A} \operatorname{pInv}_k f.$$

Remark 4. For every $Q \subseteq \widetilde{\mathcal{R}}_k$ the set $\operatorname{pPol}_k Q$ is a partial clone.

Definition 5. A relation $\varrho \in \widehat{\mathcal{R}}_k^{(h)}$ is called ∞*-strict* if $e_1^{(2)} \in \operatorname{pPol}_k \varrho$, or equivalently $e_1^{(2)}(r, s) \in \varrho$ for all $r, s \in \varrho$, or more explicitly $(t_1, \dots, t_h) \in \varrho$ with
$$t_i := \begin{cases} \infty & \text{if } s_i = \infty,\\ r_i & \text{otherwise,} \end{cases}$$
for all $(r_1, \dots, r_h), (s_1, \dots, s_h) \in \varrho$.

Since we want to describe clones with the help of relations and since the function $e_1^{(2)}$ belongs to every partial clone, we can restrict to ∞-strict relations.

Denote by $\widetilde{\mathcal{R}}_k$ the set of all ∞-strict relations on $\widetilde{E}_k$, i.e., $\widetilde{\mathcal{R}}_k = \operatorname{pInv}_k e_1^{(2)}$.

B. Diagonal Relations

Definition 6. Let ε be an arbitrary equivalence relation on $\{1, \dots, h\}$. Define
$$\widetilde{\delta}_{k;\varepsilon}^{(h)} := \left\{(a_1, \dots, a_h) \in \widetilde{E}_k^h \;\middle|\; (i,j) \in \varepsilon \Longrightarrow a_i = a_j\right\}.$$

If the relation ε is given by the non-singular equivalence classes $\varepsilon_1, \dots, \varepsilon_r$ then we write $\widetilde{\delta}_{k;\varepsilon_1,\dots,\varepsilon_r}^{(h)}$ instead of $\widetilde{\delta}_{k;\varepsilon}^{(h)}$. For example $\widetilde{\delta}_{k;\{1,2\}}^{(3)} = \left\{(x, x, y) \;\middle|\; x, y \in \widetilde{E}_k\right\}$.
These relations are called *diagonal relations*. Especially $\widetilde{E}_k^h$ is a diagonal relation for any h.

Definition 7. Let $\varrho^{(h)} \in \widetilde{\mathcal{R}}_k$. We define $I_{\infty}(r) := \{i \in \{1, \dots, h\} \mid r_i = \infty\}$ for $r = (r_1, \dots, r_h) \in \widetilde{E}_k^h$ and $I_{\infty}(\varrho) := \{I_{\infty}(r) \mid r \in \varrho\}$.

Remark 8. The relations in $\operatorname{pInv}_k \widetilde{P}_k$ are called *trivial relations*. Let $\varrho \in \widetilde{\mathcal{R}}_k^{(h)}$ and set $H := \{1, \dots, h\}$. Then $\varrho \in \operatorname{pInv}_k \widetilde{P}_k$ holds iff for all $I \in I_{\infty}(\varrho)$ there is some equivalence relation ε such that $(\operatorname{pr}_J \varrho) \setminus \widetilde{\delta}_{\varepsilon}^{(|J|)} \subseteq \widetilde{E}_k^{|J|} \setminus E_k^{|J|}$ with $J := H \setminus I$ holds. The definition of pr_J follows in Section II-E.

C. Operations on $\widetilde{\mathcal{R}}_k$

In this section we define some operations ζ, τ, pr, $\times$, $\wedge$ and $\hat{\kappa}$ on $\widehat{\mathcal{R}}_k$ which can be used to define more complex operations later. We call the operations ζ, τ, pr, $\times$, $\wedge$ and $\hat{\kappa}$ the *elementary operations* on $\widehat{\mathcal{R}}_k$.

Let $\varrho^{(h)}, \sigma^{(\mu)} \in \widehat{\mathcal{R}}_k$ with $\varrho \neq \emptyset$ and $\sigma \neq \emptyset$. Then let $\zeta\varrho, \tau\varrho \in \widehat{\mathcal{R}}_k^{(h)}$, $\operatorname{pr}\varrho \in \widehat{\mathcal{R}}_k^{(h-1)}$ for $h \geq 2$, and $\operatorname{pr}\varrho = \emptyset$ for $h = 1$, $\varrho \times \sigma \in \widehat{\mathcal{R}}_k^{(h+\mu)}$, $\varrho \wedge \sigma \in \widehat{\mathcal{R}}_k^{(h)}$ (only for $h = \mu$), $\hat{\kappa}\varrho \in \widehat{\mathcal{R}}_k^{(h)}$ defined by

$$\begin{array}{rcl}
\zeta\varrho & := & \{(a_2, a_3, \ldots, a_h, a_1) \mid (a_1, a_2, \ldots, a_h) \in \varrho\} \\
\tau\varrho & := & \{(a_2, a_1, a_3, \ldots, a_h) \mid (a_1, a_2, \ldots, a_h) \in \varrho\} \\
\operatorname{pr}\varrho & := & \{(a_2, \ldots, a_h) \mid \exists a_1 \in \widetilde{E}_k : (a_1, a_2, \ldots, a_h) \in \varrho\} \\
\varrho \times \sigma & := & \{(a_1, \ldots, a_h, b_1, \ldots, b_\mu) \mid (a_1, \ldots, a_h) \in \varrho \\
 & & \quad \text{and } (b_1, \ldots, b_\mu) \in \sigma\} \\
\varrho \wedge \sigma & := & \varrho \cap \sigma, \\
\hat{\kappa}\varrho & := & \varrho \cup \{\infty\} \text{ for } h = 1 \text{ or } \varrho = \emptyset, \\
\hat{\kappa}\varrho & := & \varrho \cup \{(a_1, a_2, \ldots, a_h) \in \widetilde{E}_k^h \mid a_1 = a_2 = \infty\} \\
 & & \text{for } h \geq 2.
\end{array}$$

Lemma 9. *For every $A \subseteq \widetilde{P}_k$, the set* $\operatorname{pInv}_k A$ *is closed with respect to ζ, τ,* pr, *$\times$, $\wedge$, $\hat{\kappa}$.*

In particular, the set $\widetilde{\mathcal{R}}_k$ of ∞-strict relations is closed under these operations, i.e., $\widetilde{\mathcal{R}}_k = \operatorname{pInv}_k e_1^{(2)}$.

Proof: Let $\varrho^{(h)}, \chi \in \operatorname{pInv}_k A$. We have to show that $\zeta\varrho, \tau\varrho, \operatorname{pr}\varrho, \varrho \times \chi, \varrho \wedge \chi, \hat{\kappa}\varrho \in \operatorname{pInv}_k A$. We only show this for $\hat{\kappa}\varrho$.

Let $f \in A^{(n)}$ and $r_1, \ldots, r_n \in \hat{\kappa}\varrho$ be arbitrary. If $r_1, \ldots, r_n \in \varrho$ then $f(r_1, \ldots, r_n) \in \varrho \subseteq \hat{\kappa}\varrho$ because $\varrho \in \operatorname{pInv}_k A$. Otherwise there is some i with $r_i \notin \varrho$, w.l.o.g. $i = 1$. Then $r_1 = (\infty, \infty, r_{31}, \ldots, r_{h1})$ and thus

$$f(r_1, r_2, \ldots, r_n) = f\begin{pmatrix} \infty & r_{12} & \cdots & r_{1n} \\ \infty & r_{22} & \cdots & r_{2n} \\ r_{31} & r_{32} & \cdots & r_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ r_{h1} & r_{h2} & \cdots & r_{hn} \end{pmatrix} = \begin{pmatrix} \infty \\ \infty \\ t_3 \\ \cdots \\ t_h \end{pmatrix}$$
$$\in \{(a_1, \ldots, a_h) \in \widetilde{E}_k^h \mid a_1 = a_2 = \infty\} \subseteq \hat{\kappa}\varrho$$

for some t_i. Thus $\hat{\kappa}\varrho$ is ∞-strict. ■

Remark 10. It holds $\operatorname{pPol}_k [Q]_{\mathrm{P}} = \operatorname{pPol}_k Q$ for every $Q \subseteq \widetilde{\mathcal{R}}_k$, and $\operatorname{pInv}_k [A] = \operatorname{pInv}_k A$ for every $A \subseteq \widetilde{P}_k$.

D. ∞-strict Relational Clones

The algebra $(\widetilde{\mathcal{R}}_k; \widetilde{\delta}_{k;\{1,2\}}^{(3)}, \zeta, \tau, \operatorname{pr}, \wedge, \times, \hat{\kappa})$ is called the *full ∞-strict relational clone* on $\widetilde{E}_k$. Every subalgebra Q of $\widetilde{\mathcal{R}}_k$ is a *∞-strict relational clone* on E_k.

Let $Q \subseteq \widetilde{\mathcal{R}}_k$. Then $[Q]_{\mathrm{P}}$ denotes the universe of the smallest ∞-strict relational clone, which contains Q. If $[Q]_{\mathrm{P}} = Q \subseteq \widetilde{\mathcal{R}}_k$ we say that Q is *closed.*

E. Derivable Operations on $\widetilde{\mathcal{R}}_k$

We derive some (term) operations from the fundamental operations of the algebra $\widetilde{\mathcal{R}}_k$. Let $\varrho^{(h)}, \sigma^{(\mu)} \in \widetilde{\mathcal{R}}_k$.

(a) *Projection onto the α_1-th, …, α_t-th coordinates*
The $\alpha_1, \ldots, \alpha_t \in H := \{1, \ldots, h\}$ are not necessarily distinct or in order. Set $J := H \setminus \{\alpha_1, \ldots, \alpha_t\}$. Let

$$\begin{aligned}\operatorname{pr}_{\alpha_1, \ldots, \alpha_t}\varrho := \{&(a_{\alpha_1}, \ldots, a_{\alpha_t}) \in \widetilde{E}_k^t \mid \\ &\exists\{a_j \mid j \in J\} \subseteq \widetilde{E}_k : (a_1, \ldots, a_h) \in \varrho\}.\end{aligned}$$

Additionally we set $\operatorname{pr}_A := \operatorname{pr}_{\alpha_1, \ldots, \alpha_t}$ for $A = \{\alpha_1, \ldots, \alpha_t\}$ with $\alpha_1 < \alpha_2 < \cdots < \alpha_t$. Furthermore $\operatorname{pr}_\emptyset \varrho = \emptyset$.

(b) *Replacement of coordinates in I with D*
We define $\operatorname{replace}_{I,D}$ for $I \subseteq H$ and $D \subseteq \widetilde{E}_k$ by

$$\begin{aligned}\operatorname{replace}_{I,D}\varrho = \{&(a_1, \ldots, a_h) \in \widetilde{E}_k^h \mid \\ &\exists(a'_1, \ldots, a'_h) \in \varrho\, \forall i \in H : \\ &(i \in I \wedge a_i \in D) \vee (i \notin I \wedge a_i = a'_i)\}.\end{aligned}$$

We note that $\operatorname{replace}_{\emptyset,D}\varrho = \varrho$, $\operatorname{replace}_{H,D}\varrho = D^{|H|}$ and $\operatorname{replace}_{I,D}\varrho = D^{|I|} \times \operatorname{pr}_{H\setminus I}\varrho$ for $I = \{1, \ldots, |I|\}$ and $\emptyset \neq I \neq H$.

(c) *Identification of coordinates Δ*
We define
$\Delta\varrho := \{(a_1, a_2, \ldots, a_{h-1}) \mid (a_1, a_1, a_2, \ldots, a_{h-1}) \in \varrho\}$.

(d) *Adding of fictitious coordinates ∇_i*
Let $\nabla_i\varrho := \{(a_1, \ldots, a_{h+1}) \in \widetilde{E}_k^{h+1} \mid (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{h+1}) \in \varrho\}$. For example ∇_1 can be derived as $\nabla_1\varrho = (\operatorname{pr}_1 \widetilde{\delta}_{k;\{1,2\}}^{(3)}) \times \varrho$.
Let $\{j_1, \ldots, j_\mu\} := J \subseteq \{1, \ldots, h\}$, $j_1 < j_2 < \cdots < j_\mu$ and $|J| = \mu$. Define unpr_J^h by

$$\operatorname{unpr}_J^h(\sigma) := \{(a_1, \ldots, a_h) \in \widetilde{E}_k^h \mid (a_{j_1}, a_{j_2}, \ldots, a_{j_\mu}) \in \sigma\}.$$

Then $\operatorname{pr}_J \operatorname{unpr}_J^h \sigma = \sigma$ and $\operatorname{unpr}_J^h \operatorname{pr}_J \varrho \supseteq \varrho$.

(e) *General ∞-extension κ_I*
Let $I \subseteq \{1, \ldots, h\}$ and $I \neq \emptyset$. Then define

$$\kappa_I\varrho := \varrho \cup \{(a_1, \ldots, a_h) \in \widetilde{E}_k^h \mid \forall i \in I : a_i = \infty\}.$$

For $I = \{1, 2\}$ and $h \geq 2$ we have $\kappa_I = \hat{\kappa}$. With permutations of coordinates one gets κ_I for all I with $|I| = 2$. Then for $I = \{i\}$ we have $\kappa_I\varrho = \operatorname{pr}_{1,\ldots,h}\kappa_{\{i,h+1\}}(\varrho \times \widetilde{E}_k)$ and for $|I| \geq 3$ we have

$$\kappa_I\varrho = \bigcap_{J \subseteq I, |J| = 2} \kappa_J\varrho.$$

(f) *Full ∞-extension κ*
Define κ by $\kappa\varrho := \varrho \cup \{(a_1, \ldots, a_h) \in \widetilde{E}_k^h \mid \exists i \in \{1, \ldots, h\} : a_i = \infty\}$. Then $\kappa\varrho = \kappa_{\{1\}}\kappa_{\{2\}} \ldots \kappa_{\{h\}}\varrho$.

(g) *Partial ∞-extension $\bar{\kappa}$*
Define $\bar{\kappa}$ by $\bar{\kappa}\varrho := \varrho \cup \{(a_1, \ldots, a_h) \in \widetilde{E}_k^h \mid \exists I \in I_\infty(\varrho)\, \forall i \in I : a_i = \infty\}$. If $I_\infty(\varrho) = \{I_1, \ldots, I_l\}$ for some $l \geq 1$ then $\bar{\kappa}\varrho = \kappa_{I_1}\kappa_{I_2} \ldots \kappa_{I_l}\varrho$.

(h) *General ∞-subextension κ_I^**
Let $I \subseteq \{1, \ldots, h\}$ and $I \neq \emptyset$. Then define

$$\begin{aligned}\kappa_I^*\varrho := \varrho \cup \{\quad &(a_1, \ldots, a_h) \in \widetilde{E}_k^h \mid \\ &(\forall i \in I : a_i = \infty) \wedge \\ &(\exists(b_1, \ldots, b_h) \in \varrho\, \forall i \notin I : a_i = b_i)\}.\end{aligned}$$

Then $\kappa_I^*\varrho = (\kappa_I\varrho) \cap (\operatorname{replace}_{I,\widetilde{E}_k}\varrho)$.

Example 11. The ∞-extensions are some new operations on relations which are needed at a few places in this paper. Let $\varrho^{(4)} \in \widetilde{\mathcal{R}}_3$ be defined by $\varrho := \{(0,1,2,0)\}$. Then

$$\hat{\kappa}\varrho = \begin{pmatrix} 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ 1 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\ 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & \infty & \infty & \infty & \infty \\ 0 & 0 & 1 & 2 & \infty & 0 & 1 & 2 & \infty & 0 & 1 & 2 & \infty & 0 & 1 & 2 & \infty \end{pmatrix},$$

$$\kappa^*_{\{1,2,3\}}\varrho = \begin{pmatrix} 0 & \infty \\ 1 & \infty \\ 2 & \infty \\ 0 & 0 \end{pmatrix}, \text{ and}$$

$$\bar{\kappa}\kappa^*_{\{1,2,3\}}\varrho = \kappa_{\{1,2,3\}}\varrho = \begin{pmatrix} 0 & \infty & \infty & \infty & \infty \\ 1 & \infty & \infty & \infty & \infty \\ 2 & \infty & \infty & \infty & \infty \\ 0 & 0 & 1 & 2 & \infty \end{pmatrix}.$$

Proposition 12. *Let* $Q \subseteq \widetilde{\mathcal{R}}_k$. *Then* $\operatorname{pInv}_k \widetilde{P}_k \subseteq [Q]_{\mathrm{P}}$.

Proof: First we have $\emptyset = \operatorname{pr}\operatorname{pr}_1 G_1(A)$, $\{\infty\} = \hat{\kappa}\emptyset$ and $\widetilde{E}_k = \hat{\kappa}\operatorname{pr}_i G_k(A)$ for some i; choose i such that the i-th row of χ_k equals $(0,1,2,\ldots,k-1)$.

Now let $\varrho^{(h)} \in \operatorname{pInv}_k \widetilde{P}_k$ be arbitrary with $\varrho \neq \emptyset$. Set $\{I(1),\ldots,I(l)\} := I_\infty(\varrho)$ and $\varepsilon(i)$ the equivalence relation associated with $I(i)$ as stated in Remark 8. Then

$$\varrho = (\kappa_{I(1)} \ldots \kappa_{I(l)}\{\infty\}^h) \cap \bigcap_{i=1}^{l} (\operatorname{unpr}^h_{I(i)} \widetilde{\delta}_{\varepsilon(i)}).$$

TODO: recursive

■

III. GALOIS CONNECTION

In this section we establish the Galois connection between partial clones and ∞-strict relational clones, which follows the proof for the Galois connection between total clones and relational clones in Lau's book [3] (pp. 131–136). Due to the limited number of pages we omit the proofs of statements which do not differ between the total case and the partial case. The main additions to the proof in that book are found in Lemma 14 (f) and in Lemma 17.

Some well-known facts about Galois connections follow.

Remark 13. For arbitrary $A, B \subseteq \widetilde{P}_k$ and arbitrary $S, T \subseteq \widetilde{\mathcal{R}}_k$, it holds:

(a) $A \subseteq B \Longrightarrow \operatorname{pInv} B \subseteq \operatorname{pInv} A$,
$S \subseteq T \Longrightarrow \operatorname{pPol} T \subseteq \operatorname{pPol} S$;

(b) $A \subseteq \operatorname{pPol}\operatorname{pInv} A$,
$S \subseteq \operatorname{pInv}\operatorname{pPol} S$;

For arbitrary $n \in \mathbb{N}$ and $k \in \mathbb{N}$, $k \geq 2$ denote by $\chi_{k;n}$ (or short χ_n) the k^n-ary relation, whose rows are just all $(x_1,\ldots,x_n) \in E_k^n$ that are arranged *lexicographically*.

For example

$$\chi_{2;3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Obviously, there is exactly one function $f_r \in \widetilde{P}_k^{(n)}$ with $f_r(\chi_n) = r$ for every column $r \in \widetilde{E}_k^{k^n}$. The relation $G_n(A) := \{r \in \widetilde{E}_k^{k^n} \mid f_r \in A^{(n)}\}$ is called the *n-th graphic of* $A \subseteq \widetilde{P}_k$.

The following lemma summarizes elementary properties of the relation $G_n(A)$.

Lemma 14. *Let* $A \subseteq \widetilde{P}_k$ *be an arbitrary partial clone. Then*

(a) $\forall n \in \mathbb{N} : G_n(A) \in \operatorname{pInv} A$;
(b) $f^{(n)} \in A^{(n)} \iff f^{(n)} \in \operatorname{pPol} G_n(A)$;
(c) $A \subseteq \ldots \subseteq \operatorname{pPol} G_n(A) \subseteq \operatorname{pPol} G_{n-1}(A) \subseteq \ldots \subseteq \operatorname{pPol} G_2(A) \subseteq \operatorname{pPol} G_1(A)$;
(d) $A = \bigcap_{n \geq 1} \operatorname{pPol} G_n(A)$;
(e) $\{\emptyset, \{\infty\}, \widetilde{E}_k\} \subseteq [\{G_n(A) \mid n \geq 1\}]_{\mathrm{P}}$;
(f) $\forall \varrho \in \operatorname{pInv} A : \varrho \in [\{G_n(A) \mid n \geq 1\}]_{\mathrm{P}}$;
(g) $\operatorname{pInv} A = [\{G_n(A) \mid n \geq 1\}]_{\mathrm{P}}$.

Proof: We only proof (e) and (f) here, since the proofs show are specific to the partial case. The other cases are similar, if not identical, to the proofs in [3].

(e): We have $\emptyset = \operatorname{pr}\operatorname{pr}_1 G_1(A)$, $\{\infty\} = \hat{\kappa}\emptyset$ and $\widetilde{E}_k = \hat{\kappa}\operatorname{pr}_i G_k(A)$ for some i; choose i such that the i-th row of χ_k equals $(0,1,2,\ldots,k-1)$.

(f): We show a construction of ϱ from the relations $G_n(A)$ by induction over the arity h. Because of (e) we can assume $\varrho \notin \{\emptyset, \{\infty\}\}$.

We start with arity $h = 0$. Then $\varrho = \emptyset$ is constructable.

Thus let $h \geq 1$ be arbitrary and assume that $\sigma \in [\{G_n(A) \mid n \geq 1\}]_{\mathrm{P}}$ for every relation $\sigma^{(\mu)} \in \operatorname{pInv} A$ with $\mu < h$.

Let $\varrho^{(h)} \in \operatorname{pInv} A$ be arbitrary. Let $\bar{\varrho} := \varrho \cap E_k^h$, i.e., the columns of ϱ without any ∞, and let $t := |\bar{\varrho}|$.

There are two cases to consider

- $\bar{\varrho} \neq \emptyset$. Then we first construct

  $$\hat{\varrho} := \{f(r_1,\ldots,r_t) \mid r_1,\ldots,r_t \in \bar{\varrho}, f \in A^{(t)}\}.$$

  As $\bar{\varrho} \subseteq \varrho$ and $\varrho \in \operatorname{pInv} A$ we get $\hat{\varrho} \subseteq \varrho$. Moreover $\hat{\varrho} \cap E_k^h = \varrho \cap E_k^h$ because $J_k^{(t)} \subseteq A^{(t)}$ and $\bar{\varrho} = \varrho \cap E_k^h$. Since $J_k \subseteq A$ we have $\chi_t \subseteq G_t(A)$ and for every row i of $\bar{\varrho}$ there is some row α_i of χ_t, i.e., $\bar{\varrho} = \operatorname{pr}_{\alpha_1,\ldots,\alpha_h} \chi_t$. Then

  $$\begin{aligned} \hat{\varrho} &= \{f(r_1,\ldots,r_t) \mid r_1,\ldots,r_t \in \operatorname{pr}_{\alpha_1,\ldots,\alpha_h} \chi_t, f \in A^{(t)}\} \\ &= \operatorname{pr}_{\alpha_1,\ldots,\alpha_h} \{f(r'_1,\ldots,r'_t) \mid r'_1,\ldots,r'_t \in \chi_t, f \in A^{(t)}\} \\ &= \operatorname{pr}_{\alpha_1,\ldots,\alpha_h} G_t(A) \end{aligned}$$

  Thus $\hat{\varrho}$ is constructable.
- $\bar{\varrho} = \emptyset$. Then there is some $r := (r_1,\ldots,r_h) \in \varrho$ and $I := \{i \in \{1,\ldots,h\} \mid r_i = \infty\}$ with $I \neq \emptyset$. Without restriction we can assume $I = \{1,\ldots,|I|\}$. Let $H := \{1,\ldots,h\}$.
  Then $\hat{\varrho} := \{\infty\}^{|I|} \times \operatorname{pr}_{H \setminus I} \varrho$ is constructable by induction because $\operatorname{pr}_{H \setminus I} \varrho \in (\operatorname{pInv} A)^{(\mu)}$ for some $\mu < h$ (if $I = H$ then $\hat{\varrho} = \{\infty\}^{|I|}$ is constructible). Furthermore, $\hat{\varrho} \subseteq \varrho$ because ϱ is ∞-strict, i.e., for all $s \in \varrho$ we have $e_1^{(2)}(s, r) \in \varrho$, and $\hat{\varrho} \cap E_k^h = \emptyset = \varrho \cap E_k^h$.

Let $H := \{1,\ldots,h\}$. Let $\varrho_1 := \hat{\varrho}$, $\{I_1,\ldots,I_{l-1}\} := I_\infty(\varrho)$ and construct for $1 \leq j \leq l-1$ the relations ϱ_{j+1} recursively.

Let $I := I_j$ and $\varrho_{j+1} := (\kappa_I \varrho_j) \cap (\widetilde{E}_k^{|I|} \times \operatorname{pr}_{H \setminus I} \varrho) = \varrho_j \cup \left(\{\infty\}^{|I|} \times \operatorname{pr}_{H \setminus I} \varrho\right)$ where we assume w.l.o.g. $I = \{1,\ldots,|I|\}$ (otherwise we can permute the rows of ϱ_j, ϱ and I first, and then permuting back the rows of ϱ_{j+1}; if $I = H$ then

$\varrho_{j+1} := \kappa_I \varrho_j$). Then $\varrho_j \subseteq \varrho_{j+1} \subseteq \varrho$ and $\varrho_{j+1} \cap E_k^h = \varrho \cap E_k^h$. Furthermore we have $s = (s_1, \ldots, s_h) \in \varrho_{j+1}$ for all $s \in \varrho$ with

$$s_i = \infty \iff i \in I.$$

Then $\varrho = \varrho_l \in [\{G_n(A) \mid n \geq 1\}]_{\mathrm{P}}$. ■

For arbitrary $A \subseteq \widetilde{P}_k$ denote by Γ_A a mapping from $\widetilde{\mathcal{R}}_k$ into $\widetilde{\mathcal{R}}_k$, which is defined for $\sigma^{(h)} \in \widetilde{\mathcal{R}}_k^{(h)}$ by

$$\Gamma_A(\sigma) := \bigcap \left\{ \varrho \in \widetilde{\mathcal{R}}_k \;\middle|\; \varrho \in \mathrm{pInv}\, A \text{ and } \sigma \subseteq \varrho \right\}.$$

Lemma 15. *Let $A \subseteq \widetilde{P}_k$ be an arbitrary partial clone and $n \in \mathbb{N}$ arbitrary. Then*

(a) $\Gamma_A(\chi_n) \in \mathrm{pInv}\, A$;
(b) $\Gamma_A(\chi_n) = G_n(A)$;
(c) $A^{(n)} = \{f_r \mid r \in \Gamma_A(\chi_n)\}$.

Proof: (a): follows from $[\mathrm{pInv}\, A]_{\mathrm{P}} = \mathrm{pInv}\, A$ by Lemma 9 and the definition of $\Gamma_A(\chi_n)$.

(b): Since every projection $e_i^{(n)}$ belongs to $A^{(n)}$, we have $\chi_n \subseteq G_n(A)$.

Let ϱ be an arbitrary k^n-ary relation of $\widetilde{\mathcal{R}}_k$ with $\chi_n \subseteq \varrho$. If $\varrho \in \mathrm{pInv}\, A$, then $f(\chi_n) \in \varrho$ for every $f \in A^{(n)}$, i.e., $G_n(A) \subseteq \varrho$. Consequently, we have shown that $G_n(A) \subseteq \Gamma_A(\chi_n)$ holds. $\Gamma_A(\chi_n) \subseteq G_n(A)$ follows from $G_n(A) \in \mathrm{pInv}\, A$ (see Lemma 14 (a)).

(c) follows from (b) and the definition of $G_n(A)$. ■

Theorem 16. *Let $A \subseteq \widetilde{P}_k$. Then*

$$[A]_{\mathrm{P}} = \mathrm{pPol}\, \mathrm{pInv}\, A.$$

Proof: By Remark 13 (b) we have $A \subseteq \mathrm{pPol}\, \mathrm{pInv}\, A$.

For the moment we assume that A is a partial clone, i.e., $A = [A]_{\mathrm{P}}$. To prove $\mathrm{pPol}\, \mathrm{pInv}\, A \subseteq A$ let $f^{(n)} \in \mathrm{pPol}\, \mathrm{pInv}\, A$ be arbitrary. Because $G_n(A) \in \mathrm{pInv}\, A$ by Lemma 14 (a) we have $f \in \mathrm{pPol}\, G_n(A)$ and then we have $f \in A^{(n)}$ by Lemma 14 (b).

Thus $[A]_{\mathrm{P}} = A = \mathrm{pPol}\, \mathrm{pInv}\, A$ holds for a partial clone A. Now let $A \subseteq \widetilde{P}_k$ be arbitrary. Then Remark 10 implies $[A]_{\mathrm{P}} = \mathrm{pPol}\, \mathrm{pInv}\, [A]_{\mathrm{P}} = \mathrm{pPol}\, \mathrm{pInv}\, A$. ■

Lemma 17. *Let $\beta^{(m)} \in \widetilde{\mathcal{R}}_k$, $f^{(t)} \in \widetilde{P}_k$ and $r_1, \ldots, r_t \in \beta$ with $f(r_1, \ldots, r_t) \notin \beta$.*

Then there is a relation $\overline{\beta}^{(\overline{m})} \in [\{\beta\}]_{\mathrm{P}} \subseteq \widetilde{\mathcal{R}}_k$ and $\overline{r}_1, \ldots, \overline{r}_t \in \overline{\beta} \cap E_k^{\overline{m}}$ with $f(\overline{r}_1, \ldots, \overline{r}_t) \notin \overline{\beta}$.

Proof: Let $M := \{1, \ldots, m\}$. Let $(r_j^1, \ldots, r_j^m) := r_j$ and $I_j := \{i \in M \mid r_j^i = \infty\}$ for all $j \in \{1, \ldots, t\}$. Moreover let $s := (s^1, \ldots, s^m) := f(r_1, \ldots, r_t)$ and $I_s := \{i \in M \mid s^i = \infty\}$. By definition of function application we know $I := \bigcup_{j=1}^{t} I_j \subseteq I_s \subseteq M$.

If $I = M$ then $I_s = M$, and thus $s = (\infty, \ldots, \infty) \in \beta$ because $r_1, \ldots, r_t \in \beta$ and β is ∞-strict. But this contradicts $s = f(r_1, \ldots, r_t) \notin \beta$.

Thus $M \setminus I \neq \emptyset$. Let $\overline{m} := |M \setminus I|$, $\overline{\beta}^{(\overline{m})} := \mathrm{pr}_{M \setminus I}\, \beta \in [\{\beta\}]_{\mathrm{P}}$ and $\overline{r}_j := \mathrm{pr}_{M \setminus I}\, r_j \in \overline{\beta}$ for all $j \in \{1, \ldots, t\}$. Since $I_j \subseteq I$ we have $\overline{r}_j \in E_k^{\overline{m}}$.

We now assume w.l.o.g. that $I = \{1, \ldots, |I|\}$. Suppose $\mathrm{pr}_{M \setminus I}\, s = f(\overline{r}_1, \ldots, \overline{r}_t) \in \overline{\beta} = \mathrm{pr}_{M \setminus I}\, \beta$. Then $s \in \{\infty\}^{|I|} \times \mathrm{pr}_{M \setminus I}\, \beta \subseteq \beta$ because β is ∞-strict in contradiction to the assumption. Thus $f(\overline{r}_1, \ldots, \overline{r}_t) \notin \overline{\beta}$. ■

Theorem 18. *Let $Q \subseteq \widetilde{\mathcal{R}}_k$. Then*

$$[Q]_{\mathrm{P}} = \mathrm{pInv}\, \mathrm{pPol}\, Q.$$

Proof: Let $A := \mathrm{pPol}\, Q$. By Remark 13 (b) we have $Q \subseteq \mathrm{pInv}\, A$.

For the moment we assume that Q is a ∞-strict relational clone, i.e., $Q = [Q]_{\mathrm{P}}$. To prove that $\mathrm{pInv}\, A \subseteq Q$ it is sufficient to show that $\Gamma_A(\chi_t) \in Q$ for all $t \in \mathbb{N}$, since $\left[\bigcup_{t \geq 1} \{\Gamma_A(\chi_t)\}\right]_{\mathrm{P}} = \mathrm{pInv}\, A$ by Lemma 14 (g) and Lemma 15 (b).

Now let $t \in \mathbb{N}$ with $t \geq 1$ be arbitrary and set

$$\gamma := \bigcap \{\varrho \in Q \mid \chi_t \subseteq \varrho\}.$$

The intersection is not empty because $\widetilde{E}_k^{k^t} \in Q$. Since Q is closed with respect to $\wedge = \cap$, we have $\chi_t \subseteq \gamma$, $\gamma \in Q$ and γ is the smallest relation in Q which contains χ_t, with respect to inclusion. Because $\gamma \in Q \subseteq \mathrm{pInv}\, A$ we have $\Gamma_A(\chi_t) \subseteq \gamma$. Consequently, our theorem is proven, if we can show $\Gamma_A(\chi_t) = \gamma$.

Suppose $\Gamma_A(\chi_t) \subset \gamma$. Then there is a column $r \in \gamma \setminus \Gamma_A(\chi_t)$. Since $A^{(t)} = \{f_s \mid s \in \Gamma_A(\chi_t)\}$ by Lemma 15 (c) we have $f_r \notin A^{(t)} = \mathrm{pPol}^{(t)}\, Q$. Thus there is some m-ary relation $\beta \in Q$ with $f_r \notin \mathrm{pPol}\, \beta$ and certain columns $r_1, \ldots, r_t \in \beta$ with $f(r_1, \ldots, r_t) \notin \beta$. By Lemma 17 we can assume $r_1, \ldots, r_t \in E_k^m$ because $\overline{\beta} \in [\{\beta\}]_{\mathrm{P}} \subseteq Q$, i.e., $\overline{\beta} \in Q$. Thus every row of the matrix $(r_1, \ldots, r_t)$ is also a row of the matrix χ_t. Let i_j be the number of the row of χ_t which agrees with the j-th row of $(r_1, \ldots, r_t)$ for all $j \in \{1, \ldots, m\}$. Let now

$$\gamma' := \mathrm{pr}_{1,2,\ldots,k^t} \left((\gamma \times \beta) \cap \widetilde{\delta}^{(k^t+m)}_{k;\{i_1,k^t+1\},\ldots,\{i_m,k^t+m\}} \right).$$

Since Q is closed, γ' belongs to Q, and by construction of γ' we have $\chi_t \subseteq \gamma' \subseteq \gamma$. Furthermore, we have $r \in \gamma \setminus \gamma'$, since $r_1, \ldots, r_t \in \beta$, $f_r(r_1, \ldots, r_t) \notin \beta$ and $f_r(\chi_t) = r \in \gamma$. But with γ' we received a contradiction to the choice of γ. Thus $\gamma = \Gamma_A(\chi_t)$ and therefore $\mathrm{pInv}\, A \subseteq Q$.

Thus $[Q]_{\mathrm{P}} = Q = \mathrm{pInv}\, \mathrm{pPol}\, Q$ holds for a ∞-strict relational clone Q. Now let $Q \subseteq \widetilde{\mathcal{R}}_k$ be arbitrary. Then Remark 10 implies $[Q]_{\mathrm{P}} = \mathrm{pInv}\, \mathrm{pPol}\, [Q]_{\mathrm{P}} = \mathrm{pInv}\, \mathrm{pPol}\, Q$. ■

Theorem 19. *Let $\mathbb{L}(\widetilde{P}_k)$ be the set of all partial clones of $\widetilde{P}_k$ and let $\mathbb{L}(\widetilde{\mathcal{R}}_k)$ be the set of all ∞-strict relational clones of $\widetilde{\mathcal{R}}_k$. Then the mappings $\mathrm{pInv} : \mathbb{L}(\widetilde{P}_k) \to \mathbb{L}(\widetilde{\mathcal{R}}_k), A \mapsto \mathrm{pInv}\, A$ and $\mathrm{pPol} : \mathbb{L}(\widetilde{\mathcal{R}}_k) \to \mathbb{L}(\widetilde{P}_k), Q \mapsto \mathrm{pPol}\, Q$ are bijective mappings, which reverse the partial order $\subseteq$. In other words: The lattices $(\mathbb{L}(\widetilde{P}_k), \subseteq)$ and $(\mathbb{L}(\widetilde{\mathcal{R}}_k), \subseteq)$ are antiisomorphic.*

Proof: By Lemma 9 and Remark 4 the mappings pInv and pPol are mappings from $\mathbb{L}(\widetilde{P}_k)$ (or $\mathbb{L}(\widetilde{\mathcal{R}}_k)$) to $\mathbb{L}(\widetilde{\mathcal{R}}_k)$ (or $\mathbb{L}(\widetilde{P}_k)$), respectively.

Let $A \in \mathbb{L}(\widetilde{P}_k)$. Then $A = \mathrm{pPol}\,\mathrm{pInv}\,A$ by Theorem 16 and $\mathrm{pInv}\,A \in \mathbb{L}(\widetilde{\mathcal{R}_k})$ by Lemma 9. Thus pPol is surjective. Let $S, T \in \mathbb{L}(\widetilde{\mathcal{R}_k})$ with $\mathrm{pPol}\,S = \mathrm{pPol}\,T$. Then $S = \mathrm{pInv}\,\mathrm{pPol}\,S = \mathrm{pInv}\,\mathrm{pPol}\,T = T$ by Theorem 18. Thus pPol is injective. Therefore pPol is a bijective mapping. Analogously, pInv is a bijective mapping.

In Remark 13 (a) it was shown that pPol and pInv reverse the partial order $\subseteq$. ■

FINAL REMARKS

The Galois connection for total clones and relational clones can be obtained from the Galois connection established in this paper if we restrict to total functions and relations on E_k. Furthermore the relations used to describe the partial clones can be restricted such that these have better properties useful in proofs.

The author could also use this Galois connection to proof the Representation Lemma of Romov [4] independently. The proof does not use $(\&, =)$-formulas of first-order calculus but instead it uses the extended set of Mal'tsev operations. Additionally at several places relations occur which do not describe strong partial clones, i.e., are not included in the original proof by Romov. The author thinks that this makes the proof more readable since these steps are explicit and expressable with concrete operations on relations.

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