A Classification of Partial Boolean Clones

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Abstract

We study intervals $\mathcal{I}(A)$ of partial clones whose total functions constitute a (total) clone A. In the Boolean case, we provide a complete classification of such intervals (according to whether the interval is finite or infinite), and determine the size of each finite interval $\mathcal{I}(A)$.

1 Introduction

Let A be a nonempty set and let T be a proper subset of A^n. For an arbitrary mapping f from T into A and for $(a_1, ..., a_n) \in A^n \setminus T$, one can use the notation $f(a_1, ..., a_n) = \infty$ with $\infty \notin A$, to indicate that $f(a_1, ..., a_n)$ is not defined. Then

$$\text{dom} f := \{(a_1, ..., a_n) \in A^n \mid f(a_1, ..., a_n) \neq \infty\}$$

is the *domain* of f. In the following, we use these notations for $A = E_k$ and for functions which are defined over subsets of E_k^n. More exact:

Let $\mathbb{N} := \{1, 2, 3, \ldots\}$ be the set of positive integers. For a fixed $k \in \mathbb{N}$, $k > 1$, set $E_k := \{0, \ldots, k-1\}$ and put $\widetilde{E}_k := E_k \cup \{\infty\}$ where $\infty \notin E_k$. For $n \in \mathbb{N}$ an *n-ary partial function* on E_k is a map $f : E_k^n \to \widetilde{E}_k$, where n is the *arity* of f. With $(x_1, ..., x_n)$ (briefly $\underset{\sim}{x}$) we denote an arbitrary n-tuple of E_k^n and usually we say that the x_i are *variables*. Furthermore, let $\mathbf{0} = (0, 0, ..., 0)$ and $\mathbf{1} := (1, 1, ..., 1)$ of appropriate length. We say that f is *total* if $\infty \notin \text{im } f := \{f(\underset{\sim}{a}) \mid \underset{\sim}{a} \in E_k^n\}$. When we introduce a partial function f we denote it by $f^{(n)}$ to specify that its arity is n. A partial function $g^{(n)}$ is a *subfunction* of $f^{(n)}$ if $g(\underset{\sim}{a}) \in \{f(\underset{\sim}{a}), \infty\}$ for all $\underset{\sim}{a} \in E_k^n$. Denote by $\widetilde{P}_k^{(n)}$ the set of all n-ary partial functions on E_k and let $\widetilde{P}_k := \bigcup_{n \in \mathbb{N}} \widetilde{P}_k^{(n)}$.

Set $P_k^{(n)} := \{f \in \widetilde{P}_k^{(n)} \mid f \text{ is total}\}$ and let $P_k := \bigcup_{n \in \mathbb{N}} P_k^{(n)}$. In the following we say "function" for "total function".

For $n \in \mathbb{N}$ and $F \subseteq \widetilde{P}_k$ set $F^{(n)} := F \cap \widetilde{P}_k^{(n)}$. For $a \in \widetilde{E}_k$ and $1 \leq i \leq n$, the *constant function* c_a^n and the *projection* e_i^n are the n-ary functions of $\widetilde{P}_k$ and P_k defined by setting $c_a^n(x_1, \ldots, x_n) := a$ and $e_i^n(x_1, \ldots, x_n) := x_i$ for all $x_1, \ldots, x_n \in E_k$. Denote by $J_k := \{e_i^n \mid n \in \mathbb{N},\ 1 \leq i \leq n\}$ the set of all projections and let C_∞ consist of all $f \in \widetilde{P}_k$ with $\text{dom } f = \emptyset$, i.e., $C_\infty = \{c_\infty^n \mid n \in \mathbb{N}\}$.

On the set P_k and $\widetilde{P}_k$ we define *superposition operations:* permutation and identification of variables, adding of fictitious variables and substitution of variables of a function by functions. The superposition operations can exactly be described by the *elementary operations* (or *Mal'cev-operations*) $\zeta, \tau, \Delta, \nabla, \star$:

For $n, m \in \mathbb{N}$ and $f^{(n)}, g^{(m)} \in \widetilde{P}_k$ let

$$(\zeta f)(x_1, x_2, ..., x_n) := f(x_2, x_3, ..., x_n, x_1),$$
$$(\tau f)(x_1, x_2, ..., x_n) := f(x_2, x_1, x_3, ..., x_n),$$
$$(\Delta f)(x_1, x_2, ..., x_{n-1}) :=$$
$$f(x_1, x_1, x_2, ..., x_{n-1}) \text{ for } n \geq 2,$$
$$\zeta f = \tau f = \Delta f = f \text{ for } n = 1,$$

$$(\nabla f)(x_1, x_2, ..., x_{n+1}) := f(x_2, x_3, ..., x_{n+1})$$

and

$$(f \star g)(x_1, ..., x_{m+n-1}) := \begin{cases} f(g(x_1, ..., x_m), x_{m+1}, ..., x_{m+n-1}) \text{ if } \\ \quad g(x_1, ..., x_m) \in E_k, \\ \infty \text{ otherwise.} \end{cases}$$

A function $f \in \widetilde{P}_k$ is called a *superposition* over F ($\subseteq \widetilde{P}_k$), if f can be obtained by a finite number of applications of the operations $\zeta, \tau, \Delta, \nabla, \star$ from the functions of F. Further, if $f \in \widetilde{P}_k^{(n)}$, $g_1, ..., g_n \in \widetilde{P}_k^{(m)}$ and the m-ary function $h \in \widetilde{P}_k$ is defined by $h(x_1, ..., x_m) := f(g_1(x_1, ..., x_m), g_2(x_1, ..., x_m), ..., g_n(x_1, ..., x_m))$, then we write briefly $h := f(g_1, ..., g_n)$. The set of all superpositions over F ($\subseteq \widetilde{P}_k$) is called *closure* of F and it is denoted by $[F]$.

A set $F \subseteq \widetilde{P}_k$ satisfying $[F] = F$ is called a *closed set* of $\widetilde{P}_k$. If $J_k \subseteq F = [F]$ then F is a *clone*. If $F := \{f_1, ..., f_r\} \subset \widetilde{P}_k$ then we write $[f_1, ..., f_t]$ instead of $[F]$.

To describe closed subsets of $\widetilde{P}_k$, h-ary relations (i.e., subsets of $\widetilde{E}_k^h$), $h \geq 1$, are suitable. We often write the elements of relations in the form of columns and we often give a relation in the form of a matrix, the columns of which are the elements of the relation.

Let $\widetilde{R}_k^{(h)}$ be the set of all h-ary relations over $\widetilde{E}_k$ and put $\widetilde{R}_k := \bigcup_{h \geq 1} \widetilde{R}_k^{(h)}$.

We say a *function* $f \in \widetilde{P}_k$ *preserves an* h*-ary relation* ϱ *over* $\widetilde{E}_k$, iff

$$f(\mathbf{r_1}, ..., \mathbf{r_n}) := \begin{pmatrix} f(r_{11}, r_{12}, ..., r_{1n}) \\ f(r_{21}, r_{22}, ..., r_{2n}) \\ \vdots \\ f(r_{h1}, r_{h2}, ..., r_{hn}) \end{pmatrix} \in \varrho$$

holds for all $\mathbf{r_1}, \mathbf{r_2}, ..., \mathbf{r_n} \in \varrho$ with $\mathbf{r_i} := (r_{1i}, r_{2i}, ..., r_{hi})$ and $i \in \{1, 2, ..., n\}$. Set $f(\underset{\sim}{a}) = \infty$ for all $\underset{\sim}{a} \in \widetilde{E}_k^n \backslash E_k^n$.

Let $pPol_k \varrho$ be the set of all functions of $\widetilde{P}_k$ that preserve the relation $\varrho \subseteq \widetilde{E}_k^h$. Furthermore, $pPOL_k \varrho := pPol_k(\varrho \cup (\widetilde{E}_k^h \backslash E_k^h))$ and $Pol_k \varrho := P_k \cap pPol_k \varrho$ for $\varrho \subseteq E_k^h$.

For the definition of the closed subsets of P_2 we use the usual symbols $\wedge, \vee, +$ and $^-$. $\wedge$ stands for conjunction, $\vee$ for disjunction, $+$ for addition modulo 2 and $^-$ for negation. Let

$M := Pol_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$,

$S := Pol_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$L := \bigcup_{n \geq 1} \{f^{(n)} \in P_2 \mid \exists a_0, ..., a_n \in E_2 : f(\underset{\sim}{x}) = a_0 + \Sigma_{n=1}^n a_i \cdot x_i\}$,

$T_{a,\mu} := Pol_2 \{0,1\}^\mu \backslash \{\overline{\mathbf{a}}\}$ for $\mu \in \mathbb{N}$ and $a \in E_2$,

$T_a := T_{a,1}, a \in E_2$,

$T_{a,\infty} := \bigcap_{\mu \geq 1} T_{a,\mu}, \ a \in E_2$,

$K := [\wedge]$ (set of all conjunctions),

$D := [\vee]$ (set of all disjunctions),

$C := [c_0, c_1]$ (set of all constant functions),

$C_a := [c_a], \ a \in \{0, 1\}$,

$I := [e_1^1]$ (set of all projections),

$\overline{I} := [^-]$.

Theorem 1 *(E. L. Post, [10])*

The set of all closed subsets of P_2 *is countably infinite. The nonempty closed subsets of* P_2 *are* $P_2,\ S,\ M,\ L,\ T_{a,\mu},\ T_{a,\mu} \cap T_{\overline{a}}, T_{a,\mu} \cap M,\ T_{a,\mu} \cap M \cap T_{\overline{a}},\ K \cup C, K \cup C_a,\ K,\ D \cup C,\ D \cup C_a,\ D,\ S \cap T_0,\ S \cap M,\ S \cap L, S \cap L \cap T_0,\ L \cap T_a,\ \overline{I} \cup C,\ I \cup C,\ \overline{I},\ I \cup C_a,\ I,\ C,\ C_a,$
where $a \in \{0, 1\}$ *and* $\mu \in \{1, 2, ..., \infty\}$.

A number of authors has found a new proof of Theorem 1 by now (see e.g. [7] or [14] and the references in [7]).

Let

$$\mathcal{I}_k(A) := \{B \subseteq \widetilde{P}_k \mid [B] = B,\ P_k \cap B = A\},$$

where A is a clone of P_k. It is the aim of this paper to show how to prove the following theorem.

Theorem 2 *Let* $A \subseteq P_2$ *be a Boolean clone. Then* $\mathcal{I}_2(A)$ *is a finite set if and only if* $T_0 \cap T_1 \cap M \subseteq A$ *or* $T_0 \cap S \subseteq A$.

2 On partial clones $C \subseteq \widetilde{P}_k$ with $C \cap P_k = Pol_k\{(0, s(0))\} \times s^o$

Let $k = 2 \cdot l$, where $l \in \mathbb{N}$. For a fixed-point-free permutation s consisting of cycles of the length 2 on E_k set

$$s^o := \{(x, s(x)) \mid x \in E_k\},$$
$$\varrho_s := \{(0, 1, x, s(x)) \mid x \in E_k\},$$
$$S := Pol_k s^o$$
$$S_{max} := pPOL_k s^o,$$
$$T := Pol_k\{(0,1)\},$$
$$T_{max} := pPOL_k\{(0,1)\},$$
and
$$(ST)_{max} := pPOL_k \varrho_s.$$

Furthermore, for $\{a_0, a_1\} \subseteq \widetilde{E}_k$ set

$$U(a_0, a_1) := \bigcup_{n \geq 1}\{f^{(n)} \in \widetilde{P}_k \mid \forall x \in \{0,1\}\ f(x, x, ..., x) = a_x\}.$$

Obviously, for all $(a_0, a_1) \in \{(\alpha, \infty), (\infty, \alpha), (\infty, \infty) \mid \alpha \in E_k\}$ we have $U(a_0, a_1) \subseteq (ST)_{max}$, $U(0,1) \not\subseteq (ST)_{max}$, $S_{max} \cap T_{max} \subset (ST)_{max}$, $(ST)_{max} \cap P_k = S \cap T$ and

$$(ST)_{max} = (S_{max} \cap T_{max}) \cup \bigcup_{\substack{(a_0, a_1) \\ \in (\widetilde{E}_k)^2 \setminus E_k^2}} U(a_0, a_1). \tag{1}$$

Lemma 3 *The partial clone $(ST)_{max}$ is the maximal element of the set $\mathcal{I}_k(S \cap T)$.*

Proof. Set w.l.o.g. $s := (0\ 1)(2\ 3)...((2l-2)\ (2l-1))$. Let $f^{(n)} \in \widetilde{P}_k \setminus (ST)_{max}$ be arbitrary. Then there are $a_1, ..., a_n, b_0, ..., b_3 \in E_k$ with

$$f\begin{pmatrix} 0 & 0 & ... & 0 \\ 1 & 1 & ... & 1 \\ a_1 & a_2 & ... & a_n \\ s(a_1) & s(a_2) & ... & s(a_n) \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

and $(b_0, b_1, b_2, b_3) \in E_k^4 \setminus \varrho_s$. It is easy to show that there are binary functions $g_1, ..., g_n \in S \cap T$ with $g_i(t, 0) := 0$ and $g_i(t, y) := a_i$ for all $i \in \{1, ..., n\}$, $t \in \{0, 2, 4, ..., 2l-2\}$ and $y \in E_k \setminus \{0\}$. Then the binary function g defined by $g := f(g_1, ..., g_n)$ belongs to $P_k \cap [\{f\} \cup (S \cap T)]$. Since

$$g\begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = f\begin{pmatrix} 0 & 0 & ... & 0 \\ 1 & 1 & ... & 1 \\ a_1 & a_2 & ... & a_n \\ s(a_1) & s(a_2) & ... & s(a_n) \end{pmatrix},$$

we have $g \notin S \cap T$. ■

In order to describe functions of $S_{max} \cap T_{max}$ we define the following relation on E_k^n:

$$\underset{\sim}{x} =_s \underset{\sim}{y} \text{ if } s^i(\underset{\sim}{x}) = \underset{\sim}{y} \text{ for some } i \in \{0, 1\},$$

where $s^i(\underset{\sim}{x}) = s^i((x_1, ..., x_n)) := (s^i(x_1), ..., s^i(x_n))$, $s^0(x) := x$, $s^{i+1}(x) := s(s^i(x))$.

Obviously, $=_s$ is an equivalence relation on E_k^n and it partitions E_k^n into equivalence classes $U_1 := \{\mathbf{0}, \mathbf{1}\}$ and U_i ($i = 2, ..., r_n$, where $r_n := k^n/2$). Choose $v_1 := \mathbf{0}$ and $v_i \in U_i$ ($i = 2, ..., r_n$) and $V_n := \{v_1, v_2, ..., v_{r_n}\}$. Since for all $f^{(n)} \in S$, $\underset{\sim}{b} \in E_k^n$ and $i \in E_2$ $f(s^i(\underset{\sim}{b})) = s^i(f(\underset{\sim}{b}))$, a function $f^{(n)} \in S \cap T$ is fully determined by its values on V_n, where $f(\mathbf{0}) = 0$.

For every $f^{(n)} \in S_{max} \cap T_{max}$ set

$$\chi(f) := \{(f(\underset{\sim}{a}), f(s(\underset{\sim}{a}))) \mid \underset{\sim}{a} \in E_k^n\}$$

and, for $(a_0, a_1) \in \{(0,1), (x, \infty), (\infty, x), (\infty, \infty) \mid x \in E_k\}$ and for $(a_0, a_1) \in R \subseteq \widetilde{E}_k^2$, set

$$F_R(a_0, a_1) := \{\ g \in S_{max} \cap T_{max} \mid \chi(g) \subseteq R,\ g(\mathbf{0}) = a_0,\ g(\mathbf{1}) = a_1\}.$$

Obviously,

$$S_{max} \cap T_{max} = \bigcup_{\substack{(a_0, a_1) \\ \in \{(0,1)\} \cup (\widetilde{E}_k^2 \setminus E_k^2)}} \bigcup_{\substack{R \\ (a_0, a_1) \in R \subseteq \widetilde{E}_k^2}} F_R(a_0, a_1) \tag{2}$$

holds.

Lemma 4 *Let $f \in \widetilde{P}_k$ with $(f(\mathbf{0}), f(\mathbf{1})) = (\alpha, \beta)$.*

1) *If $f^{(n)} \in (ST)_{max} \setminus (S_{\max} \cap T_{max})$ then $U(\alpha, \beta) \subseteq [\{f\} \cup (S \cap T)]$.*

2) *If $f \in (S_{\max} \cap T_{max})$ then $F_{\chi(f)}(\alpha, \beta) \subseteq [\{f\} \cup (S \cap T)]$.*

Proof. 1): Let $f^{(n)} \in (ST)_{max} \backslash (S_{\max} \cap T_{max})$. Then $(\alpha, \beta) \in \widetilde{E}_k^2 \backslash E_k^2$. We can assume $\alpha = \infty$. Furthermore there is a tuple $(a_1, ..., a_n) \in E_k^n \backslash \{\mathbf{0}, \mathbf{1}\}$ with

$$(a, b) := (f(a_1, ..., a_n), f(s(a_1), ..., s(a_n))) \in E_k^2 \backslash s^o.$$

Define $V_m := \{v_1, ..., v_{r_m}\}$ as in the proof of Lemma 3. One can find the functions

$$f_1^{(m)}, ..., f_n^{(m)}, r^{(m)}, t^{(2)} \in S \cap T$$

with the properties $f_i(v_j) = a_i$, $r(v_j) = 0$ for all $i \in \{1, 2, ..., n\}$ and all $v_j \in V_m \backslash \{v_1\}$ and

$$t \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} = \begin{pmatrix} c \\ c \end{pmatrix}$$

for certain $c \in E_k$ with

$$(c, \beta) \notin s^o. \tag{3}$$

Thus

$$q^{(m)} := t(r, f(f_1, ..., f_n)) \in [\{f\} \cup (S \cap T)],$$

where

$$q(\underset{\sim}{x}) = \begin{cases} \alpha & \text{for } \underset{\sim}{x} = \mathbf{0}, \\ \beta & \text{for } \underset{\sim}{x} = \mathbf{1}, \\ c & \text{otherwise.} \end{cases}$$

Let

$g^{(m)} \in U^\star(\alpha, \beta) :=$
$\bigcup_{m \geq 1} \{g^{(m)} \in U(\alpha, \beta) \mid \forall \underset{\sim}{x} \in E_k^m \backslash \{\mathbf{0}, \mathbf{1}\}\ g(\underset{\sim}{x}) \in E_k\}$.

Since a function $h^{(m+1)}$ belongs to $S \cap T$ with the properties

$$h(\beta, 1, 1, ..., 1) = \beta, \text{ if } \beta \neq \infty$$

and

$$\forall \underset{\sim}{x} \in E_k^m \backslash \{\mathbf{0}, \mathbf{1}\}\ h(c, x_1, ..., x_n) = g(x_1, ..., x_n)$$

(see (3)), we have

$$g(\underset{\sim}{x}) = h(q(\underset{\sim}{x}), \underset{\sim}{x}),$$

i.e., $U^\star(\alpha, \beta) \subseteq [\{f\} \cup (S \cap T)]$.

Let $\underset{\sim}{a} \in E_k^m \backslash \{\mathbf{0}, \mathbf{1}\}$. Next we show that a function $u_{\underset{\sim}{a}}^{(m)}$ with

$$\begin{aligned} & u_{\underset{\sim}{a}}(\underset{\sim}{a}) = \infty,\ u_{\underset{\sim}{a}}(\beta) = \beta, \text{ if } \beta \neq \infty, \\ & u_{\underset{\sim}{a}}(\underset{\sim}{x}) \in E_k \text{ for all } \underset{\sim}{x} \in E_k^m \backslash \{\mathbf{0}, \mathbf{1}\} \end{aligned} \tag{4}$$

belongs to $[\{f\} \cup (S \cap T)]$.
We can assume that $\beta \in \{1, \infty\}$. If $\beta \in E_k \backslash \{1\}$ then we can choose f' instead of f, where $f'(\underset{\sim}{x}) := t'(x_1, f(\underset{\sim}{x}))$ and $t' \in S \cap T$ with $t'(1, \beta) = 1$. Let $p_i^{(m)} \in S \cap T$ with $p_i(\underset{\sim}{a}) = 0$, $p_i(v_j) = a_i$ for $\underset{\sim}{a} \notin \{v_j, s(v_j)\}$, $i = 1, ..., n$. Then $u_{\underset{\sim}{a}} := f(p_1, ..., p_n)$ fulfills (4).

Let now $g^{(m)} \in U(\alpha, \beta)$ be arbitrary. Then we have

$$g = e_1^{(m+1)}(g', u_{\underset{\sim}{a}_1}, ... u_{\underset{\sim}{a}_d}),$$

where

$$\{\underset{\sim}{a}_1, ..., \underset{\sim}{a}_d\} = \{\underset{\sim}{x} \in E_k^m \backslash \{\mathbf{0}, \mathbf{1}\} \mid g(\underset{\sim}{x}) = \infty\}$$

and $g' \in U^\star(\alpha, \beta)$ m-ary with $g'(\underset{\sim}{x}) = g(\underset{\sim}{x})$ for all $\underset{\sim}{x}$ with $g(\underset{\sim}{x}) \in E_k$.

2): Let $g^{(m)} \in F_{\chi(f)}(\alpha, \beta)$ be arbitrary. Then, for every $v \in V_m$, $(g(v), g(s(v))) = (f(b_v), f(s(b_v)))$ for some $b_v := (b_{v,1}, b_{v,2}, ..., b_{v,n}) \in V_n$. It is easy to check that there are functions $g_i^{(m)} \in S \cap T$ $(i = 1, 2, ..., n)$ with the property $g_i(v) = b_{v,i}$ for all $i \in \{1, ..., n\}$ and $v \in V_m$. Then we have $g = f(g_1, ..., g_n)$, i.e. 2) holds. ■

From (1), (2), Lemma 3 and Lemma 4 it follows

Theorem 5 *The set $\mathcal{I}_k(S \cap T)$ is finite.*

■

3 A classification of all partial Boolean clones

Theorem 6 *Let A be a closed set of P_2 with*

$$A \subseteq B \in \{L, D \cup C, K \cup C, T_{0,\infty}, T_{1,\infty}\}.$$

Then the set $\mathcal{I}_2(A)$ has the cardinality of continuum.

Proof. The theorem has been proven in [12] for $B \in \{[P_2^{(1)}], D \cup C, K \cup C, T_{0,\infty}, T_{1,\infty}\}$, in [1] for $A = L$, and is a conclusion of [1] for $A \in \{L \cap T_0, L \cap T_1, L \cap S, L \cap T_0 \cap S\}$. ∎

Theorem 7 *Let A be a closed set of P_2 with*

$$T_0 \cap T_1 \cap M \subseteq A \ \text{or}\ T_0 \cap S \subseteq A.$$

Then $\mathcal{I}_2(A)$ is a finite set and it holds:

A	$\lvert\mathcal{I}_2(A)\rvert$
P_2	3
$T_a\ (a \in \{0,1\})$	6
M	6
S	6
$T_0 \cap T_1$	30
$T_a \cap M\ (a \in \{0,1\})$	15
$T_0 \cap T_1 \cap M$	101
$T_0 \cap S$	413

Proof. The finiteness of the set $\mathcal{I}_2(S \cap T_0)$ results from Theorem 5. The statement $|\mathcal{I}_2(S \cap T_0)| = 413$ was calculated with the help of a computer (programme description in [9]). One finds the proofs of the remaining statements in [2], [1], [11], [13], [3], [4], and [6]. ∎

We need the next theorem for the proof of Theorem 9.

Theorem 8 *Let A be a clone of P_k and I a nonempty set. Furthermore, for every $i \in I$, let Q_i be a subset of $\widetilde{P}_k$ with the following three properties:*

1) $Q_i \cap P_k = \emptyset$,

2) $[Q_i] = Q_i$,

3) $Q_i \star A \subseteq Q_i$, and $A \star Q_i \subseteq Q_i$.

Then, for every partial clone B with $B \subseteq A$, it holds:

(a) $[Q_i \cup B] = Q_i \cup B$ for all $i \in I$.

(b) $|\mathcal{I}_k(B)| \geq |I|$, if $Q_i \neq Q_j$ for all $i, j \in I$ with $i \neq j$.

Proof. (a): Let $i \in I$ and $B = [B] \subseteq A$ be arbitrary. Since B and Q_i are closed subsets of $\widetilde{P}_k$, the set $Q_i \cup B$ is closed with respect to the operations ζ, τ, Δ and ∇. Thus, we have to show that $Q_i \cup B$ is closed with respect to the operation $\star$. Let $f^{(n)}, g^{(m)} \in Q_i \cup B$ be arbitrary. Then the following two cases are possible:

Case 1: $\{f, g\} \subseteq B$.

Since B is a clone, $f \star g \in B$ holds in this case.

Case 2: $\{f, g\} \cap Q_i \neq \emptyset$.

Obviously, by assumptions 2) and 3), we have $f \star g \in Q_i$.

Therefore, (a) holds.

The statement (b) is a conclusion of 1) and (a). ∎

Theorem 9 *Let A be a clone of P_2 with*

$$A \in \{T_{0,\mu}, T_{0,\mu} \cap T_1, T_{0,\mu} \cap M, T_{0,\mu} \cap T_1 \cap M, T_{0,\mu}, T_{1,\mu} \cap T_0, T_{1,\mu} \cap M, T_{1,\mu} \cap T_0 \cap M, S \cap M\}$$

for $\mu \in \mathbb{N} \setminus \{1\}$. Then $\mathcal{I}_2(A)$ is an infinite set.

Proof. Without loss of generality let $A \subseteq T_{0,2}$. Further, we set $U_0 := \{f \in \widetilde{P}_2 \mid f(\mathbf{0}) = \infty\}$ and

$$q_0^{(n)}(x_1, ..., x_n) := \begin{cases} \infty & \text{for} \quad x_1 = ... = x_n = 0, \\ 0 & \text{otherwise,} \end{cases}$$

for $n \in \mathbb{N}$.

To prove Theorem 9 we make use of Theorem 8 and choose the sets A, I and Q_i as follows:

$$A := T_{0,2},$$
$$I := \mathbb{N},$$
$$Q_i := [\{q_0^{(i)}\} \cup T_{0,2}] \cap U_0 \text{ for } i \in \mathbb{N}.$$

Obviously, A and Q_i are closed sets for every $i \in \mathbb{N}$. Since $U_0 \cap P_k = \emptyset$, we have $Q_i \cap P_k = \emptyset$ for all $i \in \mathbb{N}$. Next we show that the above sets A and Q_i also satisfy the condition 3) from Theorem 8:

Let $f \in Q_i \star T_{0,2}$ be arbitrary. Then there are functions $f_1, f_2 \in \widetilde{P}_k$ with $f_1 \in [\{q_0^{(i)}\} \cup T_{0,2}] \cap U_0$, $f_2 \in T_{0,2}$ and $f = f_1 \star f_2$. Then $\{f_1, f_2\} \subseteq [\{q_0^{(i)}\} \cup T_{0,2}]$ and $f_1 \star f_2 \in U_0$, since $U_0 \star T_0 \subseteq U_0$. Thus f belongs to $Q_i = [\{q_0^{(i)}\} \cup T_{0,2}] \cap U_0$, i.e., $Q_i \star T_{0,2} \subseteq Q_i$. Analogously one can prove that $T_{0,2} \star Q_i \subseteq Q_i$ holds.

Therefore, the above sets A and Q_i fulfill the conditions 1)–3) given in Theorem 8.

Let $\varrho_1 := \{(0,0), (0,1), (\infty, 0), (\infty, 1), (\infty, \infty)\}$ and $\varrho_m :=$

$\{(0, a_1, ..., a_m) \in E_2^{m+1} \mid |\{i \in \{1, ..., n\} \mid a_i = 1\}| \leq 1\}$

$\cup \{(\infty, b_1, ..., b_m) \mid (b_1, ..., b_m) \in \{0, 1, \infty\}^m \setminus E_2^m\}$

for $m \geq 2$. It is proved in [8] that the following statements are valid:

$$[\{q_0^{(i)}\} \cup T_{0,2}] \cap U_0 \subseteq pPol_2\varrho_{i+1}, \quad \text{and}$$
$$[\{q_0^{(j)}\} \cup T_{0,2}] \cap U_0 \not\subseteq pPol_2\varrho_i \ \text{ for } \ i \leq j.$$

Thus, the closed sets Q_i ($i \in \mathbb{N}$) are pairwise different. Therefore, by Theorem 8, (a) and (b), we have that $\mathcal{I}_2(B)$ with $B = [B] \subseteq T_{0,2}$ is an infinite set. ■

Remarks

1) One finds some generalizations of the results given above in [3], [5] and [9].

2) Let $A \in \{T_{0,\mu}, T_{0,\mu} \cap T_1, T_{0,\mu} \cap M, T_{0,\mu} \cap T_1 \cap M, T_{0,\mu}, T_{1,\mu} \cap T_0, T_{1,\mu} \cap M, T_{1,\mu} \cap T_0 \cap M, S \cap M\}$ with $\mu \in \mathbb{N} \setminus \{1\}$. The set $\mathcal{I}_2(A)$ is infinite by Theorem 6. The authors, however, do not know whether $\mathcal{I}_2(A)$ is countable or has the cardinality of continuum. The following statement is proved in [8]: *Assume, there is a partial clone in $\mathcal{I}_2(A)$ which has an infinite basis. Then $\mathcal{I}_2(A)$ has the cardinality of continuum.*

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