A Sheffer-Criterion for partial 4-valued logic

KARSTEN SCHÖLZEL[1*]

Institute for Mathematics
University of Rostock
18051 Rostock
Germany

We determine the minimal covering of maximal partial clones in 4-valued logic. That means a necessary and sufficient condition for partial Sheffer functions in 4-valued logic in terms of the maximal partial clones is established. Furthermore several statements about members of the minimal covering of the maximal partial clones for any finite-valued logic are established.

Key words: partial functions; partial Sheffer functions; maximal clones; minimal covering

1 INTRODUCTION

Since the first two sections of this paper were nearly identical to the first two in the paper *Counting the maximal partial clones on a finite set* [13] also appearing in this issue it was suggested by the editors to remove some of the duplication. Thus kindly read the sections "Introduction" and "Definitions and Theorem of Haddad and Rosenberg" in that paper before you continue.

[*] email: karsten.schoelzel@uni-rostock.de

2 FURTHER DEFINITIONS

Definition 2.1. Let $\varrho^{(h)} \in \mathcal{R}_k$ and $A = \{a_0, \ldots, a_{l-1}\} \subseteq E_h$ with $a_i < a_j$ for all $i < j$. Then

$$\begin{aligned}\operatorname{pr}_A \varrho &:= \operatorname{pr}_{a_0,\ldots,a_{l-1}} \varrho \\ &:= \{(x_{a_0}, \ldots, x_{a_{l-1}}) \mid \exists x_0, \ldots, x_{h-1} \in E_k : (x_0, \ldots, x_{h-1}) \in \varrho\}.\end{aligned}$$

Definition 2.2. Let $\varrho^{(h)} \in \mathcal{Q}$ be arbitrary and $\varrho^\star := \operatorname{pr}_0(\varepsilon(\varrho) \setminus \iota_h^2)$, i.e., the elements of E_h which are in no singleton class of $\varepsilon(\varrho)$. We define

$$\begin{array}{rcl}\operatorname{pp} \varrho &:=& \operatorname{pr}_{\varrho^\star} \varrho, \\ \|\varrho\| &:=& |\varrho^\star|, \\ \mathcal{Q}_0 &:=& \{\chi^{(\mu)} \in \mathcal{Q} \mid \varepsilon(\chi) \text{ has no singular equivalence class}\} \\ \big(= && \{\chi^{(\mu)} \in \mathcal{Q} \mid \operatorname{pp} \chi = \chi\} = \{\chi^{(\mu)} \in \mathcal{Q} \mid \|\chi\| = \mu\}\big), \\ \mathcal{Q}_0' &:=& \{\chi^{(\mu)} \in \mathcal{Q}_0 \mid (\mu > 2) \vee (\forall x \in E_k\ \forall \pi \in S_2 : \{x\} \times E_k \not\subseteq \varrho^{[\pi]})\}, \\ \mathcal{Q}_1 &:=& \mathcal{Q} \setminus \mathcal{Q}_0.\end{array}$$

Because $\operatorname{pPOL}_k \varrho = \operatorname{pPOL}_k \varrho^{[\pi]}$ for all $\pi \in S_h$ we can assume $\operatorname{pp} \varrho = \operatorname{pr}_{E_{\|\varrho\|}} \varrho$ for all $\varrho \in \mathcal{Q}$.

The relations in $\mathcal{Q}_1$ are exactly the coherent quasi-diagonal relations ϱ where $\varepsilon(\varrho)$ has at least one singular class.

Example 2.3. Let $k = 10$ and

$$\varrho^{(5)} := \begin{pmatrix} 0 & 5 \\ 1 & 6 \\ 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{pmatrix} \cup \delta^{(5)}_{\{0,1\},\{2,3\}} \in \mathcal{Q}.$$

Then

$$\begin{array}{rcl}\varepsilon(\varrho) &=& \begin{pmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 2 & 0 & 1 & 2 & 3 & 4 \end{pmatrix}, \\ \varrho^\star &=& \{0, 1, 2, 3\}, \\ \operatorname{pp} \varrho &=& \begin{pmatrix} 0 & 5 \\ 1 & 6 \\ 2 & 7 \\ 3 & 8 \end{pmatrix} \cup \delta^{(4)}_{\{0,1\},\{2,3\}}, \\ \|\varrho\| &=& 4.\end{array}$$

Then $\varrho \in \mathcal{Q}_1$ and $\operatorname{pp} \varrho \in \mathcal{Q}_0$.

3 MINIMAL COVERING

We want to determine which maximal partial clones in the criterion in Theorem 2.7 [13] are needed to solve the completeness problem for partial Sheffer functions.

Definition 3.1. A set $\mathscr{X} \subseteq p\mathscr{M}_k$ is a *minimal covering* of $p\mathscr{M}_k$, if for every $f \in \widetilde{P}_k$ holds

$$[f]_{\mathrm{P}} = \widetilde{P}_k \iff \forall A \in \mathscr{X} : f \notin A$$

and for each $A \in \mathscr{X}$ there is some $f \in \widetilde{P}_k$ with

$$[f]_{\mathrm{P}} \neq \widetilde{P}_k \wedge (\forall B \in \mathscr{X} \setminus \{A\} : f \notin B) .$$

Theorem 3.2 (Theorem 7.13 [14])**.** *There is exactly one minimal covering $p\mathscr{C}_k$ of $p\mathscr{M}_k$ for every $k \geq 2$.*

Lemma 3.3 (Lemma 7.14 [14])**.** *Let $C \in p\mathscr{M}_k$ and $\mathscr{C} \subseteq p\mathscr{M}_k \setminus \{C\}$ such that for all $f \in C$ there is some $C' \in \mathscr{C}$ with $f \in C'$. Then $C \notin p\mathscr{C}_k$.*

Lemma 3.4 (Lemma 7.15 [14])**.** *Let $C \in p\mathscr{M}_k$. Then*

$$C \in p\mathscr{C}_k \iff (\exists f \in C\, \forall B \in p\mathscr{M}_k \setminus \{C\} : f \notin B).$$

4 A PRODUCT OF FUNCTIONS

Definition 4.1. Let $D' \in E_k^{a \times b}$ be an $a \times b$ matrix on E_k, i.e.,

$$D' = \begin{pmatrix} d_{11} & \dots & d_{1b} \\ \vdots & \ddots & \vdots \\ d_{a1} & \dots & d_{ab} \end{pmatrix}$$

with $d_{ij} \in E_k$ for all i, j.

Let $v := (v_1, \dots, v_a) \in E_k^a$. We define a function $f^{(n)} \in \widetilde{P}_k$ by $f(D') := v$. Then $n := b$, $\operatorname{dom} f := D := \{(d_{i1}, \dots, d_{ib}) \mid i \in \{1, \dots, a\}\}$ and $f(d_{i1}, \dots, d_{ib}) := v_i$ for all $i \in \{1, \dots, a\}$.

Let $\chi^{(h)} \in \mathcal{R}_k$ and $v \in E_k^h$. Then we can define $f^{(n)} \in \widetilde{P}_k$ by $f(\chi) = v$, where we assume χ to be given as a matrix as explained before and thus $n = |\chi|$.

If the domain $D := \operatorname{dom} f$ is given then D' is a matrix whose rows are the entries of D in lexicographical order.

Definition 4.2. Let $f^{(n)} \in \widetilde{P}_k$ with $D = \operatorname{dom} f$ and $g^{(m)} \in \widetilde{P}_k$ with $E = \operatorname{dom} g$. Then $D' \in E_k^{|D|\times n}$ and $E' \in E_k^{|E|\times m}$. Then $F^{(N)} := (f \otimes g) \in \widetilde{P}_k^{(n\cdot m)}$ is defined by

$$F(D' \otimes E') := F\left(\begin{array}{c|c|c} D'_{*1} & \dots & D'_{*n} \\ \hline E' & \dots & E' \end{array}\right) := \left(\begin{array}{c} f(D') \\ \hline g(E') \end{array}\right). \tag{4.1}$$

We assume E has no constant rows so F is well-defined.

Let $c = (c_1, \dots, c_N) \in \operatorname{dom} F$ be a row. Then we say it is from the E-part or g-part of F if $c = \operatorname{pr}_i(E', E', \dots, E')$ for some i. Otherwise we say it is from the D-part or f-part of F.

Likewise we set $f \otimes g_1 \otimes g_2 \otimes \dots \otimes g_l := (\dots((f \otimes g_1) \otimes g_2) \otimes \dots) \otimes g_l$ with $g_i \in \widetilde{P}_k$ for all $i \in \{1, \dots, l\}$.

Example 4.3. Let $f, g \in \widetilde{P}_k$ be given by

$$f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ g\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

where

$$D' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, D = \operatorname{dom} f = \{(0,0), (0,1)\},$$
$$E = \operatorname{dom} g = \{(0,2,3), (2,4,5)\}.$$

Then

$$(f \otimes g)\left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 0 & 2 & 3 & 0 & 2 & 3 \\ 2 & 4 & 5 & 2 & 4 & 5 \end{array}\right) = \left(\begin{array}{c} 1 \\ 2 \\ \hline 4 \\ 0 \end{array}\right)$$

and

$$\operatorname{dom}(f \otimes g) = \{(0,0,0,0,0,0), (0,0,0,1,1,1), (0,2,3,0,2,3), (2,4,5,2,4,5)\}.$$

5 KNOWN CRITERIA

Lemma 5.1 (Lemma 4 [1]). *The maximal partial clone* $(P_k \cup C_\emptyset) \in p\mathscr{C}_k$.

Lemma 5.2 (Lemmas 5, 7 [1]). *Let* $\varrho \in \mathcal{U}$. *Then* $\operatorname{pPOL}_k \varrho \in p\mathscr{C}_k$.

Lemma 5.3 (Lemma 5.3 [14])**.** *Let $\varrho^{(h)}$ be a coherent relation with $h \geq 2$ and $f^{(n)} \in \widetilde{P}_k$. Let $c_{*1}, c_{*2}, \dots, c_{*n} \in \varrho$ with $c_{1*}, \dots, c_{h*} \in \operatorname{dom}(f)$ and $c_{i'*} = c_{i''*}$ for some $i' < i''$. Then $d := f(c_{*1}, c_{*2}, \dots, c_{*n}) \in \varrho$.*

Let

$$\mathcal{S}' := \{\chi^{(\mu)} \in \mathcal{S} \mid (\mu = 3) \wedge \\ (\forall x \in \sigma(E_k^2)\, \exists a \in E_k \setminus \omega(x)\, \forall y \in \sigma(\chi) : \omega(x) \cup \{a\} \neq \omega(y))\}.$$

Lemma 5.4 (Lemma 5.5 [14])**.** *Let $\varrho \in \mathcal{Q}_0 \setminus \mathcal{Q}_0'$. Then $\mathrm{pPOL}_k\, \varrho \notin p\mathscr{C}_k$.*

Definition 5.5. Let $\varrho, \chi \in \widetilde{\mathcal{R}}_k^{\max}$ with $\varrho \neq \chi$ (i.e., $\mathrm{pPOL}_k\, \varrho \neq \mathrm{pPOL}_k\, \chi$ by definition of $\widetilde{\mathcal{R}}_k^{\max}$). We write $\varrho \ll \chi$ iff

$$\forall f \in \mathrm{pPOL}_k\, \varrho \Big(\exists g \in \mathrm{pPOL}_k\, \varrho : (g \notin \mathrm{pPOL}_k\, \chi) \wedge \\ \Big(\forall \psi \in \widetilde{\mathcal{R}}_k^{\max}\, (f \notin \mathrm{pPOL}_k\, \psi \Longrightarrow g \notin \mathrm{pPOL}_k\, \psi)\Big)\Big)$$

Lemma 5.6 (Lemma 6.2 [14])**.** *Let $X = \mathrm{pPOL}_k\, \varrho \in p\mathscr{M}_k$, $f \in X$, and $\mathscr{Y}, \mathscr{Z} \subseteq p\mathscr{M}_k$ with $f \notin Y$ for all $Y \in \mathscr{Y}$ and $\mathscr{Z} = \{\mathrm{pPOL}_k\, \psi \mid \psi \in \widetilde{\mathcal{R}}_k^{\max} \wedge \varrho \ll \psi\}$.*

Then there is some $F \in X$ with $F \notin Y$ for all $Y \in \mathscr{Y} \cup \mathscr{Z}$.

Lemma 5.7 (Lemma 6.5 [14])**.** *Let $\varrho^{(h)} \in \mathcal{A} \cup \mathcal{Q}_0' \cup \mathcal{L}$ and $\chi^{(\mu)} \in (\mathcal{Q} \cup \mathcal{L} \cup \mathcal{S}) \setminus \{\varrho\}$. Then $\varrho \ll \chi$.*

6 RELATIONS FROM $\mathcal{A} \cup \mathcal{Q}_0$

6.1 Necessary relations from $\mathcal{A} \cup \mathcal{Q}_0$

Lemma 6.1. *Let*

$$\varrho = \begin{pmatrix} 0 & 1 & 1 & 2 & 2 & 3 & 3 & 0 & 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 1 & 3 & 2 & 0 & 3 & 0 & 1 & 2 & 3 \end{pmatrix}.$$

Then $\mathrm{pPOL}_4\, \varrho \in p\mathscr{C}_4$.

Proof. We construct a function $f \in \mathrm{pPOL}_4\, \varrho$ with $f \notin \mathrm{pPOL}_4\, \chi$ for all $\chi \in \mathcal{U} \cup \mathcal{A}$. By Lemmas 5.7 and 5.6 there is some $F \in \mathrm{pPOL}_4\, \varrho$ with $F \notin \mathrm{pPOL}_4\, \chi$ for all $\chi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \{\varrho\}$, i.e., $\mathrm{pPOL}_4\, \varrho \in p\mathscr{C}_4$.

Define $\varphi : E_4 \to E_4$ by $\varphi(x) := x + 1 \pmod 4$ for all $x \in E_4$. Then $\varphi \in \mathrm{pPOL}_4\, \varrho$ and $\varphi \notin \chi$ for all $\chi \in \mathcal{U}$.

Let $\chi^{(\mu)} \in \mathcal{A}$ with $\varphi \in \mathrm{pPOL}_4\, \chi$ be arbitrary. Let $g_\chi(\chi) := (0, 0, \dots, 0)$ and g_χ not defined elsewhere. Then $g_\chi \in \mathrm{pPOL}_4\, \varrho$ and $g_\chi \notin \mathrm{pPOL}_4\, \chi$.

Let $\{\chi_1, \ldots, \chi_m\} := \{\chi \in \mathcal{U} \cup \mathcal{A} \mid \varphi \in \mathrm{pPOL}_4\, \chi\}$. Define $f_0 := \varphi$, $f_I := f_{I-1} \otimes g_{\chi_I}$ for all $I \in \{1, 2, \ldots, m\}$ and $f := f_m$. Then $f \notin \chi$ for all $\chi \in \mathcal{U} \cup \mathcal{A}$ by construction.

We show $f \in \mathrm{pPOL}_4\, \varrho$ by induction. First $f_0 = \varphi \in \mathrm{pPOL}_4\, \varrho$. Let $f_{I-1} \in \mathrm{pPOL}_4\, \varrho$ and assume $f_I^{(n)} \notin \mathrm{pPOL}_4\, \varrho$. Then there are rows c_{1*}, c_{2*} with $c_{*1}, \ldots, c_{*n} \in \varrho$ and $f(c_{*1}, \ldots, c_{*n}) \in E_4^2 \setminus \varrho$.

Because $f_{I-1} \in \mathrm{pPOL}_4\, \varrho$ and $g_{\chi_I} \in \mathrm{pPOL}_4\, \varrho$ the row c_{1*} is w.l.o.g. from the f_{I-1}-part of f_I and c_{2*} from the g_{χ_I}-part of f_I. Then

$$\begin{pmatrix} x & x & x & x \\ 0 & 1 & 2 & 3 \end{pmatrix} \subseteq \varrho$$

for some $x \in E_4$ by construction contradicting the choice of ϱ. Thus $f_I \in \mathrm{pPOL}_4\, \varrho$ and by induction $f = f_m \in \mathrm{pPOL}_4\, \varrho$. By construction $f \notin \mathrm{pPOL}_4\, \chi$ for all $\chi \in \mathcal{U} \cup \mathcal{A}$. Thus by Lemmas 5.7 and 5.6 there is some $F \in \mathrm{pPOL}_4\, \varrho$ with $F \notin \mathrm{pPOL}_4\, \chi$ for all $\chi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \{\varrho\}$, i.e., $\mathrm{pPOL}_4\, \varrho \in p\mathscr{C}_4$. □

As shown in Lemma 8 in [1] every maximal partial clone described by an equivalence relation belongs to $p\mathscr{C}_k$. Theorem 6.3 extends the statement to coherent relations which can be used to non-trivially partition the set E_k. In particular Lemma 25 from [1] is proven.

Lemma 6.2. *Let $\varrho^{(h)} = \sigma \cup \delta$ be a coherent relation on E_k and $A \subseteq E_k$ with $|A| \geq h$ and $(A^h \setminus \iota_k^h) \cap \varrho \neq \emptyset$. Then there is a relational homomorphism $\varphi : E_k \to E_k$ from ϱ to $\chi := A^h \cap \varrho$.*

Proof. Because ϱ is coherent, there is a relational homomorphism $\varphi_0 : E_k \to E_h$ from ϱ to $M(\varrho)$. Then let $\varphi_\star(\eta_h) := s \in (A^h \setminus \iota_k^h) \cap \varrho$. Then $\varphi_\star : E_h \to E_k$ is a relational homomorphism because $\varphi_\star(\eta_h^{[\pi]}) = s^{[\pi]} \in \varrho$ for all $\pi \in \Gamma_\sigma$ and $\varphi_\star(\delta \cap E_h^h) \subseteq \delta \subseteq \varrho$. Then $\varphi : E_k \to E_k$ with $\varphi(x) := \varphi_\star(\varphi_0(x))$ for all $x \in E_k$ is the requested relational homomorphism. □

Theorem 6.3. *Let $\varrho^{(h)} \in \mathcal{A} \cup \mathcal{Q}_0$ with $\varrho = \sigma_1 \cup \sigma_2$, $\omega(\sigma_1) \cap \omega(\sigma_2) = \emptyset$, and $\sigma_i \neq \emptyset$ for $i = 1, 2$. Then $\mathrm{pPOL}_k\, \varrho \in p\mathscr{C}_k$.*

Proof. Let $\omega_1 := \omega(\sigma_1)$ and $\omega_2 := E_k \setminus \omega_1$.

- If $\varrho \in \mathcal{A}$, then let $a_1 \in \sigma_1$ and $a_2 \in \sigma_2$ be arbitrary. Because $\varrho \in \widetilde{\mathcal{R}}_k^{\max}$, there are relational homomorphisms $\kappa_i : E_k \to \omega(a_i)$ from ϱ to σ_i for $i = 1, 2$ by Lemma 6.2. Then let

$$\begin{aligned} \forall x \in \omega_1 : \varphi(x) &:= \kappa_2(x) \in \omega_2 \\ \forall x \in \omega_2 : \varphi(x) &:= \kappa_1(x) \in \omega_1. \end{aligned}$$

- If $\varrho \in \mathcal{Q}_0$, then let $a_i \in \omega(\sigma_i)$ for $i = 1, 2$ be arbitrary and let

$$\begin{aligned} \forall x \in \omega_1 : \varphi(x) &:= a_2 \in \omega_2 \\ \forall x \in \omega_2 : \varphi(x) &:= a_1 \in \omega_1. \end{aligned}$$

Then φ is well-defined, $\varphi \in \mathrm{pPOL}_k\,\varrho$ and $\varphi \notin \mathrm{pPOL}_k\{x\}$ for all $x \in E_k$ by construction.

Let $\{\chi_1, \ldots, \chi_m\} := X := \{\chi \in \mathcal{U} \cup \mathcal{A} \mid \varphi \in \mathrm{pPOL}_k\,\chi\}$.

Let $\chi^{(\mu)} \in X$ be arbitrary. Define g_χ in the following way

- If $\mu < h$ then let $g_\chi(\chi) := w \in E_k^\mu \setminus \chi$.
- If $\mu = h$ and $\chi \subseteq \varrho$ then $g_\chi(\chi) := w \in \varrho \setminus \chi$.
- If $\mu = h$ and $\chi \not\subseteq \varrho$ then $g_\chi(\chi) := w \in E_k^\mu \setminus \chi$.
- If $\mu > h$ there is a relational homomorphism $\varphi' : E_k \to E_k$ from ϱ to $\sigma_1 \cup \delta(\varrho)$. Let $g_\chi(\chi) := \varphi'(w) \in \iota_k^\mu$.

Then $g_\chi \in \mathrm{pPOL}_k\,\varrho$ and $g_\chi \notin \mathrm{pPOL}_k\,\chi$.

Define $f_0 := \varphi$, $f_I := f_{I-1} \otimes g_{\chi_I}$ for all $I \in \{1, 2, \ldots, m\}$ and $f := f_m$. Then $f \notin \chi$ for all $\chi \in \mathcal{U} \cup \mathcal{A}$ by construction.

We show $f \in \mathrm{pPOL}_k\,\varrho$ by induction. First $f_0 = \varphi \in \mathrm{pPOL}_k\,\varrho$. Let $f_{I-1} \in \mathrm{pPOL}_k\,\varrho$ and assume $f_I^{(n)} \notin \mathrm{pPOL}_k\,\varrho$. Then there exist rows $c_{1*}, \ldots, c_{h*}$ with $c_{*1}, \ldots, c_{*n} \in \varrho$ and $f(c_{*1}, \ldots, c_{*n}) \in E_k^h \setminus \varrho$.

Because $f_{I-1} \in \mathrm{pPOL}_k\,\varrho$ there is at least one row $c_{i_g *}$ from the g_{χ_I}-part of f_I and because $g_{\chi_I} \in \mathrm{pPOL}_k\,\varrho$ there is at least one row $c_{i_f *}$ from the f_{I-1}-part of f_I.

Because $\varphi \in \mathrm{pPOL}_k\,\chi_I$ there are columns c_{*j} and $c_{*j'}$ with

$$\begin{pmatrix} c_{i_f,j} & c_{i_f,j'} \\ c_{i_g,j} & c_{i_g,j'} \end{pmatrix} = \begin{pmatrix} x & x \\ y & \varphi(y) \end{pmatrix}$$

and w.l.o.g. $x \in \omega_1$. Because $y \in \omega_1$ implies $\varphi(y) \in \omega_2$ and $y \in \omega_2$ implies $\varphi(y) \in \omega_1$, we get $\omega(c_{*j}) \not\subseteq \omega_1$ or $\omega(c_{*j'}) \not\subseteq \omega_1$ in contradiction to $x \in \omega_1$ and the structure of ϱ.

Thus $f_I \in \mathrm{pPOL}_k\,\varrho$ and by induction $f = f_m \in \mathrm{pPOL}_k\,\varrho$. By construction $f \notin \mathrm{pPOL}_k\,\chi$ for all $\chi \in \mathcal{U} \cup \mathcal{A}$. Thus by Lemmas 5.7 and 5.6 there is some $F \in \mathrm{pPOL}_k\,\varrho$ with $F \notin \mathrm{pPOL}_k\,\chi$ for all $\chi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \{\varrho\}$, i.e., $\mathrm{pPOL}_k\,\varrho \in p\mathscr{C}_k$. □

Theorem 6.4. *Let $\varrho^{(h)} \in \mathcal{Q}_0$ and $\varepsilon(\varrho)$ has at least two equivalence classes. Then* $\mathrm{pPOL}_k\,\varrho \in p\mathscr{C}_k$.

Proof. We construct a function $f \in \mathrm{pPOL}_k\,\varrho$ with $f \notin \mathrm{pPOL}_k\,\chi$ for all $\chi \in \mathcal{U} \cup \mathcal{A}$. By Lemmas 5.7 and 5.6 there is some $F \in \mathrm{pPOL}_k\,\varrho$ with $F \notin \mathrm{pPOL}_k\,\chi$ for all $\chi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \{\varrho\}$, i.e., $\mathrm{pPOL}_k\,\varrho \in p\mathscr{C}_k$.

Let $\{A_1, A_2, \ldots, A_l\} := \{\{y \in E_h \mid (x, y) \in \varepsilon(\varrho)\} \mid x \in E_h\}$, i.e., the set of equivalence classes of $\varepsilon(\varrho)$. Then $l \geq 2$ because $\varepsilon(\varrho)$ has at least two equivalence classes, and $|A_i| \geq 2$ for all i because $\varrho \in \mathcal{Q}_0$. Let $a_i \in A_i$ be arbitrary for each i.

Because ϱ is coherent, there is some relational homomorphism $\varphi_0 : E_k \to E_h$ from $\sigma(\varrho)$ to $M(\varrho)$ and some $s_0 \in \sigma(\varrho)$ with $\varphi_0(s_0) = \eta_h$.

Define $\varphi_1 : E_h \to E_h$ by

$$\varphi_1(x) = \begin{cases} a_{i+1} & \text{if } x \in A_i \text{ and } i \in \{1, 2, \ldots, l-1\}, \\ a_1 & \text{if } x \in A_l. \end{cases}$$

Then φ_1 is a relational homomorphism from $M(\varrho)$ to $M(\varrho)$ and $\varphi_1(t) \in \delta(M(\varrho)) \subseteq \delta(\varrho)$ for all $t \in M(\varrho)$.

Define $\varphi_2 : E_h \to E_k$ by $\varphi_2(\eta_h) = s_0$. Then φ_2 is a relational homomorphism from $M(\varrho)$ to ϱ.

Define $\varphi : E_k \to E_k$ by $\varphi(x) := \varphi_2(\varphi_1(\varphi_0(x)))$ for all $x \in E_k$. By construction φ is a relational homomorphism from ϱ to ϱ, i.e., $\varphi \in \mathrm{pPOL}_k\,\varrho$. We first show $\varphi(x) \neq x$ for all $x \in E_k$. If $x \in E_k \setminus \omega(s_0)$ then $\varphi(x) \neq x$ because $\varphi(E_k) \subseteq \omega(s_0)$. Because $\varphi_1(y) \neq y$ for all $y \in E_h$ and $\varphi_2(\varphi_0(x)) = x$ for all $x \in \omega(s_0)$ we get $\varphi(x) \neq x$ for all $x \in \omega(s_0)$. Thus $\varphi \notin \mathrm{pPOL}_k\{x\}$ for all $x \in E_k$.

Let $\chi^{(\mu)} \in \mathcal{U} \cup \mathcal{A}$ with $\varphi \in \mathrm{pPOL}_k\,\chi$ be arbitrary. Then $\mu \leq \frac{h}{2}$ because $|\omega(\varphi(s))| \leq l = |\{a_1, \ldots, a_l\}| \leq \frac{h}{2}$ for all $s \in \chi$. Let $g_\chi(\chi) := v \in E_k^\mu \setminus \chi$ and g_χ not defined elsewhere. Then $g_\chi \in \mathrm{pPOL}_k\,\varrho$ by Lemma 5.3 with $\mu < h$, and $g_\chi \notin \mathrm{pPOL}_k\,\chi$.

Let $\{\chi_1, \ldots, \chi_m\} := \{\chi \in \mathcal{U} \cup \mathcal{A} \mid \varphi \in \mathrm{pPOL}_k\,\chi\}$. Define $f_0 := \varphi$, $f_I := f_{I-1} \otimes g_{\chi_I}$ for all $I \in \{1, 2, \ldots, m\}$ and $f := f_m$. Then $f \notin \chi$ for all $\chi \in \mathcal{U} \cup \mathcal{A}$ by construction.

We show $f \in \mathrm{pPOL}_k\,\varrho$ by induction. First $f_0 = \varphi \in \mathrm{pPOL}_k\,\varrho$. Let $f_{I-1} \in \mathrm{pPOL}_k\,\varrho$ and assume $f_I^{(n)} \notin \mathrm{pPOL}_k\,\varrho$. Then there exist rows $c_{1*}, \ldots, c_{h*}$ with $c_{*1}, \ldots, c_{*n} \in \varrho$ and $f(c_{*1}, \ldots, c_{*n}) \in E_k^h \setminus \varrho$.

Because $f_{I-1} \in \mathrm{pPOL}_k\,\varrho$ there is at least one row $c_{i_g *}$ from the g_{χ_I}-part of f_I and because $g_{\chi_I} \in \mathrm{pPOL}_k\,\varrho$ there is at least one row $c_{i_f *}$ from the f_{I-1}-part of f_I. Let i_f be arbitrary with this condition. Let $c_{i_1 *}$ be an arbitrary third row.

- $c_{i_1 *}$ is from the f_{I-1}-part of f_I. Then $c_{i_1 *}$ and $c_{i_f *}$ differ at least one column by Lemma 5.3. By construction and because $\varphi \in \mathrm{pPOL}_k\,\chi_I$

we get

$$\left\{ \begin{pmatrix} x \\ y \\ \varphi^i(z) \end{pmatrix} \middle| i \in E_l \right\} \subseteq \left\{ \begin{pmatrix} c_{i_f j} \\ c_{i_1 j} \\ c_{i_g j} \end{pmatrix} \middle| j \in \{1, \dots, n\} \right\}$$

with $x \neq y$. From $\{\varphi^1(z), \dots, \varphi^l(z)\} = \{\varphi_2(a_1), \dots, \varphi_2(a_l)\}$ we get:

- if $x' := \varphi(x) \neq \varphi(y) =: y'$ then there are two columns $c_{*j'}$ and $c_{*j''}$ with

$$\begin{pmatrix} \varphi(c_{i_f,j'}) \\ \varphi(c_{i_1,j'}) \\ \varphi(c_{i_g,j'}) \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ x' \end{pmatrix} \wedge \begin{pmatrix} \varphi(c_{i_f,j''}) \\ \varphi(c_{i_1,j''}) \\ \varphi(c_{i_g,j''}) \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ y' \end{pmatrix}$$

  and $x' \neq y'$,
- if $x' := \varphi(x) = \varphi(y)$ then there is some column $c_{*j'}$ with

$$\begin{pmatrix} \varphi(c_{i_f,j+j'}) \\ \varphi(c_{i_1,j+j'}) \\ \varphi(c_{i_g,j+j'}) \end{pmatrix} = \begin{pmatrix} x' \\ x' \\ z' \end{pmatrix}$$

  and $x' \neq z'$.

Thus the rows c_{i_1*} and c_{i_g*} do not belong to the same equivalence class of $\varepsilon(\varrho)$ because $\varphi(c_{*j+j'}), \varphi(c_{*j+j''}) \in \varrho$.

- c_{i_1*} is from the g_{χ_I}-part of f_I. Then c_{i_1*} and c_{i_g*} differ at every column because $\chi_I \in \mathcal{A}$. By construction and because $\varphi \in \mathrm{pPOL}_k\, \chi_I$ we get

$$\left\{ \begin{pmatrix} x \\ \varphi^i(y) \\ \varphi^i(z) \end{pmatrix} \middle| i \in E_l \right\} \subseteq \left\{ \begin{pmatrix} c_{i_f j} \\ c_{i_1 j} \\ c_{i_g j} \end{pmatrix} \middle| j \in \{1, \dots, n\} \right\}$$

  with $y \neq z$. Because $\{\varphi^1(y), \dots, \varphi^l(y)\} = \{\varphi_2(a_1), \dots, \varphi_2(a_l)\}$ there is a column c_{*j}

$$\begin{pmatrix} \varphi(c_{i_f,j}) \\ \varphi(c_{i_1,j}) \\ \varphi(c_{i_g,j}) \end{pmatrix} = \begin{pmatrix} x' \\ x' \\ z' \end{pmatrix}$$

  and $x' \neq z'$. Thus the rows c_{i_1*} and c_{i_g*} do not belong to the same equivalence class of $\varepsilon(\varrho)$.

Thus c_{i_g*} belongs to a singular class of $\varepsilon(\varrho)$ in contradiction to $\varrho \in \mathcal{Q}_0$. Thus $f_I \in \mathrm{pPOL}_k\,\varrho$ and by induction $f = f_m \in \mathrm{pPOL}_k\,\varrho$. By construction $f \notin \mathrm{pPOL}_k\,\chi$ for all $\chi \in \mathcal{U} \cup \mathcal{A}$. Thus by Lemmas 5.7 and 5.6 there is some $F \in \mathrm{pPOL}_k\,\varrho$ with $F \notin \mathrm{pPOL}_k\,\chi$ for all $\chi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \{\varrho\}$, i.e., $\mathrm{pPOL}_k\,\varrho \in p\mathscr{C}_k$. □

In the Lemmas 9 and 10 in [1] it is shown that $\mathrm{pPOL}_k\,\varrho \in p\mathscr{C}_k$ for every relation $\varrho \in p\mathfrak{S}_h$ with $2 \leq h \leq k$. The following Theorem 6.5 extends this to a more general set of relations, and in particular the Lemmas 24, 26, 27 and 28 in [1] follow from it.

Theorem 6.5. *Let $\varrho^{(h)} \in \mathcal{A} \cup \mathcal{Q}_0$. If $\varrho \in \mathcal{Q}_0$ then $h \geq 3$ and $\delta(\varrho) = \delta_{E_h}^{(h)}$. Let $\alpha = \alpha_1\alpha_2\cdots\alpha_l \in \Gamma_{\sigma(\varrho)}$ be a fix-point-free permutation with disjunct cycles $\alpha_1, \alpha_2, \ldots, \alpha_l$ and*

$$\forall \emptyset \neq T \subset \{1, 2, \ldots, l\} : \prod_{t \in T} \alpha_t \notin \Gamma_{\sigma(\varrho)}.$$

Then $\mathrm{pPOL}_k\,\varrho \in p\mathscr{C}_k$.

Proof. We construct a function $f \in \mathrm{pPOL}_k\,\varrho$ with $f \notin \mathrm{pPOL}_k\,\chi$ for all $\chi \in \mathcal{U} \cup \mathcal{A}$. By Lemmas 5.7 and 5.6 there is some $F \in \mathrm{pPOL}_k\,\varrho$ with $F \notin \mathrm{pPOL}_k\,\chi$ for all $\chi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \{\varrho\}$, i.e., $\mathrm{pPOL}_k\,\varrho \in p\mathscr{C}_k$.

Because ϱ is coherent, there is some relational homomorphism $\varphi_0 : E_k \to E_h$ from $\sigma(\varrho)$ to $M(\varrho)$ and some $s_0 \in \sigma(\varrho)$ with $\varphi_0(s_0) = \eta_h$.

The permutation $\alpha \in \Gamma_{\sigma(\varrho)}$ is a relational homomorphism from $M(\varrho)$ to $M(\varrho)$.

Define $\varphi_2 : E_h \to E_k$ by $\varphi_2(\eta_h) = s_0$. Then φ_2 is a relational homomorphism from $M(\varrho)$ to ϱ.

Define $\varphi : E_k \to E_k$ by $\varphi(x) := \varphi_2(\alpha(\varphi_0(x)))$ for all $x \in E_k$. By construction φ is a relational homomorphism from ϱ to ϱ, i.e., $\varphi \in \mathrm{pPOL}_k\,\varrho$. We first show $\varphi(x) \neq x$ for all $x \in E_k$. If $x \in E_k \setminus \omega(s_0)$ then $\varphi(x) \neq x$ because $\varphi(E_k) \subseteq \omega(s_0)$. Because $\alpha(y) \neq y$ for all $y \in E_h$ and $\varphi_2(\varphi_0(x)) = x$ for all $x \in \omega(s_0)$ we get $\varphi(x) \neq x$ for all $x \in \omega(s_0)$. Thus $\varphi \notin \mathrm{pPOL}_k\{x\}$ for all $x \in E_k$.

Let $X := \{\chi \in \mathcal{U} \cup \mathcal{A} \mid \varphi \in \mathrm{pPOL}_k\,\chi\}$. Let $\chi^{(\mu)} \in X$ be arbitrary. Then $\mu \leq h$ because $|\omega(\varphi(s))| \leq |\omega(\varphi(s_0))| = h$ for all $s \in \chi$.

If

- $\mu < h$, or
- $\mu = h$ and $\chi \nsubseteq \varrho^{[\pi]}$ for all $\pi \in S_h$,

then let $g_\chi(\chi) := v \in E_k^\mu \setminus \chi$ and g_χ not defined elsewhere. Then $g_\chi \in \mathrm{pPOL}_k\,\varrho$ by Lemma 5.3 with $\mu < h$, and $g_\chi \notin \mathrm{pPOL}_k\,\chi$.

If $\mu = h$ and $\chi \subset \varrho^{[\pi]}$ for some $\pi \in S_h$, w.l.o.g. $\pi = \mathrm{id}$, then let $g_\chi(\chi) := v \in \varrho \setminus \chi$ and g_χ not defined elsewhere. Then $g_\chi \notin \mathrm{pPOL}_k\,\chi$. Assume $g_\chi^{(n)} \notin \mathrm{pPOL}_k\,\varrho$. Then there are rows $c_{1*}, \ldots, c_{h*}$ with $c_{*1}, \ldots, c_{*n} \in \varrho$ and $g_\chi(c_{*1}, \ldots, c_{*n}) =: d \notin \varrho$. Then $c_{*1}, \ldots, c_{*n} \in \chi^{[\pi']}$ for some $\pi' \in \Gamma_{\sigma(\varrho)}$ because the rows are pairwise different by Lemma 5.3. But then $d = v^{[\pi']} \in \varrho$ contradicting the assumption. Thus $g_\chi \in \mathrm{pPOL}_k\,\varrho$.

Let $\{\chi_1, \ldots, \chi_m\} := X$. Define $f_0 := \varphi$, $f_I := f_{I-1} \otimes g_{\chi_I}$ for all $I \in \{1, 2, \ldots, m\}$ and $f := f_m$. Then $f \notin \chi$ for all $\chi \in \mathcal{U} \cup \mathcal{A}$ by construction.

We show $f \in \mathrm{pPOL}_k\,\varrho$ by induction. First $f_0 = \varphi \in \mathrm{pPOL}_k\,\varrho$. Let $f_{I-1} \in \mathrm{pPOL}_k\,\varrho$ and assume $f_I^{(n)} \notin \mathrm{pPOL}_k\,\varrho$. Then there exist rows $c_{1*}, \ldots, c_{h*}$ with $c_{*1}, \ldots, c_{*n} \in \varrho$ and $f(c_{*1}, \ldots, c_{*n}) \in E_k^h \setminus \varrho$.

Because $f_{I-1} \in \mathrm{pPOL}_k\,\varrho$ there is at least one row c_{i_g*} from the g_{χ_I}-part of f_I and because $g_{\chi_I} \in \mathrm{pPOL}_k\,\varrho$ there is at least one row c_{i_f*} from the f_{I-1}-part of f_I. Let w.l.o.g. the rows of ϱ be sorted such that the rows $c_{1*}, \ldots, c_{p*}$ belong to the f_{I-1}-part of f_I and the rows $c_{p+1*}, \ldots, c_{h*}$ to the g_{χ_I}-part of f_I. Then $1 \le p \le h-1$.

Let

$$B_0 := \begin{pmatrix} c_{1,1} \\ c_{2,1} \\ \cdots \\ c_{p,1} \end{pmatrix}, B_1 := \varphi(B_0),$$

$$C_0 := \begin{pmatrix} c_{p+1,1} \\ c_{p+2,1} \\ \cdots \\ c_{h,1} \end{pmatrix}, C_1 := \varphi(C_0), C_2 := \varphi(C_1).$$

Because $\varphi \in \mathrm{pPOL}_k\,\chi_I$ and the way f_I is constructed we have $\{B_0\} \times \{C_0, C_1\} \subseteq \varrho$, and $C_1, C_2 \in E_k^{h-p} \setminus \iota_k^{h-p}$. Because $\varphi \in \mathrm{pPOL}_k\,\varrho$ we get $\{B_1\} \times \{C_1, C_2\} \subseteq \varrho$.

- $1 \le p \le h-2$. Then $\{B_1\} \times \{C_1, C_2\} \subseteq \sigma(\varrho)$ because $C_1, C_2 \in E_k^{h-p} \setminus \iota_k^{h-p}$ and $\delta(\varrho) \in \{\delta_{E_h}^{(h)}, \emptyset\}$. Because $\omega(B_1, C_1) = \omega(B_1, C_2) = \omega(s_0)$ and $\varphi(s_0) = s_0^{[\alpha]}$ by construction there is some $\emptyset \subset T \subset \{1, \ldots, t\}$ with $\beta = \prod_{t \in T} \alpha_t$ and

$$\begin{pmatrix} B_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} B_1 \\ C_1 \end{pmatrix}^{[\beta]} \in \varrho.$$

Thus $\beta \in \Gamma_{\sigma(\varrho)}$ in contradiction to the assumption on α.

- $p = h - 1$ and $h = 2$. Because $\omega(B_1, C_1) \subseteq \omega(s_0)$, $\omega(B_1, C_2) \subseteq \omega(s_0)$ and $\varphi(x) \neq x$ for all $x \in E_k$ we get

$$\begin{pmatrix} B_1 \\ C_1 \end{pmatrix} \in \iota_k^2 \vee \begin{pmatrix} B_1 \\ C_2 \end{pmatrix} \in \iota_k^2$$

in contradiction to $\varrho \in \mathcal{A}$.

- $p = h - 1$ and $h > 2$. Because $\omega(B_1, C_1) \subseteq \omega(s_0)$, $\omega(B_1, C_2) \subseteq \omega(s_0)$ and $\varphi(x) \neq x$ for all $x \in E_k$ we get

$$\begin{pmatrix} B_1 \\ C_1 \end{pmatrix} \in \iota_k^2 \vee \begin{pmatrix} B_1 \\ C_2 \end{pmatrix} \in \iota_k^2$$

But then

$$\begin{pmatrix} B_1 \\ C_1 \end{pmatrix} \in \delta(\varrho) \in \{\emptyset, \delta_{E_h}^{(h)}\}$$

and thus a contradiction to $\delta(\varrho) = \emptyset$, or

$$\begin{pmatrix} B_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} x \\ \dots \\ x \\ y \end{pmatrix}$$

for some $x \neq y$ in contradiction to $\delta(\varrho) = \delta_{E_h}^{(h)}$.

Thus $f_I \in \mathrm{pPOL}_k\, \varrho$ and by induction $f = f_m \in \mathrm{pPOL}_k\, \varrho$. By construction $f \notin \mathrm{pPOL}_k\, \chi$ for all $\chi \in \mathcal{U} \cup \mathcal{A}$. Thus by Lemmas 5.7 and 5.6 there is some $F \in \mathrm{pPOL}_k\, \varrho$ with $F \notin \mathrm{pPOL}_k\, \chi$ for all $\chi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \{\varrho\}$, i.e., $\mathrm{pPOL}_k\, \varrho \in p\mathscr{C}_k$. $\square$

6.2 Superfluous relations from $\mathcal{A} \cup \mathcal{Q}_0$

Lemma 6.6. *Let* $\varrho = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 2 & 2 \end{pmatrix}$. *Then* $\mathrm{pPOL}_4\, \varrho \notin p\mathscr{C}_4$.

Proof. Let $f^{(n)} \in \mathrm{pPOL}_4\, \varrho$. Assume $f \notin \mathrm{pPOL}_4\{0\}$ and $f \notin \mathrm{pPOL}_4\{1, 2\}$. Then there are rows c_{0*}, c_{1*} with $c_{*1}, \dots, c_{*n} \in \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$ and $f(c_{0*}) =: d_0 \in E_4 \setminus \{0\}$ and $f(c_{1*}) =: d_1 \in E_4 \setminus \{1, 2\}$. But then $(d_0, d_1) \notin \varrho$ contradicting the assumption. Thus $\mathrm{pPOL}_4\, \varrho \notin p\mathscr{C}_4$ by Lemma 3.3. $\square$

Lemma 6.7. *Let* $\varrho = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 2 & 2 \end{pmatrix} \cup \delta^{(2)}_{\{0,1\}}$. *Then* $\mathrm{pPOL}_4\, \varrho \notin p\mathscr{C}_4$.

Proof. Let $f^{(n)} \in \mathrm{pPOL}_4\, \varrho$ and assume $f \notin \mathrm{pPOL}_4\{x\}$ for all $x \in \{0, 2, 3\}$ and $f \notin \mathrm{pPOL}_4\{0, 1, 2\}$. Then there are rows $c_{0*}, c_{1*}, c_{2*}, c_{3*}$ with

$$c_{*1}, \ldots, c_{*n} \in \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}.$$

Let $d_i := f(c_{i*})$. Then $d_1 = 3$ and thus $d_0 = 3$ because $(d_0, d_1) \in \varrho$. This implies $d_2 = 3$ because $(d_0, d_2) \in \varrho$. But $(d_3, d_2) \in \varrho$ and thus $d_3 = 3$ in contradiction to $f \notin \mathrm{pPOL}_4\{3\}$. Thus $\mathrm{pPOL}_4\, \varrho \notin p\mathscr{C}_4$ by Lemma 3.3. □

Theorem 6.8. *Let* $\varrho^{(h)} \in \mathcal{A} \cup \mathcal{Q}_0$, $\sigma_1^{(l)}, \sigma_2^{(h-l)} \in \mathcal{U} \cup \mathcal{A}$ *with* $\sigma(\varrho) = \sigma_1 \times \sigma_2$, $\delta(\varrho) \in \{\emptyset, \delta_{E_h}\}$, *and if* $\varrho \in \mathcal{Q}_0$ *then* $\omega(\sigma_1) \cup \omega(\sigma_2) = E_k$. *Then* $\mathrm{pPOL}_k\, \varrho \notin p\mathscr{C}_k$.

Proof. Let $U := \{\sigma_1, \sigma_2\} \cup \{\{a\} \mid a \in E_k\}$. We show

$$\forall f \in \mathrm{pPOL}_k\, \varrho\, \exists \chi \in U : f \in \mathrm{pPOL}_k\, \chi. \tag{6.1}$$

Assume, (6.1) does not hold. Then there is some $f^{(n)} \in \mathrm{pPOL}_k\, \varrho$ with

$$\forall \chi \in U : f \notin \mathrm{pPOL}_k\, \chi.$$

Then there exist $s_{*1}, s_{*2}, \ldots, s_{*n} \in \sigma_1$ and $t_{*1}, t_{*2}, \ldots, t_{*n} \in \sigma_2$ with $f(s_{*1}, s_{*2}, \ldots, s_{*n}) \in E_k^l \setminus \sigma_1$ and $f(t_{*1}, t_{*2}, \ldots, t_{*n}) \in E_k^{h-l} \setminus \sigma_2$. By $f \in \mathrm{pPOL}_k\, \varrho$ holds

$$\mathbf{y} := f \begin{pmatrix} s_{*1} & s_{*2} & \ldots & s_{*n} \\ t_{*1} & t_{*2} & \ldots & t_{*n} \end{pmatrix} \in \delta(\varrho) \in \{\emptyset, \delta_{E_h}\}$$

Thus either $f \notin \mathrm{pPOL}_k\, \varrho$, contradicting the assumption, or $\mathbf{y} = (y, y, \ldots, y)$ holds for some $y \in E_k$ if $\delta \neq \emptyset$.

If $y \in \omega(\sigma_1)$ then there is an $s \in \sigma_1$ with $y \in \omega(s)$, and $y \notin \omega(\sigma_2)$ because $\sigma_1 \times \sigma_2 \subseteq \sigma(E_k^h)$. Thus

$$f \begin{pmatrix} s & s & \ldots & s \\ t_{*1} & t_{*2} & \ldots & t_{*n} \end{pmatrix} = \mathbf{y} \in \delta(\varrho) = \delta_{E_h}$$

and then also $f \in \mathrm{pPOL}_k\{y\}$, in contradiction to the assumption. If $y \in \omega(\sigma_2)$ then we get a contradiction in a symmetric way. Thus (6.1) holds and Lemma 3.3 applies. □

The following defintion generalizes the term "partial order with greatest resp. least element" (denoted by $\mathfrak{M}_{k,e}$ resp. $\mathfrak{M}_{k,o}$ in [1]) to the term "centered partial order" (denoted by $\mathfrak{M}_{k,c}$). In particular $\mathfrak{M}_{k,e} \cup \mathfrak{M}_{k,o} \subseteq \mathfrak{M}_{k,c}$ holds. Thus the statements of Lemma 13(a-c) in [1] are extended by the following Theorem, which also covers Lemma 20 in [1].

Definition 6.9. A partial order ϱ on E_k is called *centered* by $c \in E_k$, if

$$\forall x \in E_k : (x, c) \in \varrho \vee (c, x) \in \varrho$$

holds. Let $\mathfrak{M}_{k,c}$ all partial orders on E_k, which are centered by c.

Let

$$\varrho := \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 & 1 & 2 & 3 \end{pmatrix}$$

be a relation on E_4. The transitive closure of ϱ

$$\leq := \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 2 & 3 & 0 & 1 & 2 & 3 \end{pmatrix}$$

is a centered relation on E_4. It is $\leq \in \mathfrak{M}_{4,0}$ and $\leq \in \mathfrak{M}_{4,1}$, i.e., the centering element is not uniquely determined in general.

On the other hand the partial order ϱ' on E_4 given by

$$\varrho' := \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 2 & 3 \\ 2 & 3 & 2 & 3 & 0 & 1 & 2 & 3 \end{pmatrix}$$

is not centered. Assume, $\varrho' \in \mathfrak{M}_{4,c}$ for some $c \in E_4$, and w.l.o.g. let $c = 0$. Then $(0, 1) \notin \varrho'$, in contradiction to the assumption.

Theorem 6.10. *Let ϱ be a binary relation on E_k. If the transitive closure of ϱ is in $\mathfrak{M}_{k,c}$, then follows*

$$\forall f \in \mathrm{pPOL}_k\, \varrho\; \exists a \in E_k : f \in \mathrm{pPOL}_k\{a\} \tag{6.2}$$

and $\mathrm{pPOL}_k\, \varrho \notin p\mathscr{C}_k$ *by Lemma 3.3.*

Proof. Let $f \in \mathrm{pPOL}_k\, \varrho$ be arbitrary and let $f'(x) := f(x, \ldots, x)$ for all $x \in E_k$. Assume $f \notin \mathrm{pPOL}_k\{a\}$ for all $a \in E_k$. Then $f' \in \mathrm{Pol}_k^{(1)}\, \varrho$ and $f'(x) \neq x$ for all $x \in E_k$.

Let $\leq$ be the transitive closure of ϱ. Let $p := f'(c)$, and w.l.o.g. $c < p$, otherwise consider f for the relation

$$\varrho' := \{(y, x) \mid (x, y) \in \varrho\} = \varrho^{[(0\,1)]}.$$

An $(m+1)$-tuple $K = (x_0, \ldots, x_m)$ is called *valid chain* in respect to ϱ, if

$$\forall 0 < i \leq m : (x_{i-1}, x_i) \in \varrho \setminus \iota_k^2$$

and $m \geq 1$ as well as $f'(x_0) = x_m$ hold.

Because $\leq$ is the transitive closure of ϱ and $c < p$ hold, there is a valid chain $K = (x_0, \ldots, x_m)$ in respect to ϱ with

$$x_0 = c \text{ and } x_m = p.$$

Let

$$M := M(K) := \{x_0, x_1, \ldots, x_m\} \cup \{x \in E_k : x_m < x\}$$

We show

$$\exists x \in M : f'(x) = x \tag{6.3}$$

by induction over $|M|$.

If $|M| = 1$, then $x_0 = x_m = f'(x_0)$. Thus (6.3).

Let $|M| > 1$ and for any valid chain K' with $M(K') \subset M$ (6.3) holds. Consider $d := f'(x_1)$. Because $f \in \mathrm{pPOL}_k\,\varrho$ and $(x_0, x_1) \in \varrho$ we get $(x_m, d) = (f'(x_0), f'(x_1)) \in \varrho$.

- If $d = x_m$ and $m = 1$, then $f'(x_1) = d = x_1$. Thus (6.3) holds.

- If $d = x_m$ and $m > 1$, then let $K' := (x'_0, \ldots, x'_{m'}) := (x_1, \ldots, x_m)$ and $M' := \{x_1, \ldots, x_m\} \cup \{x \in E_k : x_m < x\}$ with $m' = m-1 \geq 1$. Then K' is a valid chain and by $M(K') = M' \subset M$ follows (6.3).

- If $d \neq x_m$, then $x_m < d$ holds, and thus $d \in M$. Let $K' := (x'_0, \ldots, x'_{m'}) := (x_1, \ldots, x_m, d)$ and $M' := \{x_1, \ldots, x_m, d\} \cup \{x \in E_k : d < x\}$ with $m' = m$. Then K' is a valid chain and by $M(K') = M' \subseteq M \setminus \{x_0\}$ follows (6.3).

Thus (6.3) holds for the function f, i.e., there is some $a \in M \subseteq E_k$ with $f \in \mathrm{pPOL}_k\{a\}$ in contradiction to the assumption. Thus (6.2) is true and $\mathrm{pPOL}_k\,\varrho \notin p\mathscr{C}_k$ by Lemma 3.3. □

Definition 6.11. Let $\varrho \in R_k$ be a binary coherent reflexive and symmetric relation. Then let

$$\mathcal{G}(\varrho) := (\mathcal{V}, \mathcal{E}) \text{ with } \mathcal{V} = E_k,\ \mathcal{E} := \sigma(\varrho)$$

be the assigned graph. Let

$$\forall x \in \mathcal{V} = E_k : d(x) := |\{(x, y) \in \mathcal{E} \mid y \in \mathcal{V} \setminus \{x\}\}|,$$

i.e., $d(x)$ is the degree of the vertex x.

Lemma 6.12. *Let $\varrho \in R_k$ be a binary coherent reflexive and symmetric relation and $f^{(l)} \in \mathrm{pPOL}_k\, \varrho$ with $f'(x) := f(x, \ldots, x)$, $f'(U) = U$ for some $U \leq \mathcal{G}(\varrho)$ and $U \cong K_n$ for an $n \in \mathbb{N}$. Additionaly let $\mathcal{G}(\varrho)$ be connected. Then there is some $\emptyset \subset C \subset E_k$ with*

$$f \in \mathrm{pPOL}_k\, C.$$

Proof. Let $\mathcal{G} := \mathcal{G}(\varrho)$ and w.l.o.g.

$$U = \left(E_n, \sigma(E_n^2)\right).$$

Assume,

$$\forall \emptyset \subset C \subset E_k : f \notin \mathrm{pPOL}_k\, C \tag{6.4}$$

holds. We first show

$$\forall x \in E_n\, \forall y \in E_k \setminus E_n : (x, y) \in \mathcal{G}. \tag{6.5}$$

Let

$$M := \{y \in E_k \mid \forall x \in E_n \setminus \{y\} : (x, y) \in \mathcal{G}\}.$$

Assume $|M| < k$. Then there are $m_1, \ldots, m_l \in M$ with

$$m' := f(m_1, \ldots, m_l) \notin M.$$

Then

$$\forall x \in E_k\, \forall i \in \{1, 2, \ldots, l\} : \begin{pmatrix} x \\ m_i \end{pmatrix} \in \varrho$$

holds, and thus

$$\forall x \in E_k : \begin{pmatrix} f'(x) \\ m' \end{pmatrix} \in \varrho.$$

From $f'(U) = U$ follows

$$\forall x \in E_n : (x, m') \in \mathcal{G},$$

thus $m' \in M$ in contradiction to the assumption on M. Thus (6.5) holds.

Let $U' \leq \mathcal{G}$ be a maximal subgraph of $\mathcal{G}$ with $U \leq U'$ and $U' \cong K_{n'}$ as well as

$$\forall x \in \mathcal{V}(U')\, \forall y \in E_k \setminus \mathcal{V}(U') : (x, y) \in \mathcal{G}. \tag{6.6}$$

Because $U \cong K_n$ and (6.6), we have $n \leq n' \leq k - 2$. W.l.o.g.

$$U' = \big(E_{n'}, \sigma(E_{n'}^2)\big)$$

holds. Then there are $p_1, \ldots, p_l \in E_{n'}$ with

$$p' := f(p_1, \ldots, p_l) \notin E_{n'}.$$

Let $N := E_k \setminus (E_{n'} \cup \{p'\})$. Then

$$\forall x \in N\, \exists q_{x,1}, \ldots, q_{x,l} \in E_k \setminus \{x\} : f(q_{x,1}, \ldots, q_{x,l}) = x$$

holds. From

$$\forall x \in N\, \forall i \in \{1, 2, \ldots, l\} : \begin{pmatrix} q_{x,i} \\ p_i \end{pmatrix} \in \varrho$$

follows

$$\forall x \in E_k \setminus \{p'\} : (x, p') \in \mathcal{G},$$

thus $p' \in E_{n'}$ in contradiction to the assumption and (6.4). □

Theorem 6.13. *Let $\varrho \in R_k$ be a binary coherent reflexive and symmetric relation and $\mathcal{G} := \mathcal{G}(\varrho)$ a tree. Then*

$$\forall f \in \mathrm{pPOL}_k\, \varrho\, \exists C \in \mathcal{U} : f \in \mathrm{pPOL}_k\, C \tag{6.7}$$

and $\mathrm{pPOL}_k\, \varrho \notin p\mathscr{C}_k$ *by Lemma 3.3.*

Proof. Let $f^{(n)} \in \mathrm{pPOL}_k\, \varrho$ be arbitrary and let $f'(x) := f(x, \ldots, x)$. Assume $f \notin \mathrm{pPOL}_k\{a\}$ for all $a \in E_k$. Otherwise (6.7) holds for this f and nothing is left to show.

Let $H \leq \mathcal{G}$ be a tree with $f'(H) = H$ and

$$|\mathcal{V}(H)| = \min\{|\mathcal{V}(H')| \mid f'(H') = H' \leq \mathcal{G}, H' \text{ tree}\}. \tag{6.8}$$

If $|\mathcal{V}(H)| \in \{1, 2\}$ then $H \cong K_1$ or $H \cong K_2$. Thus the claim follows from Lemma 6.12.

Let $|\mathcal{V}(H)| \geq 3$. f' is a graph automorphism of H. In particular

$$\forall x \in \mathcal{V}(H) : d(f'(x)) = d(x).$$

Let

$$I := \{x \in \mathcal{V}(H) \mid d(x) > 1\},$$

i.e. I is the set of the inner vertices. Then

$$f'(I) = I.$$

Let

$$H' := (I, E') \text{ with } E' := \{(x,y) \in \mathcal{E}(H) \mid x, y \in I\}.$$

Then H' is a tree and $H' \leq H \leq \mathcal{G}$. But $|\mathcal{V}(H')| < |\mathcal{V}(H)|$ contradicts (6.8), i.e., the assumption that H is minimal. □

7 RELATIONS FROM $\mathcal{S}$

Lemma 7.1 (Lemma 5.4 [14]). *Let $\varrho \in \mathcal{S} \setminus \mathcal{S}'$. Then $\mathrm{pPOL}_k\,\varrho \notin p\mathscr{C}_k$.*

Theorem 7.2. *Let $\varrho \in \mathcal{S}'$. Let $\varphi_0, \varphi_1, \ldots, \varphi_L \in \mathrm{Pol}_k^{(1)}$ with $L \geq 0$,*

$$\forall v \in \sigma(E_k^2)\,\forall z \in E_k : \{v\} \times \{\varphi_0^l(z) \mid l \geq 0\} \not\subseteq \varrho \tag{7.1}$$

$$\forall v \in \sigma(E_k^2)\,\forall z \in E_k : \{\varphi(v) \mid \varphi \in \Psi_L\} \times \{z\} \not\subseteq \varrho \tag{7.2}$$

$$\forall v \in \sigma(E_k^2)\,\forall z \in E_k\,\forall i \in \{1, \ldots, L\} :$$
$$(\{\varphi(v) \mid \varphi \in \Psi_{i-1}\} \times \{z\} \subseteq \varrho \Longrightarrow \{\varphi_i(v)\} \times \{\varphi_0(z)\} \subseteq \varrho) \tag{7.3}$$

and

$$\Psi_i := \left\{\psi_1 \circ \psi_2 \circ \cdots \circ \psi_l \in \widetilde{P}_k^{(1)} \mid l \geq 1, \{\psi_1, \ldots, \psi_l\} \subseteq \{\mathrm{id}, \varphi_0, \ldots, \varphi_i\}\right\}$$

with $\psi \circ \psi' := \psi[\psi']$ for all $\psi, \psi' \in \widetilde{P}_k^{(1)}$. Then $\mathrm{pPOL}_k\,\varrho \in p\mathscr{C}_k$.

Proof. Let $\psi_i := \{\varphi(\eta_k) \mid \varphi \in \Psi_{i-1}\}$ and $f_i(\theta_i) := \varphi(\eta_k)$ for all $i \in \{0, 1, \ldots, L\}$. Then $f_i \in \mathrm{pPOL}_k\,\varrho$ and $f_0 = \varphi_0$. Let $X := \{\chi_1, \ldots, \chi_M\} := \{\chi \in \widetilde{\mathcal{R}}_k^{\max} \mid \Psi_L \subseteq \mathrm{pPOL}_k\,\chi\} \setminus \{\varrho\}$. Define $f_\chi =: f_{L+j}$ for all $j \in \{1, \ldots, M\}$ and $\chi := \chi_j$ by

- $\mu \leq 3, \chi \subset \varrho$: $f_\chi(\chi) := w$ with $w \in \varrho \setminus \chi$,
- $\mu \leq 3, \chi \not\subseteq \varrho$: $f_\chi(\chi) := w$ with $w \in E_k^\mu \setminus \chi$,
- $\mu > 3, \chi \in \mathcal{S}$: $f_\chi(\chi) := w$ with $w \in E_k^\mu \setminus \chi$,
- $\mu > 3, \chi \in \mathcal{A} \cup \mathcal{Q} \cup \mathcal{L}$: $f_\chi(\chi) := (1, 0, \ldots, 0)$, where w.l.o.g. $(0,1) \in \varepsilon(\chi)$ if $\chi \in \mathcal{Q}$.

Thus $f_\chi \notin \mathrm{pPOL}_k\,\chi$ for all $\chi \in X$.

We now show $f_\chi := f_{L+j} \in \mathrm{pPOL}_k\,\varrho$ for all $j \in \{1, \dots, M\}$ and $\chi := \chi_j$.

- Let $\chi \notin \mathcal{S}$ and $\mu \neq 3$. Then the image of f_χ has at most two different values and thus $f_\chi \in \mathrm{pPOL}_k\,\varrho$.

- Let $\chi \in \mathcal{S}$ and $\mu > 3$. Assume $f_\chi \notin \mathrm{pPOL}_k\,\varrho$. Then exist three different rows c_{1*}, c_{2*}, c_{3*} with $c_{*1}, \dots, c_{*N} \in \varrho$ and $f_\chi(c_{*1}, \dots, c_{*N}) \notin \varrho$. But then $E_k^3 = \mathrm{pr}_A\,\chi = \{c_{*1}, \dots, c_{*N}\} \subseteq \varrho$ for some A with $|A| = 3$ in contradiction to ϱ coherent.

- Let $\mu = 3$. Assume $f_\chi \notin \mathrm{pPOL}_k\,\varrho$. Then there are three different rows c_{1*}, c_{2*}, c_{3*} with $c_{*1}, \dots, c_{*N} \in \varrho$ and $f_\chi(c_{*1}, \dots, c_{*N}) \notin \varrho$. But then $\chi^{[\pi]} = \{c_{*1}, \dots, c_{*N}\} \subseteq \varrho$ for some $\pi \in S_3$. Because $\varrho \in \mathcal{S}$ this holds already for $\pi = \mathrm{id}$. If $\chi \nsubseteq \varrho$ this contradicts the assumption. If $\chi \subset \varrho$, then $f_\chi(c_{*1}, \dots, c_{*N}) = w \in \varrho$ contradicting the assumption.

Let $F_0 := f_0$, $F_i := F_{i-1} \otimes f_i$ for $i \in \{1, \dots, L+M\}$. We show $F_L \in \mathrm{pPOL}_k\,\varrho$ by induction starting by $F_0 = f_0 \in \mathrm{pPOL}_k\,\varrho$.

Assume $F_i \notin \mathrm{pPOL}_k\,\varrho$. Then there are three different rows c_{1*}, c_{2*}, c_{3*} with $c_{*1}, \dots, c_{*N} \in \varrho$ and $f_\chi(c_{*1}, \dots, c_{*N}) \notin \varrho$. Because $F_{i-1}, f_i \in \mathrm{pPOL}_k\,\varrho$ there have to be rows from the F_{i-1}-part of F_i and from the f_i-part of F_i.

There are two cases because ϱ is totally symmetric.

- c_{1*}, c_{2*} are from the F_{i-1}-part and c_{3*} is from the f_i-part. Then there is some $z \in E_k$ with $\{\varphi_0^l(z) \mid l \geq 0\} \subseteq c_{3*}$. Because c_{1*} and c_{2*} are different, we get $\{v\} \times \{\varphi_0^l(z) \mid l \geq 0\} \subseteq \{c_{*1}, \dots, c_{*N}\} \subseteq \varrho$ for some $v \in \sigma(E_k^2)$ in contradiction to (7.1).

- c_{1*}, c_{2*} are from the f_i-part and c_{3*} is from the F_{i-1}-part.

  If c_{3*} is not a constant row then there is some $z \in E_k$ with $\{\varphi_0^l(z) \mid l \geq 0\} \subseteq c_{3*}$ by construction of F_{i-1}. Because c_{1*} and c_{2*} are different, we get $\{v\} \times \{\varphi_0^l(z) \mid l \geq 0\} \subseteq \{c_{*1}, \dots, c_{*N}\} \subseteq \varrho$ for some $v \in \sigma(E_k^2)$ in contradiction to (7.1). Thus c_{3*} has to be a constant row.

  If $i > L$ then $f_i = f_\chi$ for some $\chi \in X$, i.e., $\Psi_L \subseteq \mathrm{pPOL}_k\,\chi$. Because c_{1*} and c_{2*} are different there is some $v \in \sigma(E_k^2)$ and $z \in E_k$ with $\{\varphi(v) \mid \varphi \in \Psi_L\} \times \{z\} \subseteq \{c_{*1}, \dots, c_{*N}\} \subseteq \varrho\}$ in contradiction to (7.2). Thus $i \leq L$.

Now there are $v \in \sigma(E_k^2)$ and $z \in E_k$ with $\{\varphi(v) \mid \varphi \in \Psi_{i-1}\} \times \{z\} \subseteq \{c_{*1}, \ldots, c_{*N}\} \subseteq \varrho$ and

$$F_i(c_{*1}, \ldots, c_{*N}) = \begin{pmatrix} f_i(\mathrm{pr}_{0,1}\, c_{*1}, \ldots, \mathrm{pr}_{0,1}\, c_{*N}) \\ \varphi_0(z) \end{pmatrix} = \begin{pmatrix} \varphi_i(v) \\ \varphi_0(z) \end{pmatrix} \in \varrho$$

by (7.3) in contradiction to the assumption $F_i \notin \mathrm{pPOL}_k\, \varrho$.

Thus $F_i \in \mathrm{pPOL}_k\, \varrho$ and in consequence $F_{L+M} \in \mathrm{pPOL}_k\, \varrho$.

Because $F_{L+M} \notin \mathrm{pPOL}_k\, \chi$ for all $\chi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \{\varrho\}$ we have $\mathrm{pPOL}_k\, \varrho \in p\mathscr{C}_k$. □

Lemma 7.3. *Let* $k \geq 3$. *Then* $\mathrm{pPOL}_k\, \iota_k^3 \in p\mathscr{C}_k$.

Proof. Apply Theorem 7.2 with $L = 0$ and $\varphi_0(x) := x + 1 \pmod{k}$. □

Lemma 7.4. *Let* $\varrho = \{(0,1,2)^{[\pi]} \mid \pi \in S_3\} \cup \iota_4^3 \in \mathcal{S}'$. *Then* $\mathrm{pPOL}_4\, \varrho \in p\mathscr{C}_4$.

Proof. Apply Theorem 7.2 with $L = 1$, $\varphi_0(\eta_4) := (2,2,3,0)$ and $\varphi_1(\eta_4) := (2,3,3,0)$. □

8 RELATIONS FROM $\mathcal{L}$

Theorem 8.1. *Let* $\varrho := \varrho_i \in \mathcal{L}$ *for some* $i \in \{1,2\}$ *with* $k \geq 3$.
Then $\mathrm{pPOL}_k\, \varrho \in p\mathscr{C}_k$.

Proof. Let $\varphi : E_k \to E_k$ with $\varphi(x) = x + 1 \pmod{k}$ for all $x \in E_k$.

Let $f_1, f_2 : E_k^k \to E_k$, $g_1, g_2 : E_k \to E_k$ with

$$\begin{aligned} g_1(\eta_k) &:= f_1\left(\eta_k, \varphi(\eta_k), \varphi^2(\eta_k), \ldots, \varphi^{k-1}(\eta_k)\right) \\ &:= (1, 0, 2, 3, \ldots, k-1), \\ g_2(\eta_k) &:= f_2\left(\eta_k, \varphi(\eta_k), \varphi^2(\eta_k), \ldots, \varphi^{k-1}(\eta_k)\right) \\ &:= (0, 1, 1, \ldots, 1) \end{aligned}$$

Let $X := \{\chi \in \mathcal{U} \cup \mathcal{A} \mid \{\varphi, f_1, f_2\} \subseteq \mathrm{pPOL}_k\, \chi\}$. Then $g_i \in \mathrm{pPOL}_k\, \chi$ for all $\chi \in X$ because $\varphi \in \mathrm{pPOL}_k\, \chi$ and $f_i \in \mathrm{pPOL}_k\, \chi$ imply $g_i \in \mathrm{pPOL}_k\, \chi$ for $i \in \{1,2\}$.

Let $\chi^{(\mu)} \in X$ be arbitrary. Then there is some $v \in \sigma(\chi)$. Because φ and g_1 are the permutations $(0,1,2,\dots,k-1)$ and $(0,1)$, respectively, on E_k and thus generate S_k, we get

$$E_k^\mu \setminus \iota_k^\mu = \{\pi(v) \mid \pi \in S_k\} \subseteq \chi.$$

- $\mu = 1$. Then $\iota_k^1 = \emptyset$ and thus $\chi = E_k$ in contradiction to $\chi \in \mathcal{U}$.
- $\mu = 2$. Then $(1,2) \in \chi$ and thus $g_2((1,2)) = (1,1)$ and thus $\iota_k^2 \subseteq \chi$ implying $\chi = E_k^2$ in contradiction to $\chi \in \mathcal{A}$.
- $\mu \geq 3$. Then

$$g_2 \begin{pmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ \mu \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \chi$$

  in contradiction to $\chi \in \mathcal{A}$.

Thus $X = \emptyset$.

Let $F_0 := f_0 := \varphi$, $F_i := F_{i-1} \otimes f_i$ for $i \in \{1,2\}$. We show $F_2 \in \mathrm{pPOL}_k\, \varrho$ by induction starting by $F_0 = f_0 \in \mathrm{pPOL}_k\, \varrho$.

Assume $F_i \notin \mathrm{pPOL}_k\, \varrho$. Then there are four different rows $c_{1*}, c_{2*}, c_{3*}, c_{4*}$ with $c_{*1}, \dots, c_{*N} \in \varrho$ and $f_\chi(c_{*1}, \dots, c_{*N}) \notin \varrho$. Because $F_{i-1}, f_i \in \mathrm{pPOL}_k\, \varrho$ there have to be rows from the F_{i-1}-part of F_i and from the f_i-part of F_i.

- Only one row is from the F_{i-1}-part of F_i, w.l.o.g. this is c_{1*}. Then $c_{*1} \in \{(x,x,y,y), (x,y,x,y), (x,y,y,x) \mid x, y \in E_k\}$, w.l.o.g. $c_{*1} = (x,x,y,y)$. Then we have $\{(x, \varphi^l(x), \varphi^l(y), \varphi^l(y)) \mid l \in E_k\} \subseteq \{c_{*1}, \dots, c_{*N}\}$ by construction of f_i. But $|\{x, \varphi^l(x), \varphi^l(y)\}| = 3$ for some $l \in E_k$ contradicting $c_{*i} \in \varrho$ for all i.
- Two rows are from the F_{i-1}-part of F_i, w.l.o.g. these are c_{1*} and c_{2*}. Then w.l.o.g. $c_{1*} = (x,y,x,y)$ with $x \neq y$. Then $(x, y, \varphi(x), \varphi(y)) \in \{c_{*1}, \dots, c_{*N}\} \subseteq \varrho$ in contradiction to $|\{x, y, \varphi(x), \varphi(y)\}| \geq 3$.
- Only one row is from the f_i-part of F_i, w.l.o.g. this is c_{4*}. Then w.l.o.g. $c_{1*} = (x,y,x,y)$ with $x \neq y$. Then $\{(x,y,x,\varphi^l(y)) \mid l \in E_k\} \subseteq \{c_{*1}, \dots, c_{*N}\} \subseteq \varrho$ in contradiction to $|\{x, y, \varphi^l(y)\}| = 3$ for some $l \in E_k$.

Thus $F_2 \in \mathrm{pPOL}_k\,\varrho$ and $F_2 \notin \mathrm{pPOL}_k\,\chi$ for all $\chi \in \mathcal{U} \cup \mathcal{A}$. Then there is some $F \in \mathrm{pPOL}_k\,\varrho$ with $F \notin \mathrm{pPOL}_k\,\chi$ for all $\chi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \{\varrho\}$ by Lemmas 5.7 and 5.6. Thus $\mathrm{pPOL}_k\,\varrho \in p\mathscr{C}_k$. □

Lemma 8.2. *Let $\varrho = \sigma \cup \varrho_i$ be a coherent relation on E_4 with $\sigma \neq \emptyset$ areflexive and $i = 1, 2$. Then*

$$\forall f \in \mathrm{pPOL}_4\,\varrho\, \exists \chi^{(\mu)} \in \widetilde{\mathcal{R}}_k^{\max} : \mu \leq 2 \wedge f \in \mathrm{pPOL}_4\,\chi \tag{8.1}$$

holds and $\mathrm{pPOL}_4\,\varrho \notin p\mathscr{C}_4$ *by Lemma 3.3.*

Proof. W.l.o.g. $\eta_4 = (0,1,2,3) \in \varrho$ holds. Let $f^{(n)} \in \mathrm{pPOL}_4\,\varrho$ be arbitrary. Assume, (8.1) does not hold. Then $f \notin \mathrm{pPOL}_4\{x\}$ holds for every $x \in E_4$, in particular f' has no fixpoints.

- If $|f'(E_4)| = 1$ then f' has a fixpoint, in contradiction to the assumption.
- If $|f'(E_4)| = 2$ then there are $a, b \in E_4, a \neq b$ with $f'(a) = b$ and $f'(b) = a$. Let

  $$\chi := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \begin{pmatrix} a \\ x \\ y \\ b \end{pmatrix} \in \varrho \right\}.$$

  Then χ is a binary symmetric areflexive relation with

  $$\chi \subseteq \begin{pmatrix} a & b & a' & b' \\ b & a & b' & a' \end{pmatrix}$$

  and $\{a, b, a', b'\} = E_4$. Thus χ is coherent. From the conditions follow $f \notin \mathrm{pPOL}_4\,\chi$. Thus there are rows c_{1*}, c_{2*} with

  $$f \begin{pmatrix} c_{1*} \\ c_{2*} \end{pmatrix} =: \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \in E_4^2 \setminus \chi \tag{8.2}$$

  and $c_{*j} \in \chi$ for all $j \in \{1, 2, \ldots, n\}$. From

  $$C_{*j} := \begin{pmatrix} a \\ c_{1,j} \\ c_{2,j} \\ b \end{pmatrix} \in \varrho$$

and $f \in \mathrm{pPOL}_4\, \varrho$ follows

$$f(C_{*1}, C_{*2}, \ldots, C_{*n}) = \begin{pmatrix} b \\ d_1 \\ d_2 \\ a \end{pmatrix} \in \varrho.$$

Because of $(0\,3) \in \Gamma_\sigma$ we have

$$\begin{pmatrix} a \\ d_1 \\ d_2 \\ b \end{pmatrix} \in \varrho,$$

i.e., $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \in \chi$, in contradiction to (8.2).

- If $|f'(E_4)| = 3$ then $|\omega(f'(\eta_4))| = 3$, thus $f'(\eta_4) \notin \varrho$, in contradiction to the assumption.

- If $|f'(E_4)| = 4$ then f' is a fix-point-free permutation of E_4, i.e., either $f' = (a\,b)(a'\,b') \in S_4$ or $f' = (a\,a'\,b\,b') \in S_4$ with $\{a, a', b, b'\} = E_4$. The first subcase is like the case $|f'(E_4)|$ with $f'(a) = b$ and $f'(b) = a$ with $a \neq b$.

  The second subcase gives

$$\chi := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \begin{pmatrix} a \\ x \\ y \\ b \end{pmatrix} \in \varrho \right\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; \begin{pmatrix} a' \\ x \\ y \\ b' \end{pmatrix} \in \varrho \right\}.$$

  Then χ binary symmetric areflexive relation with

$$\chi \subseteq \begin{pmatrix} a & b & a' & b' \\ b & a & b' & a' \end{pmatrix},$$

  i.e., it is coherent. From the conditions follow $f \notin \mathrm{pPOL}_4\, \chi$. Thus there are rows c_{1*}, c_{2*} with

$$f\begin{pmatrix} c_{1*} \\ c_{2*} \end{pmatrix} =: \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \in E_4^2 \setminus \chi \tag{8.3}$$

and $c_{*j} \in \chi$ for all $j \in \{1, 2, \ldots, n\}$. Because of $(0\,3) \in \Gamma_\sigma$ the tuples

$$C_{*j} := \begin{pmatrix} a \\ c_{1,j} \\ c_{2,j} \\ b \end{pmatrix} \in \varrho$$

with

$$f(C_{*1}, C_{*2}, \ldots, C_{*n}) = \begin{pmatrix} a' \\ d_1 \\ d_2 \\ b' \end{pmatrix} \in \varrho$$

give $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \in \chi$, a contradiction to (8.3). From

$$C_{*j} := \begin{pmatrix} a \\ c_{1,j} \\ c_{2,j} \\ b \end{pmatrix} \in \varrho$$

and $f \in \mathrm{pPOL}_4\,\varrho$ follows

$$f(C_{*1}, C_{*2}, \ldots, C_{*n}) = \begin{pmatrix} a' \\ d_1 \\ d_2 \\ b' \end{pmatrix} \in \varrho.$$

Thus $\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \in \chi$ holds, in contradiction to (8.3). □

9 RELATIONS FROM $\mathcal{Q}_1$

Definition 9.1. Let $\varrho^{(h)} \in \mathcal{Q}_1$. We call ϱ *irreducible* if

$$\forall \emptyset \subset A \subset E_h\, \forall v \in E_k^{h-|A|} \setminus \iota_k^{h-|A|}\, \forall \pi \in S_h : (\mathrm{pr}_A\,\varrho) \times \{v\} \not\subseteq \varrho^{[\pi]}.$$

Otherwise we call it *reducible*.

Lemma 9.2 (Lemma 7.16 [14])**.** *Let $\varrho^{(h)} \in \mathcal{Q}_1$ be reducible. Then $\mathrm{pPOL}_k\,\varrho$ is not in the minimal covering $\mathrm{p}\mathscr{C}_k$ of $\mathrm{p}\mathscr{M}_k$.*

For the application of Lemma 9.2 suitable parameters A and v have to be found. In the case $k = 4$ we only have $A = \{0, 1\}$ for $\delta(\varrho) \in \{\delta^{(3)}_{\{0,1\}}, \delta^{(4)}_{\{0,1\}}\}$ and $A = \{0, 1, 2\}$ for $\delta(\varrho) = \delta^{(4)}_{\{0,1,2\}}$. Then v is determined easily.

Lemma 9.3. *Let $\varrho \in \mathcal{Q}_1$ be one of the relations enlisted in Table 1 in the Appendix with the numbers 67, 74, 82, 86, 69, 71, 76, 79, 84, 88, 90, or 72. Then* $\mathrm{pPOL}_4\, \varrho \in p\mathscr{C}_4$.

Proof. All these relations are irreducible. Here are some parameters for each $\varrho^{(3)}$ we will use later.

ϱ	$\varphi_0(\eta_4)$	$\varphi_1(\eta_4)$	v_1	$\varphi_2(\eta_4)$	v_2	Z
$67, 74, 82, 86$	$(3, 3, 3, 2)$	$(0, 0, 1, 2)$	3	$(0, 2, 3, 3)$	2	$2, 3$
69	$(3, 3, 3, 2)$	$(0, 0, 1, 2)$	3	$(1, 2, 3, 3)$	2	$2, 3$
$71, 76, 79, 84, 88, 90$	$(2, 2, 0, 0)$	$(2, 2, 0, 1)$	0	$(2, 3, 0, 0)$	2	$0, 2$
72	$(2, 2, 0, 0)$	$(2, 2, 1, 0)$	0	$(3, 2, 0, 0)$	2	$0, 2$

Let $\varrho^{(3)}$ be fixed. We have $Z = \varphi_0(E_4)$ and $|Z| = 2$. Let $\sigma_i := \{s \in \sigma(\mathrm{pp}\, \varrho) \mid \{s\} \times \{v_i\} \subseteq \sigma(\varrho)\}$ and let $s_i \in \sigma_i$ be arbitrary for $i \in \{1, 2\}$.

Let $\chi \in X$, $v \in \sigma(\mathrm{pp}\, \varrho)$ and $\pi \in S_h$ be arbitrary. Then we have $\{v\} \times \{\varphi_0(w), \varphi_0^2(w)\} \not\subseteq \varrho^{[\pi]}$.

Define

$$
\begin{aligned}
f_0 &:= \varphi_0,\\
f_1(\eta_4, \varphi_0(\eta_4), \varphi_0^2(\eta_4)) &:= \varphi_1(\eta_4), \text{ and}\\
f_2(\{\eta_4\} \cup \delta^{(4)}_{\{0,1\}}) &:= \varphi_2(\eta_4).
\end{aligned}
$$

One can check that $f_i \in \mathrm{pPOL}_4\, \varrho$ for each $i \in \{0, 1, 2\}$.

Let $X := \{\chi \in \widetilde{\mathcal{R}}_k^{\max} \mid \{f_0, f_1, f_2\} \subseteq \mathrm{pPOL}_4\, \chi\}$. Let $\chi^{(\mu)} \in X$ be arbitrary. We define $f_\chi \in \mathrm{pPOL}_4\, \varrho$ with $f_\chi \notin \mathrm{pPOL}_4\, \chi$.

- $\mu = 1$: Let $f_\chi(\chi) := w \in E_4 \setminus \chi$.
- $\mu = 2$: If $\chi \subseteq \sigma_i \cup \delta^{(2)}_{\{0,1\}}$ for some $i \in \{1, 2\}$ then let $f_\chi(\chi) := \varphi_i(s_i)$. Then $f_\chi \notin \mathrm{pPOL}_4\, \chi$ because $\varphi_i(s_i) \notin \sigma_i \cup \delta^{(2)}_{\{0,1\}}$. Otherwise let $f_\chi(\chi) := w \in E_4^2 \setminus \chi$.
- $\mu = 3$: If $\sigma(\chi) \neq \emptyset$, i.e., there is some $t \in \sigma(\chi)$. Then $\delta^{(3)}_{\{0,1\}} \subset \chi$ because $f_0, f_1 \in \mathrm{pPOL}_4\, \chi \in X$ and thereby $\varphi_0(t), \varphi_1(t) \in \chi$. Thus $\chi \in \mathcal{Q}_1 \cup \mathcal{S}$.

If $\chi \in \mathcal{Q}_1$ and $\operatorname{pp}\chi \subseteq \sigma_i \cup \delta^{(2)}_{\{0,1\}}$ for some $i \in \{1,2\}$ then let $f_\chi(\chi) := \begin{pmatrix} \varphi_i(s_i) \\ \varphi_0(v_i) \end{pmatrix}$ for $s \in \sigma(\chi)$. Then $f_\chi \notin \mathrm{pPOL}_4\,\chi$ because $\varphi_i(s') \notin \sigma_i \cup \delta^{(2)}_{\{0,1\}}$ for all $s' \in \sigma_i$.

If previous case does not apply but $\chi \subseteq \varrho$ then let $f_\chi(\chi) := w \in \varrho \setminus \chi$.

Otherwise let $f_\chi(\chi) := w \in E_4^\mu \setminus \chi$.

- $\mu = 4$: If $\sigma(\chi) \neq \emptyset$ we can assume $\eta_4 \in \chi$. Then $\varphi_1(\eta_4) \in \chi$ because $f_0, f_1 \in \mathrm{pPOL}_4\,\chi$. This implies $\delta^{(4)}_{\{0,1\}} \subseteq \chi$ because χ is coherent. But this implies $\varphi_2(\eta_4) \in \chi$ because $f_2 \in \mathrm{pPOL}_4\,\chi$ and thus $\delta^{(4)}_{\{2,3\}} \subseteq \chi$. Then $\iota_4^4 \subseteq \chi$, i.e., $\chi \in \mathcal{S}$, because χ is coherent. But then $\chi = \{\eta_4^{[\pi]} \mid \pi \in S_4\} \cup \iota_4^4 = E_4^4$ in contradiction to χ coherent. Thus $\sigma(\chi) = \emptyset$i, i.e., $\chi \in \{\iota_4^4, \varrho_1, \varrho_2\}$.

  Let $f_\chi(\chi) := \eta_4$.

We first show $f_\chi \in \mathrm{pPOL}_4\,\varrho$. Assume otherwise. Then there are rows c_{1*}, c_{2*}, c_{3*} with $c_{*1}, \ldots, c_{*N} \in \varrho$ and $f_\chi(c_{*1}, \ldots, c_{*N}) =: d \notin \varrho$.

- $\mu < 3$: Then $f_\chi \in \mathrm{pPOL}_4\,\varrho$ because of Lemma 5.3.

- $\mu = 3$: If $\chi \in \mathcal{Q}_1$ and $\operatorname{pp}\chi \subseteq \sigma_i \cup \delta^{(2)}_{\{0,1\}}$ for some $i \in \{1,2\}$. If additionally $\chi \subset \varrho$ then $c_{*1}, \ldots, c_{*N} \subseteq \chi$ and $d = \begin{pmatrix} \varphi(s_i) \\ \varphi_0(v_i) \end{pmatrix} \in \varrho$ in contradiction to the assumption. If additionally $\chi \not\subseteq \varrho$ then there is some $c_{*j} \notin \varrho$ in contradiction to the assumption.

  If previous case does not apply but $\chi \subseteq \varrho$ then $d = w \in \varrho$ in contradiction to the assumption.

  Otherwise $\chi \not\subseteq \varrho$, i.e., there is some $c_{*j} \notin \varrho$ in contradiction to the assumption.

- $\mu = 4$: Then $\chi \in \{\iota_4^4, \varrho_1, \varrho_2\}$ and $\operatorname{pr}_A \chi \not\subseteq \varrho$ for all $A \subseteq E_4$ with $|A| = 3$, i.e., there is some $c_{*i} \notin \varrho$ contradicting the assumption.

Thus $f_\chi \in \mathrm{pPOL}_4\,\varrho$ for every $\chi \in X$.

Now let $\{\chi_3, \ldots, \chi_L\} := X$ and $f_j := f_{\chi_j}$ for $j \in \{3, \ldots, L\}$. Define $F_0 := f_0$ and $F_j := F_{j-1} \otimes f_j$ for all $j \in \{1, \ldots, L\}$. We show $F_L \in \mathrm{pPOL}_4\,\varrho$ inductively over j.

The induction starts with $F_0 := f_0 \in \mathrm{pPOL}_4\,\varrho$. Assume $F_j \notin \mathrm{pPOL}_4\,\varrho$. Then there are rows c_{1*}, c_{2*}, c_{3*} with $c_{*1}, \dots, c_{*N} \in \varrho$ and

$$F_j \begin{pmatrix} c_{1*} \\ c_{2*} \\ c_{3*} \end{pmatrix} =: \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} =: d \in E_4^3 \setminus \varrho.$$

We first consider several cases with regard to which part of F_j the lines come from.

- The rows can neither come completely the F_{j-1}-part nor the f_j-part of F_j because $f_j, F_{j-1} \in \mathrm{pPOL}_4\,\varrho$ holds.
- The row c_{1*} is from the F_{j-1}-part and c_{2*}, c_{3*} from the f_j-part. Because of Lemma 5.3 the rows c_{2*} and c_{3*} are different and thus there is some column $c_{*j} = (x, y, z)$ with $y \neq z$. Because of the specific construction of f_1 and f_2 and because $f_0 = \varphi_0 \in \mathrm{pPOL}_4\,\chi$ we get

$$\begin{pmatrix} x & x & x \\ y & \varphi_0(y) & \varphi_0^2(y) \\ z & \varphi_0(z) & \varphi_0^2(z) \end{pmatrix} \subseteq \{c_{*1}, \dots, c_{*N}\} \subseteq \varrho.$$

  If $\varphi_0(y) = \varphi_0(z) =: y'$ then $(x, y', y'), (x, y'', y'') \in \varrho$ with $y'' := \varphi_0(y') \neq y'$. But this implies $\delta^{(3)}_{\{1,2\}} \subseteq \varrho$ in contradiction to $\delta(\varrho) = \delta^{(3)}_{\{0,1\}}$. Thus $y' := \varphi_0(y) \neq \varphi_0(z) =: z'$. Because $y', z' \in Z$ and $|Z| = 2$ we conclude $\varphi_0(z') = y'$, $\varphi_0(y') = z'$ and thus we have $(x, y', z'), (x, z', y') \in \varrho$. If $x = y'$ or $x = z'$ then $\delta^{(3)}_{\{0,2\}} \subseteq \varrho$ in contradiction to $\delta(\varrho) = \delta^{(3)}_{\{0,1\}}$. This implies $|\{x, y', z'\}| = 3$, i.e., $(x, y', z'), (x, y', z')^{[(1\,2)]} \in \sigma(\varrho)$ in contradiction to $(1\,2) \notin \Gamma_{\sigma(\varrho)}$. Thus this case can not occur.
- The row c_{2*} is from the F_{j-1}-part and c_{1*}, c_{3*} from the f_j-part. This case is identical to proof as the case above, i.e., it can not occur.

Thus the rows c_{1*}, c_{2*} are from the f_j-part and c_{3*} is from the F_{j-1}-part of F_j.

Assume the row c_{3*} is not constant. Then it occurs in the domain of some f_j with $j \neq 0$, i.e., for some $x \in c_{3*}$ then also $Z = \{\varphi_0(x), \varphi_0^2(x)\} \subseteq c_{3*}$. But then $\{(x, y)\} \times Z \subseteq \varrho$ with $x \neq y$ which can be shown to be false for each relation by looking at the relation ϱ. Thus $c_{3*} = (c_3, \dots, c_3)$, i.e., a constant row and $F_j(c_{3*}) = f_0(c_3) = \varphi_0(c_3)$.

Because $c_{1*} \neq c_{2*}$ there is a column $c_{*j} = (x, y, c_3)$ with $x \neq y$. If $\varphi_0(x) \neq \varphi_0(y)$ then there are columns $(x', y', c_3) = (\varphi_0(x), \varphi_0(y), c_3) \in \varrho$ and $(x'', y'', c_3) = (\varphi_0(x'), \varphi_0(y'), c_3) = (\varphi_0^2(x), \varphi_0^2(y), c_3) \in \varrho$ with $\{x', y'\} = Z = \{x'', y''\}$. But this cannot happen for the given relations. Thus $\varphi_0(x) = \varphi_0(y)$.

- If $C := \mathrm{pr}_{0,1}\{c_{*1}, \ldots, c_{*N}\} \subseteq \sigma_i \cup \delta^{(2)}_{\{0,1\}}$ for some $i \in \{1, 2\}$ then we have

$$d = \begin{pmatrix} \varphi_i(s_i) \\ \varphi_0(v_i) \end{pmatrix} \in \sigma_{i'} \times \{v_i'\} \subseteq \varrho$$

  with $i' \in \{1, 2\}$ and $i \neq i'$ by construction of f_χ in this case. This contradicts $d \notin \varrho$.

- If $C := \mathrm{pr}_{0,1}\{c_{*1}, \ldots, c_{*N}\} \nsubseteq \sigma_i \cup \delta^{(2)}_{\{0,1\}}$ for all $i \in \{1, 2\}$ then $C \subseteq \sigma_1 \cup \sigma_2 \cup \delta^{(2)}_{\{0,1\}}$ because $\{c_{*1}, \ldots, c_{*N}\} \subseteq \varrho$. Then there are $t_1 \in \sigma_1$ and $t_2 \in \sigma_2$ with

$$\begin{pmatrix} t_1 \\ c_3 \end{pmatrix}, \begin{pmatrix} t_2 \\ c_3 \end{pmatrix} \in \varrho.$$

  But this cannot happen for the given ϱ and corresponding σ_1 and σ_2. Thus this contradicts $\{c_{*1}, \ldots, c_{*N}\} \subseteq \varrho$.

Thus we can conclude $F_j \in \mathrm{pPOL}_4\, \varrho$ and finally $F := F_L \in \mathrm{pPOL}_4\, \varrho$ Because $F \notin \mathrm{pPOL}_4\, \chi$ for all $\chi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \{\varrho\}$ this proofs $\mathrm{pPOL}_4\, \varrho \in p\mathscr{C}_4$ because of Lemma 3.4. ☐

Lemma 9.4. *Let $\varrho \in \mathcal{Q}_1$ be the relation with the number 124. Then* $\mathrm{pPOL}_4\, \varrho \in p\mathscr{C}_4$.

Proof. This proof is similar to the previous one. Here are some parameters for each $\varrho^{(4)}$ we will use later.

ϱ	$\varphi_0(\eta_4)$	$\varphi_1(\eta_4) = \varphi_2(\eta_4)$	v_1	v_2	Z
124	$(3, 3, 3, 2)$	$(1, 0, 3, 2)$	23	32	$2, 3$

We have $Z = \varphi_0(E_4)$ and $|Z| = 2$. Let $\sigma_1 := \{(0, 1)\}$ and $\sigma_2 := \{(1, 0)\}$ and let $s_i \in \sigma_i$ be arbitrary for $i \in \{1, 2\}$.

Let $\chi \in X$, $v \in \sigma(\mathrm{pp}\, \varrho)$ and $\pi \in S_h$ be arbitrary. Then we have $\{v\} \times \{\varphi_0(w), \varphi_0^2(w)\} \nsubseteq \varrho^{[\pi]}$.

Define $f_0 := \varphi_0$, $f_1(\eta_4, \varphi_0(\eta_4), \varphi_0^2(\eta_4)) := \varphi_1(\eta_4)$. One can check that $f_i \in \mathrm{pPOL}_4\, \varrho$ for each $i \in \{0, 1\}$.

Let $X := \{\chi \in \widetilde{\mathcal{R}}_k^{\max} \mid \{f_0, f_1\} \subseteq \mathrm{pPOL}_4\, \chi\}$. Let $\chi^{(\mu)} \in X$ be arbitrary. We define $f_\chi \in \mathrm{pPOL}_4\, \varrho$ with $f_\chi \notin \mathrm{pPOL}_4\, \chi$.

- $\mu = 1$: Let $f_\chi(\chi) := w \in E_4 \setminus \chi$.
- $\mu = 2$: If $\chi \subseteq \sigma_i \cup \delta^{(2)}_{\{0,1\}}$ for some $i \in \{1,2\}$ then let $f_\chi(\chi) := \varphi_i(s_i)$. Then $f_\chi \notin \mathrm{pPOL}_4\, \chi$ because $\varphi_i(s_i) \notin \sigma_i \cup \delta^{(2)}_{\{0,1\}}$. Otherwise let $f_\chi(\chi) := w \in E_4^2 \setminus \chi$.
- $\mu \geq 3$: If $\sigma(\chi) \neq \emptyset$, i.e., there is some $t \in \sigma(\chi)$. Then $\delta^{(4)}_{\{0,1\}} \subset \chi$ because $f_0, f_1 \in \mathrm{pPOL}_4\, \chi \in X$ and thereby $\varphi_0(t), \varphi_1(t), \varphi_0(\varphi_1(t)) \in \chi$. Thus $\chi \in \mathcal{Q}_1 \cup \mathcal{S}$.

  If $\chi \in \mathcal{Q}_1$ and $\mathrm{pp}\, \chi \subseteq \sigma_i \cup \delta^{(2)}_{\{0,1\}}$ for some $i \in \{1,2\}$ then let $f_\chi(\chi) := \begin{pmatrix} \varphi_i(s_i) \\ \varphi_0(v_i) \end{pmatrix}$ for $s \in \sigma(\chi)$. Then $f_\chi \notin \mathrm{pPOL}_4\, \chi$ because $\varphi_i(s') \notin \sigma_i \cup \delta^{(2)}_{\{0,1\}}$ for all $s' \in \sigma_i$.

  If previous case does not apply but $\chi \subseteq \varrho$ then let $f_\chi(\chi) := w \in \varrho \setminus \chi$.

  Otherwise let $f_\chi(\chi) := w \in E_4^\mu \setminus \chi$.

We first show $f_\chi \in \mathrm{pPOL}_4\, \varrho$. Assume otherwise. Then there are rows c_{1*}, c_{2*}, c_{3*} with $c_{*1}, \ldots, c_{*N} \in \varrho$ and $f_\chi(c_{*1}, \ldots, c_{*N}) =: d \notin \varrho$.

- $\mu < 4$: Then $f_\chi \in \mathrm{pPOL}_4\, \varrho$ because of Lemma 5.3.
- $\mu = 4$: If $\chi \in \mathcal{Q}_1$ and $\mathrm{pp}\, \chi \subseteq \sigma_i \cup \delta^{(2)}_{\{0,1\}}$ for some $i \in \{1,2\}$. If additionally $\chi \subset \varrho$ then $c_{*1}, \ldots, c_{*N} \subseteq \chi$ and $d = \begin{pmatrix} \varphi(s_i) \\ \varphi_0(v_i) \end{pmatrix} \in \varrho$ in contradiction to the assumption. If additionally $\chi \nsubseteq \varrho$ then there is some $c_{*j} \notin \varrho$ in contradiction to the assumption.

  If previous case does not apply but $\chi \subseteq \varrho$ then $d = w \in \varrho$ in contradiction to the assumption.

  Otherwise $\chi \nsubseteq \varrho$, i.e., there is some $c_{*j} \notin \varrho$ in contradiction to the assumption.

Thus $f_\chi \in \mathrm{pPOL}_4\, \varrho$ for every $\chi \in X$.

Now let $\{\chi_3, \ldots, \chi_L\} := X$ and $f_j := f_{\chi_j}$ for $j \in \{3, \ldots, L\}$. Define $F_0 := f_0$, $F_2 := F_1$ and $F_j := F_{j-1} \otimes f_j$ for all $j \in \{1, 3, 4, \ldots, L\}$. We show $F_L \in \mathrm{pPOL}_4\, \varrho$ inductively over j.

The induction starts with $F_0 := f_0 \in \mathrm{pPOL}_4\,\varrho$. Assume $F_j \notin \mathrm{pPOL}_4\,\varrho$. Then there are rows $c_{1*}, c_{2*}, c_{3*}, c_{4*}$ with $c_{*1}, \ldots, c_{*N} \in \varrho$ and

$$F_j \begin{pmatrix} c_{1*} \\ c_{2*} \\ c_{3*} \\ c_{4*} \end{pmatrix} =: \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix} =: d \in E_4^3 \setminus \varrho.$$

We first consider several cases with regard to which part of F_j the lines come from.

- The rows can neither come completely the F_{j-1}-part nor the f_j-part of F_j because $f_j, F_{j-1} \in \mathrm{pPOL}_4\,\varrho$ holds.

- The row c_{1*} is from the F_{j-1}-part and c_{2*} is from the f_j-part. Because of the specific construction of f_1 and because $f_0 = \varphi_0 \in \mathrm{pPOL}_4\,\chi$ we get

  $$\begin{pmatrix} x & x & x \\ y & \varphi_0(y) & \varphi_0^2(y) \\ z & z' & z'' \\ w & w' & w'' \end{pmatrix} \subseteq \{c_{*1}, \ldots, c_{*N}\} \subseteq \varrho.$$

  But then

  $$\{x\} \times \{2,3\} = \{x\} \times \{\varphi_0(y), \varphi_0^2(y)\} \subseteq \mathrm{pp}\,\varrho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cup \delta_{\{0,1\}}^{(2)},$$

  which is false. Thus this case can not occur.

- The row c_{2*} is from the F_{j-1}-part and c_{1*} is from the f_j-part. This case can be handled similar to the case above is identical to proof as the case above, i.e., it can not occur.

- The rows c_{1*}, c_{2*}, c_{3*} are from the F_{j-1}-part and c_{4*} is from the f_j-part. Because of Lemma 5.3 there is some column $c_{*j} = (x, y, z, w)$ with $|\{x, y, z\}| = 3$. Because of the specific construction of f_1 and because $f_0 = \varphi_0 \in \mathrm{pPOL}_4\,\chi$ we get

  $$\begin{pmatrix} x & x \\ y & y \\ z & z \\ w & \varphi_0(w) \end{pmatrix} \subseteq \{c_{*1}, \ldots, c_{*N}\} \subseteq \varrho.$$

  Then $|\{x, y, z, w\}| = 3$ or $|\{x, y, z, \varphi_0(w)\}| = 3$ because $w \neq \varphi_0(w)$ and $x, y, z, w, \varphi_0(w) \in E_4$. But this implies $\delta_A^{(4)} \subseteq \varrho$ for some A with

$|\{0,1\} \cap A| = 1$ in contradiction to $\delta(\varrho) = \delta^{(4)}_{\{0,1\}}$. Thus this case can not occur.

- The rows c_{1*}, c_{2*}, c_{4*} are from the F_{j-1}-part and c_{3*} is from the f_j-part. This case is identical to proof as the case above, i.e., it can not occur.

Thus the rows c_{1*}, c_{2*} are from the f_j-part and c_{3*}, c_{4*} are from the F_{j-1}-part of F_j.

Assume the row c_{3*} is not constant. Then it occurs in the domain of some f_j with $j \neq 0$, i.e., for some $x \in c_{3*}$ then also $Z = \{\varphi_0(x), \varphi_0^2(x)\} \subseteq c_{3*}$. But then $\{(x,y)\} \times Z \subseteq \mathrm{pr}_{0,1,2}\,\varrho$ with $x \neq y$ which is false for relation ϱ. Thus $c_{3*} = (c_3, \ldots, c_3)$, i.e., a constant row and $d_3 = F_j(c_{3*}) = f_0(c_3) = \varphi_0(c_3)$, and similar $c_{4*} = (c_4, \ldots, c_4)$, i.e., a constant row and $d_4 = F_j(c_{4*}) = f_0(c_4) = \varphi_0(c_4)$,

Because $c_{1*} \neq c_{2*}$ there is a column $c_{*j} = (x, y, c_3)$ with $x \neq y$. If $\varphi_0(x) \neq \varphi_0(y)$ then there are columns $(x', y', c_3) = (\varphi_0(x), \varphi_0(y), c_3) \in \varrho$ and $(x'', y'', c_3) = (\varphi_0(x'), \varphi_0(y'), c_3) = (\varphi_0^2(x), \varphi_0^2(y), c_3) \in \varrho$ with $\{x', y'\} = Z = \{x'', y''\}$. But this cannot happen for the given relation. Thus $\varphi_0(x) = \varphi_0(y)$.

- If $C := \mathrm{pr}_{0,1}\{c_{*1}, \ldots, c_{*N}\} \subseteq \sigma_i \cup \delta^{(2)}_{\{0,1\}}$ for some $i \in \{1,2\}$ then we have
$$d = \begin{pmatrix} \varphi_i(s_i) \\ \varphi_0(v_i) \end{pmatrix} \in \sigma_{i'} \times \{v_i'\} \subseteq \varrho$$
with $i' \in \{1,2\}$ and $i \neq i'$ by construction of f_χ in this case. This contradicts $d \notin \varrho$.

- If $C := \mathrm{pr}_{0,1}\{c_{*1}, \ldots, c_{*N}\} \nsubseteq \sigma_i \cup \delta^{(2)}_{\{0,1\}}$ for all $i \in \{1,2\}$ then $C = \sigma_1 \cup \sigma_2 \cup \delta^{(2)}_{\{0,1\}}$ because $\{c_{*1}, \ldots, c_{*N}\} \subseteq \varrho$. Then there are $t_1 \in \sigma_1$ and $t_2 \in \sigma_2$ with
$$\begin{pmatrix} t_1 \\ c_3 \\ c_4 \end{pmatrix}, \begin{pmatrix} t_2 \\ c_3 \\ c_4 \end{pmatrix} \in \varrho.$$
But this cannot happen for the given ϱ and this is in contradiction to $\{c_{*1}, \ldots, c_{*N}\} \subseteq \varrho$.

Thus we can conclude $F_j \in \mathrm{pPOL}_4\,\varrho$ and finally $F := F_L \in \mathrm{pPOL}_4\,\varrho$ Because $F \notin \mathrm{pPOL}_4\,\chi$ for all $\chi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \{\varrho\}$ this proofs $\mathrm{pPOL}_4\,\varrho \in p\mathscr{C}_4$ because of Lemma 3.4. $\square$

10 CONCLUSION

The list in the appendix shows with the help of the theorems and lemmas in the previous sections in this paper the following theorem

Theorem 10.1. *The minimal covering* $p\mathscr{C}_4$ *of the maximal partial clones in* $p\mathscr{M}_4$ *has exactly* 449 *elements and all these elements are given in the list in the appendix.*

In addition to the determination of $p\mathscr{C}_4$ several general theorems such as for example the ones in Section 6 can be used for the determination of elements of $p\mathscr{C}_k$ for arbitrary $k \geq 3$.

REFERENCES

[1] L. Haddad and D. Lau. (2007). Some criteria for partial Sheffer functions in k-valued logic. *Mult.-Valued Logic and Soft Computing*, 13(4-6):415–446.

[2] L. Haddad and I.G. Rosenberg. (1989). Maximal partial clones determined by the areflexive relations. *Discrete Appl. Math.*, 24(1-3):133–143.

[3] L. Haddad and I.G. Rosenberg. (1991). Partial Sheffer operations. *Eur. J. Comb.*, 12(1):9–21.

[4] L. Haddad and I.G. Rosenberg. (1992). Completeness theory for finite partial algebras. *Algebra Univers.*, 29(3):378–401.

[5] L. Haddad, I.G. Rosenberg, and D. Schweigert. (1990). A maximal partial clone and a Slupecki-type criterion. *Acta Sci. Math.*, 54(1-2):89–98.

[6] L. Haddad and G.E. Simons. (2002). Maximal partial clones of 4-valued logic. *Mult.-Valued Logic and Soft Computing*, 8(4):531–562.

[7] D. Lau. (2006). *Function algebras on finite sets. A basic course on many-valued logic and clone theory.* Springer Monographs in Mathematics. Berlin: Springer. xiv, 668 p.

[8] B.A. Romov. The algebras of partial functions and their invariants. *Cybernetics*, 17:157–167, 1981.

[9] I.G. Rosenberg. (1965). La structure des fonctions de plusieurs variables sur un ensemble fini. *C. R. Acad. Sci., Paris*, 260:3817–3819.

[10] I.G. Rosenberg. (1970). Über die funktionale Vollständigkeit in den mehrwertigen Logiken. *Rozpravy Československe Akad. Ved. Řada Mat. Přirod. Věd*, 80:3–93.

[11] I.G. Rosenberg. (1984). Composition of functions on finite sets, completeness and relations, a short survey. *In D. Rine (ed.) Multiple-valued Logic and Computer Science, 2nd edition, North-Holland, Amsterdam*, pages 150–192.

[12] K. Schölzel. (May 2009). The Minimal Covering of Maximal Partial Clones in 4-valued Logic. In *Multiple-Valued Logic, 39th IEEE International Symposium on*, pages 126–131.

[13] K. Schölzel. (2010). Counting the maximal partial clones on a finite set. *Multiple-Valued Logic and Soft Computing*. to appear.

[14] K. Schölzel. (2010). Uniqueness of minimal coverings of maximal partial clones. *Algebra Universalis*. to appear.

[15] D.L. Webb. (1935). Generation of an n-valued logic by one binary operation. *Bull. Amer. Math. Soc.*, 41.

A LIST OF ALL COHERENT RELATIONS

The following table include all coherent relations $\varrho^{(h)}$ for $k = 4$.

Define the relation class $R(\varrho)$ by

$$R(\varrho) := \{\{f(v) \mid v \in \varrho\} \mid f \in S_k\}.$$

The column *i*so is the number of different maximal clones described by the relations in $R(\varrho)$, i.e., $|\{\mathrm{pPOL}_4\,\chi \mid \chi \in R(\varrho)\}|$. The column *T*/L gives the number of the theorem or lemma which can be used to determine, whether the described maximal clone belongs to every minimal covering of $p\mathscr{M}_4$ or not. This information is also given in the last column. Thus the clone described by a relation with *In* is in every minimal covering, and the ones with *Out* are not.

Relation where $|\sigma(\varrho)|$ is big can be written as

$$v(G) := \left\{ v^{[\pi]} \;\middle|\; \pi \in G \right\}$$

with $v \in E_4^h$ and $G \leq S_h$ for some h.

Nr.	$\delta(\varrho)$	$\sigma(\varrho)$	iso	T/L	I/O
1	$\emptyset$	0	4	5.2	In
2	$\emptyset$	$0, 1$	6	5.2	In
3	$\emptyset$	$0, 1, 2$	4	5.2	In
4	$\emptyset$	01	6	6.8	Out
5	$\emptyset$	$01, 02$	12	6.8	Out
6	$\emptyset$	$01, 23$	6	6.3	In
7	$\emptyset$	$01, 02, 03$	4	6.8	Out
8	$\emptyset$	$01, 02, 32$	12	6.6	Out
9	$\emptyset$	$01, 02, 31, 32$	3	6.8	Out
10	$\emptyset$	$01, 10$	6	6.5	In
11	$\emptyset$	$01, 10, 02, 20$	12	6.5	In
12	$\emptyset$	$01, 10, 23, 32$	3	6.5	In
13	$\emptyset$	$01, 10, 02, 20, 03, 30$	4	6.5	In
14	$\emptyset$	$01, 10, 12, 21, 23, 32$	12	6.5	In
15	$\emptyset$	$01, 10, 02, 20, 13, 31, 23, 32$	3	6.5	In
16	$\delta^2_{\{0,1\}}$	01	6	6.3	In
17	$\delta^2_{\{0,1\}}$	$01, 02$	12	6.3	In
18	$\delta^2_{\{0,1\}}$	$01, 12$	12	6.3	In
19	$\delta^2_{\{0,1\}}$	$01, 23$	6	6.3	In
20	$\delta^2_{\{0,1\}}$	$01, 02, 03$	4	6.10	Out
21	$\delta^2_{\{0,1\}}$	$01, 02, 30$	12	6.10	Out
22	$\delta^2_{\{0,1\}}$	$01, 02, 12$	12	6.3	In
23	$\delta^2_{\{0,1\}}$	$01, 02, 13$	24	6.10	Out
24	$\delta^2_{\{0,1\}}$	$01, 02, 32$	12	6.7	Out
25	$\delta^2_{\{0,1\}}$	$01, 12, 23$	12	6.10	Out
26	$\delta^2_{\{0,1\}}$	$01, 02, 03, 12$	24	6.10	Out
27	$\delta^2_{\{0,1\}}$	$01, 02, 30, 32$	24	6.10	Out

TABLE 1: All coherent relations for $k = 4$

Nr.	$\delta(\varrho)$	$\sigma(\varrho)$	iso	T/L	I/O
28	$\delta^2_{\{0,1\}}$	$01, 02, 31, 32$	3	6.8	Out
29	$\delta^2_{\{0,1\}}$	$01, 02, 12, 23$	24	6.10	Out
30	$\delta^2_{\{0,1\}}$	$01, 03, 12, 23$	12	6.10	Out
31	$\delta^2_{\{0,1\}}$	$01, 02, 13, 23$	6	6.10	Out
32	$\delta^2_{\{0,1\}}$	$01, 02, 03, 12, 13$	12	6.10	Out
33	$\delta^2_{\{0,1\}}$	$01, 02, 03, 12, 31$	24	6.10	Out
34	$\delta^2_{\{0,1\}}$	$01, 02, 03, 13, 23$	6	6.10	Out
35	$\delta^2_{\{0,1\}}$	$01, 02, 12, 13, 23$	12	6.10	Out
36	$\delta^2_{\{0,1\}}$	$01, 02, 03, 12, 13, 23$	12	6.10	Out
37	$\delta^2_{\{0,1\}}$	$01, 10$	6	6.3	In
38	$\delta^2_{\{0,1\}}$	$01, 10, 02, 20$	12	6.3	In
39	$\delta^2_{\{0,1\}}$	$01, 10, 23, 32$	3	6.3	In
40	$\delta^2_{\{0,1\}}$	$01, 10, 02, 20, 12, 21$	4	6.3	In
41	$\delta^2_{\{0,1\}}$	$01, 10, 02, 20, 03, 30$	4	6.13	Out
42	$\delta^2_{\{0,1\}}$	$01, 10, 12, 21, 23, 32$	12	6.13	Out
43	$\delta^2_{\{0,1\}}$	$01, 10, 02, 20, 03, 30, 12, 21$	12	5.4	Out
44	$\delta^2_{\{0,1\}}$	$01, 10, 02, 20, 13, 31, 23, 32$	3	6.1	In
45	$\delta^2_{\{0,1\}}$	$01, 10, 02, 20, 03, 30, 12, 21, 13, 31$	6	5.4	Out
46	$\emptyset$	012	4	6.8	Out
47	$\emptyset$	$012, 021$	12	6.8	Out
48	$\emptyset$	$012, 120, 201$	4	6.5	In
49	$\emptyset$	$012(S_3)$	4	6.5	In
50	$\emptyset$	$012, 013$	6	6.8	Out
51	$\emptyset$	$012, 013, 102, 103$	6	6.8	Out
52	$\emptyset$	$012, 013, 021, 031$	12	6.8	Out
53	$\emptyset$	$012, 120, 201, 013, 130, 301$	6	6.5	In
54	$\emptyset$	$012(S_3), 013(S_3)$	6	6.5	In
55	$\delta^3_{\{0,1,2\}}$	012	4	6.3	In
56	$\delta^3_{\{0,1,2\}}$	$012, 021$	12	6.3	In
57	$\delta^3_{\{0,1,2\}}$	$012, 120, 201$	4	6.3	In
58	$\delta^3_{\{0,1,2\}}$	$012(S_3)$	4	6.3	In
59	$\delta^3_{\{0,1,2\}}$	$012, 013$	6	6.8	Out
60	$\delta^3_{\{0,1,2\}}$	$012, 013, 102, 103$	6	6.8	Out
61	$\delta^3_{\{0,1,2\}}$	$012, 013, 021, 023$	12	6.8	Out
62	$\delta^3_{\{0,1,2\}}$	$012, 120, 201, 013, 130, 301$	6	6.5	In
63	$\delta^3_{\{0,1,2\}}$	$012(S_3), 013(S_3)$	6	6.5	In
64	$\delta^3_{\{0,1\}}$	012	12	9.2	Out
65	$\delta^3_{\{0,1\}}$	$012, 013$	6	9.2	Out
66	$\delta^3_{\{0,1\}}$	$012, 023$	24	9.3	In
67	$\delta^3_{\{0,1\}}$	$012, 032$	12	9.2	Out
68	$\delta^3_{\{0,1\}}$	$012, 123$	24	9.3	In
69	$\delta^3_{\{0,1\}}$	$012, 132$	12	9.2	Out
70	$\delta^3_{\{0,1\}}$	$012, 230$	12	9.3	In
71	$\delta^3_{\{0,1\}}$	$012, 320$	12	9.3	In
72	$\delta^3_{\{0,1\}}$	$012, 031, 032$	24	9.2	Out
73	$\delta^3_{\{0,1\}}$	$012, 023, 123$	24	9.3	In
74	$\delta^3_{\{0,1\}}$	$012, 032, 312$	12	9.2	Out
75	$\delta^3_{\{0,1\}}$	$012, 230, 231$	24	9.3	In
76	$\delta^3_{\{0,1\}}$	$012, 130, 132$	24	9.2	Out
77	$\delta^3_{\{0,1\}}$	$012, 013, 023, 123$	24	9.2	Out
78	$\delta^3_{\{0,1\}}$	$012, 013, 230, 231$	6	9.3	In
79	$\delta^3_{\{0,1\}}$	$012, 102$	12	9.2	Out

TABLE 1: All coherent relations for $k = 4$

Nr.	$\delta(\varrho)$	$\sigma(\varrho)$	iso	T/L	I/O
80	$\delta^3_{\{0,1\}}$	$012, 102, 013, 103$	6	9.2	Out
81	$\delta^3_{\{0,1\}}$	$012, 102, 023, 203$	24	9.3	In
82	$\delta^3_{\{0,1\}}$	$012, 102, 032, 302$	12	9.2	Out
83	$\delta^3_{\{0,1\}}$	$012, 102, 230, 320$	12	9.3	In
84	$\delta^3_{\{0,1\}}$	$012, 102, 031, 301, 032, 302$	24	9.2	Out
85	$\delta^3_{\{0,1\}}$	$012, 102, 023, 203, 123, 213$	12	9.3	In
86	$\delta^3_{\{0,1\}}$	$012, 102, 032, 302, 312, 132$	4	9.2	Out
87	$\delta^3_{\{0,1\}}$	$012, 102, 230, 320, 231, 321$	12	9.3	In
88	$\delta^3_{\{0,1\}}$	$012, 102, 013, 103, 023, 203, 123, 213$	12	9.2	Out
89	$\delta^3_{\{0,1\}}$	$012, 102, 013, 103, 230, 320, 231, 321$	3	9.3	In
90	ι^3_4	$\emptyset$	1	7.3	In
91	ι^3_4	$012(S_3)$	4	7.4	In
92	ι^3_4	$012(S_3), 013(S_3)$	6	7.1	Out
93	ι^3_4	$012(S_3), 013(S_3), 023(S_3)$	4	7.1	Out
94	$\emptyset$	0123	1	6.8	Out
95	$\emptyset$	$0123, 1023$	6	6.8	Out
96	$\emptyset$	$0123, 1032$	3	6.5	In
97	$\emptyset$	$0123, 1203, 2013$	4	6.8	Out
98	$\emptyset$	$0123, 1230, 2301, 3012$	3	6.5	In
99	$\emptyset$	$0123, 0132, 1023, 1032$	3	6.8	Out
100	$\emptyset$	$0123, 1032, 3210, 2301$	1	6.5	In
101	$\emptyset$	$0123, 0132, 0213, 0231, 0312, 0321$	4	6.8	Out
102	$\emptyset$	$0123, 1230, 2301, 3012, 2103, 3210, 0321, 1032$	3	6.5	In
103	$\emptyset$	$0123(A_4)$	1	6.5	In
104	$\emptyset$	$0123(S_4)$	1	6.5	In
105	$\delta^4_{\{0,1,2,3\}}$	0123	1	6.8	Out
106	$\delta^4_{\{0,1,2,3\}}$	$0123, 1023$	6	6.8	Out
107	$\delta^4_{\{0,1,2,3\}}$	$0123, 1032$	3	6.5	In
108	$\delta^4_{\{0,1,2,3\}}$	$0123, 1203, 2013$	4	6.8	Out
109	$\delta^4_{\{0,1,2,3\}}$	$0123, 1230, 2301, 3012$	3	6.5	In
110	$\delta^4_{\{0,1,2,3\}}$	$0123, 0132, 1023, 1032$	3	6.8	Out
111	$\delta^4_{\{0,1,2,3\}}$	$0123, 1032, 3210, 2301$	1	6.5	In
112	$\delta^4_{\{0,1,2,3\}}$	$0123, 0132, 0213, 0231, 0312, 0321$	4	6.8	Out
113	$\delta^4_{\{0,1,2,3\}}$	$0123, 1230, 2301, 3012, 2103, 3210, 0321, 1032$	3	6.5	In
114	$\delta^4_{\{0,1,2,3\}}$	$0123(A_4)$	1	6.5	In
115	$\delta^4_{\{0,1,2,3\}}$	$0123(S_4)$	1	6.5	In
116	$\delta^4_{\{0,1,2\}}$	0123	4	9.2	Out
117	$\delta^4_{\{0,1,2\}}$	$0123, 1023$	12	9.2	Out
118	$\delta^4_{\{0,1,2\}}$	$0123, 1203, 2013$	4	9.2	Out
119	$\delta^4_{\{0,1,2\}}$	$0123, 0213, 1023, 1203, 2013, 2103$	4	9.2	Out
120	$\delta^4_{\{0,1\}}$	0123	6	9.2	Out
121	$\delta^4_{\{0,1\}}$	$0123, 1023$	6	9.2	Out
122	$\delta^4_{\{0,1\}}$	$0123, 0132$	6	9.2	Out
123	$\delta^4_{\{0,1\}}$	$0123, 1032$	6	9.4	In
124	$\delta^4_{\{0,1\}}$	$0123, 0132, 1023, 1032$	6	9.2	Out
125	$\delta^4_{\{0,1\},\{2,3\}}$	0123	3	6.4	In
126	$\delta^4_{\{0,1\},\{2,3\}}$	$0123, 1023$	6	6.4	In
127	$\delta^4_{\{0,1\},\{2,3\}}$	$0123, 1032$	3	6.4	In
128	$\delta^4_{\{0,1\},\{2,3\}}$	$0123, 2301$	6	6.4	In
129	$\delta^4_{\{0,1\},\{2,3\}}$	$0123, 0132, 1023, 1032$	3	6.4	In
130	$\delta^4_{\{0,1\},\{2,3\}}$	$0123, 1032, 2310, 3201$	3	6.4	In
131	$\delta^4_{\{0,1\},\{2,3\}}$	$0123, 1032, 2301, 3210$	3	6.4	In

TABLE 1: All coherent relations for $k = 4$

Nr.	$\delta(\varrho)$	$\sigma(\varrho)$	iso	T/L	I/O
132	$\delta^4_{\{0,1\},\{2,3\}}$	$0123, 0132, 1023, 1032, 2301, 2310, 3201, 3210$	3	6.4	In
133	ι^4_4	$\emptyset$	1	7.1	Out
134	ϱ_1	$\emptyset$	1	8.1	In
135	ϱ_2	$\emptyset$	1	8.1	In
136	ϱ_1	$0123(S_4)$	1	8.2	Out
137	ϱ_2	$0123, 1230, 2301, 3012, 2103, 3210, 0321, 1032$	3	8.2	Out

TABLE 1: All coherent relations for $k = 4$