Counting the maximal partial clones on a finite set

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We show that different coherent relations specify different maximal partial clones. Then we describe a computer program to find all coherent relations and thus all maximal partial clones on 4-element, 5-element, and 6-element sets.

Key words: completeness theorem; partial functions; maximal partial clones

1 INTRODUCTION

In many-valued logic finite basic sets are considered. We only have to consider the set $E_k := \{0, 1, \ldots, k-1\}$ with $k \geq 3$ being fixed in the rest of this paper.

The set $P_k := \{f^{(n)} \mid f^{(n)} : E_k^n \to E_k, n \geq 1\}$ is the set of all total functions on E_k. Let $D \subseteq E_k^n$, $n \geq 1$ and $f^{(n)} : D \to E_k$. Then f is called an n-ary partial function on E_k with domain D. We also write $\operatorname{dom}(f) = D$. Let $\widetilde{P}_k^{(n)}$ be the set of all n-ary partial functions on E_k and

$$\widetilde{P}_k := \bigcup_{n \geq 1} \widetilde{P}_k^{(n)}.$$

Let $C_\emptyset := \left\{ f \in \widetilde{P}_k \;\middle|\; \operatorname{dom}(f) = \emptyset \right\}$.

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The n-ary function $e_i^{(n)}$ defined by $e_i^{(n)}(x_1,\ldots,x_n) := x_i$ is called the projection onto the i-th coordinate with $i \in \{1,\ldots,n\}$. Let the set of all projections be $J_k := \left\{ e_i^{(n)} \,\middle|\, n \in \mathbb{N}, 1 \leq i \leq n \right\}$.

Let $f[g_1,\ldots,g_n] \in \widetilde{P}_k^{(m)}$ be the *composition* as given in [2] with $f \in \widetilde{P}_k^{(n)}$ and $g_1,\ldots,g_n \in \widetilde{P}_k^{(m)}$, i.e., for any $x \in E_k^n$

$$x \in \operatorname{dom}(f[g_1,\ldots,g_n]) \iff$$
$$\left(x \in \bigcap_{i=1}^{n} \operatorname{dom}(g_i) \right) \wedge (g_1(x),\ldots,g_n(x)) \in \operatorname{dom}(f)$$

and

$$f[g_1,\ldots,g_n](x_1,\ldots,x_m) := f(g_1(x_1,\ldots,x_m),\ldots,g_n(x_1,\ldots,x_m))$$

for all $(x_1,\ldots,x_m) \in \operatorname{dom}(f[g_1,\ldots,g_n])$.

A *partial clone* (*clone*) on E_k is a composition closed subset of $\widetilde{P}_k$ (P_k) containing J_k.

The set of all partial clones on E_k (clones on E_k), ordered by inclusion, forms an algebraic lattice $\mathbb{L}\widetilde{P}_k$ ($\mathbb{L}P_k$), whose smallest element is the set of all projections and greatest element is $\widetilde{P}_k$ (P_k), respectively. A maximal partial clone (a maximal clone) on E_k is a co-atom of $\widetilde{P}_k$ and P_k, respectively. Thus a partial clone (clone) M is a maximal partial clone (maximal clone) if the inclusions $M \subset C \subset \widetilde{P}_K$ ($M \subset C \subset P_k$) hold for no partial clone (hold for no clone) C on E_k.

For $F \subseteq \widetilde{P}_k$ ($F \subseteq P_k$), we denote by $[F]_{\mathrm{P}}$ ($[F]$) the partial clone (clone) on E_k generated by F, i.e., the intersection of all partial clones (clones) containing the set F on E_k. Clearly $[F]_{\mathrm{P}}$ ($[F]$) is the least partial clone (clone) on E_k containing F.

A set F of partial functions (functions) on E_k is *complete* if $[F]_{\mathrm{P}} = \widetilde{P}_k$ and $[F] = P_k$, respectively. It is well known that a set $F \subseteq \widetilde{P}_k$ ($F \subseteq P_k$) is complete if and only if F is contained in no maximal partial clone (maximal clone) on E_k (see, e.g., [6] for the partial case and [9], Theorem 1.5.4.1, for the total case). Therefore maximal clones play a fundamental role for completeness. They are fully described in [12, 13] (see also [14]).

Similarly, maximal partial clones play a very important role for the completeness problem of finite partial algebras. Several descriptions of all maximal partial clones on a finite set can be found in the literature, we refer the reader to [3, 5] for the classification of all maximal partial clones.

In 1913, Sheffer described functions $f \in P_2$ for which every function on E_2 can be expressed in terms of f only. Call a function $f \in P_k$ a Sheffer function if every function on E_k can be obtained by composition from f and the projections. Thus f is a Sheffer function if $[f] = P_k$.

For example Webb [16] has shown that the function $f(x, y) := \min(x, y) + 1 \pmod{k}$ is a Sheffer function for P_k. Sheffer functions have been well studied. We refer the reader to [14] for a list of references on the subject.

Partial Sheffer functions are defined similarly. A partial function f on E_k is a partial Sheffer function if every partial function on E_k can be obtained by composition from f and the projections, i.e., if $[f]_{\mathrm{P}} = \widetilde{P}_k$. On the other hand, very little is known about partial Sheffer functions for $\widetilde{P}_k$, essentially due to the difficulty of the problem. Indeed the family of all maximal partial clones on E_k is far more complex than the family of all maximal clones on E_k. For example, there are 58 maximal partial clones and 18 maximal clones on a 3-element set, and there are 1102 maximal partial clones ([7, 15]; complete list in the appendix) and 82 maximal clones ([9] Theorem 5.4.2) on a 4-element set. Results on partial Sheffer functions can be found in the papers [4, 11] and [2].

The completeness problem for partial Sheffer functions is the question if for a given partial function $f \in \widetilde{P}_k$ the identity $[f]_{\mathrm{P}} = \widetilde{P}_k$ holds. That means, criteria are investigated to decide if a partial function is a partial Sheffer function.

2 DEFINITIONS AND THEOREM OF HADDAD AND ROSENBERG

Relations are useful to describe the clones of $\widetilde{P}_k$. We often write the elements of relations as columns and a relation can then be given as a matrix. For example the relation $\varrho = \{(0, 1, 2), (1, 2, 0), (3, 4, 5), (2, 3, 1)\}$ can also be written as

$$\varrho = \begin{pmatrix} 0 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 2 & 0 & 5 & 1 \end{pmatrix}.$$

Let a matrix be given by $C = (c_{ij})_{h \times n}$. Then c_{i*} is the i-th row of the matrix with $i \in \{1, \ldots, h\}$, i.e., $c_{i*} = (c_{i1}, c_{i2}, \ldots, c_{in})$, and c_{*j} is the j-th column of the matrix with $j \in \{1, \ldots, n\}$, i.e., $c_{*j} = (c_{1j}, c_{2j}, \ldots, c_{hj})^{\mathrm{T}}$.

Let $\mathcal{R}_k^{(h)}$ be the set of all h-ary relations on E_k and $\mathcal{R}_k := \bigcup_{h \geq 1} \mathcal{R}_k^{(h)}$.

An n-ary function $f^{(n)} \in \widetilde{P}_k$ *preserves* an h-ary relation $\varrho^{(h)} \in \mathcal{R}_k$ iff for all $c_{*1}, c_{*2}, \ldots, c_{*n} \in \varrho$ with $c_{1*}, \ldots, c_{h*} \in \operatorname{dom}(f)$ holds

$$f(c_{*1}, \ldots, c_{*n}) := \begin{pmatrix} f(c_{1*}) \\ f(c_{2*}) \\ \vdots \\ f(c_{h*}) \end{pmatrix} := \begin{pmatrix} f(c_{11}, c_{12}, \ldots, c_{1n}) \\ f(c_{21}, c_{22}, \ldots, c_{2n}) \\ \vdots \\ f(c_{h1}, c_{h2}, \ldots, c_{hn}) \end{pmatrix} \in \varrho.$$

Let $\mathrm{pPOL}_k\, \varrho$ be the set of all functions $f \in \widetilde{P}_k$ which preserve the relation $\varrho \in \mathcal{R}_k$.

Let $f \in \widetilde{P}_k^{(1)}$ be a unary function. Define $f^0 := e_1^{(1)}$ and $f^n := f[f^{n-1}]$ for all $n \geq 1$.

For each $m \in \mathbb{N}$ let $\eta_m := (0, 1, \ldots, m-1)^{\mathrm{T}}$.

Define $\omega(v)$ to be the set of entries of any $v = (v_1, \ldots, v_h) \in E_k^h$, i.e., $\omega(v) = \omega((v_1, \ldots, v_h)) := \{v_1, \ldots, v_h\}$. Additionally let $\omega(\varrho) = \bigcup_{v \in \varrho} \omega(v)$.

Definition 2.1. Let for all h with $1 \leq h \leq k$

$$\begin{aligned} \varrho_1 &:= \left\{(a, a, b, b), (a, b, a, b) \mid a, b \in E_k\right\}, \\ \varrho_2 &:= \left\{(a, a, b, b), (a, b, a, b), (a, b, b, a) \mid a, b \in E_k\right\}, \\ \iota_k^h &:= \left\{(x_1, \ldots, x_h) \in E_k^h \;\middle|\; |\{x_1, \ldots, x_h\}| \leq h-1\right\}. \end{aligned}$$

Definition 2.2. Let ε be an arbitrary equivalence relation on E_h. Define $\delta_{k,\varepsilon}^{(h)} := \left\{(a_0, \ldots, a_{h-1}) \in E_k^h \;\middle|\; (i,j) \in \varepsilon \Longrightarrow a_i = a_j\right\}$. If h or k can be deduced from the context we just write δ_ε or $\delta_\varepsilon^{(h)}$ or $\delta_{k,\varepsilon}$. If the relation ε is given by the non-singular equivalence classes $\varepsilon_1, \ldots, \varepsilon_r$ then we write $\delta_{k;\varepsilon_1,\ldots,\varepsilon_r}^{(h)}$ or $\delta_{\varepsilon_1,\ldots,\varepsilon_r}$ instead of $\delta_{k,\varepsilon}^{(h)}$. For example if ε has only the equivalence class E_h then $\delta_{k;E_h}^{(h)} = \left\{(x, x, \ldots, x) \in E_k^h \;\middle|\; x \in E_k\right\}$.

Definition 2.3. Let $\varrho^{(h)} \subseteq E_k^h$. Then we write $\sigma(\varrho) := \varrho \setminus \iota_k^h$ and $\delta(\varrho) := \varrho \cap \iota_k^h = \varrho \setminus \sigma(\varrho)$. If $\delta = \delta_\varepsilon$ for some equivalence relation ε then we write $\varepsilon(\varrho) := \varepsilon$.

Definition 2.4. Let $\varrho^{(h)} \subseteq E_k^h$. Then ϱ is

- *areflexive*, if $h \geq 2$ and $\delta(\varrho) = \emptyset$, i.e., for each $(x_1, \ldots, x_h) \in \varrho$ we have $x_i \neq x_j$ for all $1 \leq i < j \leq h$.

- *quasi-diagonal*, if $\sigma(\varrho)$ is a non-empty relation, $\delta(\varrho) = \delta_\varepsilon$ with $\varepsilon \neq \iota_h^2$ an equivalence relation.

Definition 2.5. Let $\varrho^{(h)} \subseteq E_k^h$, $\sigma := \sigma(\varrho)$ and $\delta := \delta(\varrho)$.

If $r = (r_0, r_1, \ldots, r_{n-1}) \in E_k^n$ is a tuple and $\pi \in S_n$ then we write $r^{[\pi]} := (r_{\pi(0)}, r_{\pi(1)}, \ldots, r_{\pi(n-1)})$. Let $\Gamma_\sigma := \{\pi \in S_h \mid \sigma \cap \sigma^{[\pi]} \neq \emptyset\}$, where S_h is the symmetric group on E_h and $\sigma^{[\pi]} := \{s^{[\pi]} \mid s \in \sigma\}$.

The *model* of ϱ is the h-ary relation

$$M(\varrho) := \left\{\eta_h^{[\pi]} \,\middle|\, \pi \in \Gamma_\sigma\right\} \cup (\delta \cap E_h^h)$$

on E_h.

The relation ϱ is *coherent*, if the following conditions hold:

1. $\varrho \neq E_k^h$, $\varrho \neq \emptyset$,
2. (a) ϱ is a unary relation, i.e., $h = 1$, or
   (b) ϱ is areflexive with $2 \leq h \leq k$, or
   (c) ϱ is quasi-diagonal with $2 \leq h \leq k$, or
   (d) $\delta = \iota_k^h$ with $3 \leq h \leq k$, or
   (e) $\delta = \varrho_i$ with $i \in \{1, 2\}$ (see Definition 2.1) and $h = 4$,
3. $r^{[\pi]} \in \sigma$ for all $r \in \sigma$ and all $\pi \in \Gamma_\sigma$,
4. for every σ' with $\emptyset \neq \sigma' \subseteq \sigma$ there is a relational homomorphism $\varphi : E_k \to E_h$ mapping σ' into $M(\varrho)$, such that $\varphi(r) = \eta_h$ for some $r \in \sigma'$, i.e., $(\varphi(r_0), \ldots, \varphi(r_{h-1})) = (0, \ldots, h-1)$ for some $r = (r_0, \ldots, r_{h-1}) \in \sigma'$,
5. (a) if $\delta = \iota_k^h$ and $h \geq 3$ then $\Gamma_\sigma = S_h$,
   (b) if $\delta = \varrho_1$ then $\Gamma_\sigma = \langle(0231), (12)\rangle$ (Γ_σ is the permutation group generated by the cycles (0231) and (12)),
   (c) if $\delta = \varrho_2$ then $\Gamma_\sigma = S_4$.

Let $\widetilde{\mathcal{R}}_k^{\max}$ be the set of all coherent relations such that for each coherent relation $\varrho^{(h)}$ exactly one relation of the set $\{\varrho^{[\pi]} \mid \pi \in S_h\}$ occurs in $\widetilde{\mathcal{R}}_k^{\max}$. Let

$$p\mathscr{M}_k := \{P_k \cup C_\emptyset\} \cup \left\{\mathrm{pPOL}_k\, \varrho \,\middle|\, \varrho \in \widetilde{\mathcal{R}}_k^{\max}\right\}.$$

Theorem 2.6 (Haddad and Rosenberg; [3, 5]). *Let $k \geq 2$. For each $A \subset \widetilde{P}_k$ with $A = [A]_{\mathrm{P}}$ there is a maximal partial clone M_A with $A \subseteq M_A$. A clone M is a maximal partial clone of $\widetilde{P}_k$ if and only if $M \in p\mathscr{M}_k$, i.e., $p\mathscr{M}_k$ is the set of all maximal partial clones of $\widetilde{P}_k$.*

Theorem 2.7 (Completeness criterion for $\widetilde{P}_k$; [5]). *Let $C \subseteq \widetilde{P}_k$. Then $[C]_{\mathrm{P}} = \widetilde{P}_k$ if and only if $C \nsubseteq M$ for all $M \in \mathrm{p}\mathscr{M}_k$.*

Definition 2.8. The set of coherent relations $\widetilde{\mathcal{R}}_k^{\max}$ can be divided into the following sets:

$$\begin{aligned}
\mathcal{U} &:= \{\chi^{(\mu)} \in \widetilde{\mathcal{R}}_k^{\max} \mid \mu = 1\}, \\
\mathcal{A} &:= \{\chi^{(\mu)} \in \widetilde{\mathcal{R}}_k^{\max} \mid \mu \geq 2 \wedge \chi \text{ is areflexive}\}, \\
\mathcal{Q} &:= \{\chi^{(\mu)} \in \widetilde{\mathcal{R}}_k^{\max} \mid \mu \geq 2 \wedge \chi \text{ is quasi-diagonal}\}, \\
\mathcal{S} &:= \{\chi^{(\mu)} \in \widetilde{\mathcal{R}}_k^{\max} \mid \mu \geq 3 \wedge \delta(\chi) = \iota_k^{\mu}\}, \\
\mathcal{L} &:= \{\chi^{(\mu)} \in \widetilde{\mathcal{R}}_k^{\max} \mid \mu = 4 \wedge \delta(\chi) \in \{\varrho_1, \varrho_2\}\}.
\end{aligned}$$

3 DIFFERENT RELATIONS – DIFFERENT CLONES

Lemma 3.1. *Let ϱ and χ be two h-ary relations and $\pi \in S_h$ a permutation with $\varrho^{[\pi]} = \chi$. Then $\mathrm{pPOL}_k\, \varrho = \mathrm{pPOL}_k\, \chi$.*

Proof. It suffices to show

$$\mathrm{pPOL}_k\, \varrho \supseteq \mathrm{pPOL}_k\, \chi.$$

The other direction is symmetric if ϱ and χ are exchanged and π^{-1} instead of π is used.

Let $f \in \mathrm{pPOL}_k\, \chi$ and $r_{*1}, r_{*2}, \ldots, r_{*n} \in \varrho$. Then

$$\forall i \in \{1, \ldots, n\} : (r_{*i})^{[\pi]} \in \chi$$

holds and thus

$$f\left((r_{*1})^{[\pi]}, (r_{*2})^{[\pi]}, \ldots, (r_{*n})^{[\pi]}\right) \in \chi = \varrho^{[\pi]}.$$

Then

$$\begin{aligned}
f(r_{*1}, r_{*2}, \ldots, r_{*n}) &= \left(f\left((r_{*1})^{[\pi]}, (r_{*2})^{[\pi]}, \ldots, (r_{*n})^{[\pi]}\right)\right)^{[\pi^{-1}]} \\
&\in \chi^{[\pi^{-1}]} = \left(\varrho^{[\pi]}\right)^{[\pi^{-1}]} = \varrho
\end{aligned}$$

follows and we have $f \in \mathrm{pPOL}_k\, \varrho$. □

Thus we call two coherent relations $\varrho^{(h)}$ and $\chi^{(h)}$ *equivalent*, if $\pi \in S_h$ with

$$\varrho^{[\pi]} = \chi$$

exists. If two clones resp. the corresponding relations are compared, it suffices to choose one representative of each equivalence class. Let $\varrho^{(h)}$ and $\chi^{(h)}$ be two h-ary relations. If $\varrho \cap \chi$ is considered, then let χ be one of the relations $\chi^{[\pi]}$ with $\pi \in S_h$, such that $|\varrho \cap \chi^{[\pi]}|$ is maximal. Thus χ is chosen such that

$$\chi \in \left\{ \chi' \in S \;\middle|\; |\varrho \cap \chi'| = \max_{\chi'' \in S} |\varrho \cap \chi''| \right\}$$

holds with

$$S = \left\{ \chi^{[\pi]} \;\middle|\; \pi \in S_h \right\}.$$

Let $W := \{\varrho_1, \varrho_2\} \cup \left\{ \iota_k^h \;\middle|\; 3 \leq h \leq k \right\}$.

Lemma 3.2. *Let $\varrho^{(h)}$ and $\chi^{(\mu)}$ be coherent relations on E_k with $\varrho \neq \chi$ and $\varrho, \chi \notin W$. Then $\mathrm{pPOL}_k\, \varrho \neq \mathrm{pPOL}_k\, \chi$ holds.*

Proof. We consider three cases:

- $\mu < h$: Let $s = (s_1, \ldots, s_\mu) \in \chi \setminus \iota_k^\mu$ and $v = (v_1, \ldots, v_\mu) \in E_k^\mu \setminus \chi$ be arbitrary. Then define a function $f^{(1)} : E_k \to E_k$ by $f(s_i) := v_i$ for all $i \in \{0, \ldots, \mu - 1\}$ and $\mathrm{dom}(f) := \{s_0, \ldots, s_{\mu-1}\}$. Then f is well-defined, because $s_i \neq s_j$ holds for $i \neq j$. Additionally $f \notin \mathrm{pPOL}_k\, \chi$ follows from $f(s) = v \notin \chi$ and $s \in \chi$.

  Assume $f \notin \mathrm{pPOL}_k\, \varrho$. Then there is an $r \in \varrho$ with $f(r) \notin \varrho$. But f is defined only at μ different values, thus r can only have μ different entries. That means, we have $r \in \delta_\varepsilon$ with ε equivalence relation on E_h and $\varepsilon \neq \iota_h^2$. Then follows $\delta_\varepsilon \subset \varrho$ by definition of a coherent relation. Then $r_i = r_j \implies f(r_i) = f(r_j)$ implies $f(r) \in \delta_\varepsilon \subseteq \varrho$, in contradiction to the assumption, i.e., $f \in \mathrm{pPOL}_k\, \varrho$.

- $\mu > h$. Because the case $\mu > h$ is symmetrical to the case above we have $\mathrm{pPOL}_k\, \varrho \neq \mathrm{pPOL}_k\, \chi$ for $\mu \neq h$.

- $\mu = h$. Let the rows of χ be permuted such that $|\varrho \cap \chi|$ becomes maximal, as motivated by Lemma 3.1. Then the following cases need to be checked.

  - $\varrho \cap \chi = \emptyset$. Let $s \in \chi \setminus \iota_k^h$ and $v \in E_k^h \setminus \chi$ be arbitrary. Then define a function $f^{(1)} : E_k \to E_k$ by $f(s_i) := v_i$ for all $i \in \{0, \ldots, \mu - 1\}$ and $\mathrm{dom}(f) := \{s_0, \ldots, s_{\mu-1}\}$. Then f is well-defined because $s_i \neq s_j$ holds for $i \neq j$. Additionally we have $f \notin \mathrm{pPOL}_k\, \chi$ because $f(s) = v \notin \chi$ and $s \in \chi$.

Assume $f \notin \mathrm{pPOL}_k\,\varrho$. Then there is $r \in \varrho$ with $f(r) \notin \varrho$. The tuple r has less than h different entries. Otherwise r would be a permutation of s and thus there would be a permutation of rows of χ such that $\varrho \cap \chi \neq \emptyset$. But then a contradiction follows as in the case $\mu < h$ above, i.e., $\mathrm{pPOL}_k\,\varrho \neq \mathrm{pPOL}_k\,\chi$ holds.

– $\varrho \cap \chi \neq \emptyset, \varrho \not\subset \chi$. Let $s \in \chi \setminus \iota_k^h$ and $v \in \varrho \setminus \chi$ be arbitrary. Define a function $f^{(1)} : E_k \to E_k$ by $f(s_i) := v_i$ for all $i \in \{0, \ldots, \mu - 1\}$ and $\mathrm{dom}(f) := \{s_0, \ldots, s_{\mu-1}\}$. Then f is well-defined, because $s_i \neq s_j$ holds for $i \neq j$. Additionally we have $f \notin \mathrm{pPOL}_k\,\chi$ because $f(s) = v \notin \chi$ and $s \in \chi$.

  Assume $f \notin \mathrm{pPOL}_k\,\varrho$. Then there is $r \in \varrho$ with $f(r) \notin \varrho$. If r has less than h different entries then a contradiction follows as in the case $\mu < h$ above. Thus r has exactly h different entries, i.e., $r = s^{[\alpha]}$ holds for some $\alpha \in \Gamma_\varrho$. Because ϱ is coherent we also have

$$f\left(r\right) = f\left(s^{[\alpha]}\right) = v^{[\alpha]} \in \varrho.$$

  This is a contradiction. Thus $f \in \mathrm{pPOL}_k\,\varrho$ holds and this implies $\mathrm{pPOL}_k\,\varrho \neq \mathrm{pPOL}_k\,\chi$.

– The case $\varrho \subset \chi$ is the same as above if the roles of χ and ϱ are exchanged. □

Lemma 3.3. *Let $\varrho^{(h)} \in W$ and $\chi^{(\mu)} \notin W$ be coherent relations. Then* $\mathrm{pPOL}_k\,\varrho \neq \mathrm{pPOL}_k\,\chi$ *holds.*

Proof. Let $s \in \chi \setminus \iota_k^\mu$ and $v \in E_k^\mu \setminus \chi$ be arbitrary. Define a function $f^{(1)} : E_k \to E_k$ by $f(s_i) := v_i$ for all $i \in \{0, \ldots, \mu - 1\}$ and $\mathrm{dom}(f) := \{s_0, \ldots, s_{\mu-1}\}$. Then f is well-defined because $s_i \neq s_j$ holds for $i \neq j$. Additionally we have $f \notin \mathrm{pPOL}_k\,\chi$ because $f(s) = v \notin \chi$ and $s \in \chi$.

Assume $f \notin \mathrm{pPOL}_k\,\varrho$. Then there is some $r \in \varrho$ with $f(r) \notin \varrho$. From $\varrho = \bigcup_{\varepsilon \in E} \delta_\varepsilon$ with E a set of equivalence relations on E_h follows $r \in \delta_\varepsilon$ for some $\varepsilon \neq \iota_h^2$. This implies $\delta_\varepsilon \subset \varrho$. From $r_i = r_j \Longrightarrow f(r_i) = f(r_j)$ follows $f(r) \in \delta_\varepsilon \subseteq \varrho$, in contradiction to the assumption. Thus $f \in \mathrm{pPOL}_k\,\varrho$. □

Lemma 3.4. *Let* $3 \leq \mu < h \leq k$. *Then* $\mathrm{pPOL}_k\,\iota_k^\mu \neq \mathrm{pPOL}_k\,\iota_k^h$ *holds.*

Proof. Define the function $f^{(h)}$ by

x_{*1}	x_{*2}	x_{*3}	$\dots$	x_{*h}	$f(x)$
0	0	0	$\dots$	0	0
0	1	1	$\dots$	1	1
1	1	2	$\dots$	2	2
2	2	2	$\dots$	3	3
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$h-3$	$h-3$	$h-3$	$\dots$	$h-2$	$h-2$
$h-2$	$h-2$	$h-2$	$\dots$	$h-2$	$h-1$
otherwise					undefined

We have $f \notin \mathrm{pPOL}_k\, \iota_k^h$, because the columns x_{*1} to x_{*h} are in ι_k^h, but $f(x)$ is not.

Assume $f \notin \mathrm{pPOL}_k\, \iota_k^\mu$. Then there are columns $c_{*1}, \dots, c_{*h} \in \iota_k^\mu$ with $f(c_{*1}, \dots, c_{*h}) =: d \notin \iota_k^\mu$. If the rows $c_{1*}, \dots, c_{\mu *}$ all different then there is a column c_{*j} with $|\omega(c_{*j})| = \mu$, i.e., c_{*j} has μ different elements, in contradiction to $c_{*j} \in \iota_k^\mu$. Thus two rows are equal, i.e., two entries of d are equal in contradiction to $d \notin \iota_k^\mu$. Thus $f \in \mathrm{pPOL}_k\, \iota_k^\mu$. $\square$

Lemma 3.5. *Let* $\varrho = \varrho_i$ *for some* $i \in \{1,2\}$ *and* $\chi = \iota_k^\mu$ *for some* $\mu \in \{3,4,\dots,k\}$. *Then* $\mathrm{pPOL}_k\, \varrho \neq \mathrm{pPOL}_k\, \chi$.

Proof. Define the function $f^{(2)}$ by

$$f\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and not defined otherwise. Then $f \notin \mathrm{pPOL}_k\, \varrho$ because

$$(0,0,1,1), (0,1,0,1) \in \varrho$$

but $(0,0,0,1) \notin \varrho$. Because $|\{0,1\}| = 2$ we get $f \in \mathrm{pPOL}_k\, \chi$. $\square$

Lemma 3.6. $\mathrm{pPOL}_k\, \varrho_1 \neq \mathrm{pPOL}_k\, \varrho_2$.

Proof. Define the function $f^{(3)}$ by

$$f\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and not defined otherwise. Then $f \notin \mathrm{pPOL}_k\,\varrho_2$ because

$$(0,0,1,1),(0,1,0,1),(0,1,1,0) \in \varrho_2$$

but $(0,0,0,1) \notin \varrho_2$. Because $\{(0,0,1,1),(0,1,0,1),(0,1,1,0)\} \not\subseteq \varrho_1^{[\pi]}$ for all $\pi \in S_4$ we have $f \in \mathrm{pPOL}_k\,\varrho_1$. $\square$

Theorem 3.7. *The relations in $\widetilde{\mathcal{R}}_k^{\max}$ describe distinct maximal partial clones.*

Proof. Combining the Lemmas 3.2, 3.3, 3.4, 3.5 and 3.6 implies the theorem. $\square$

4 SHORTCUTS FOR FINDING COHERENT RELATIONS

We want to find all coherent relations $\widetilde{\mathcal{R}}_k^{\max}$, and determine $|p\mathscr{M}_k|$. The trivial approach is to enumerate all subsets of E_k^h for $h \in \{1,\ldots,k\}$, check if they are coherent and remove duplicates with respect to equivalence of the relations, see page 6. But this approach needs too much memory and time because all coherent relations have to be in memory at the same time, all have to be checked for coherence, and relations generating the same clone have to be identified. Thus we need to use a better method.

Definition 4.1. Let $v := (v_1,\ldots,v_h) \in E_k^h$, $w := (w_1,\ldots,w_h) \in E_k^h$. We write $v \prec w$ if

$$\exists i \in \{1,2,\ldots,h\} : (\forall j < i : v_j = w_j) \wedge v_i < w_i.$$

Let $v \preceq w$ if $v \prec w$ or $v = w$. Then $\preceq$ defines a lexicographical order on E_k^h.

Let $V = \{V_1,\ldots,V_m\} \subseteq E_k^h$ and $W = \{W_1,\ldots,W_n\} \subseteq E_k^h$ with $V_i \prec V_j$ and $W_i \prec W_j$ for all $i < j$. We write $V \prec W$ if

$$\exists i \in \{1,2,\ldots,\min(m,n)\} : (\forall j < i : V_j = W_j) \wedge V_i < W_i,$$

or

$$m < n \wedge (\forall i \in \{1,2\ldots,m\} : V_i = W_i).$$

Let $V \preceq W$ if $V \prec W$ or $V = W$. Thus $\preceq$ defines a lexicographical order on the subsets of E_k^h.

Let $\min_\prec V := x \in V$ with $x \preceq y$ for all $y \in V$.

Definition 4.2. Let $\varrho^{(h)} \in \widetilde{\mathcal{R}}_k^{\max}$.

Define the *quasi-relation-class* $\operatorname{qclass}(\varrho)$ by

$$\operatorname{qclass}(\varrho) := \{\{f(v) \mid v \in \varrho\} \mid f \in S_k\}$$

and we call ϱ a quasi-minimal relation if $\varrho = \min_\prec \operatorname{qclass}(\varrho)$.

Define the *relation-class* $\operatorname{class}(\varrho)$ by

$$\operatorname{class}(\varrho) := \{\{(f(v))^{[\pi]} \mid v \in \varrho\} \mid f \in S_k, \pi \in \Gamma(\varrho)\}$$

where

$$\Gamma(\varrho) := \{\pi \in S_h \mid \pi(\varepsilon(\varrho)) = \varepsilon(\varrho)\},$$

i.e., all permutations which leave $\delta(\varrho)$ fixed. We call ϱ a minimal relation if $\varrho = \min_\prec \operatorname{class}(\varrho)$.

Every minimal relation is a quasi-minimal relation because $\operatorname{qclass}(\varrho) \subseteq \operatorname{class}(\varrho)$.

The general idea to find all coherent relations $\varrho^{(h)}$ in reasonable time is to

- traverse over all model $M(\varrho)$ with $\varepsilon(\varrho)$ given in such a way, that all equivalence classes are continuous intervals on E_h and sorted in decreasing order of size. The areflexive relations are represented by $\varepsilon(\varrho) = \emptyset$. The relations in $\mathcal{S}$ are treated in a special way and no equivalence relation is given.

- For a given model $M(\varrho)$ check all relations χ with $M(\chi) = M(\varrho)$, i.e., traverse over all $\chi \subset E_k^h$ with $M(\chi) = M(\varrho)$. Let

  $$A_\varrho := \left\{ \min_\prec \left\{ v^{[\pi]} \,\middle|\, \pi \in \Gamma_{\sigma(\varrho)} \right\} \,\middle|\, v \in \sigma(E_k^h) \right\},$$

  the set of all areflexive h-ary tuples on E_k modulo the group associated with the model $M(\varrho)$. Then any subset of A_ϱ uniquely identifies a relation with the given model $M(\varrho)$. The subsets are determined recursively starting at the empty set and adding tuples $v_1, v_2, \dots$ from A_ϱ such that $v_i \prec v_j$ for all $i < j$. Thus every subset of A_ϱ is given exactly once and we can make cut offs with the following two statements.

- We only print a relation ϱ if it is a minimal relation in the sense given above, and the size

  $$|\{\operatorname{pPOL}_k \chi \mid \chi \in \operatorname{class}(\varrho)\}| = |\{\min_\prec \operatorname{qclass}(\varrho^{[\pi]}) \mid \pi \in \Gamma(\varrho)\}|.$$

  Because there is exactly one minimal relation in each $\operatorname{class}(\varrho)$ we can deduce the number of all coherent relations.

Lemma 4.3. *Let $\varrho \in \widetilde{\mathcal{R}}_k^{\max}$. Let $\chi \subset \varrho$ with*

(1) $\emptyset \subset \sigma(\chi) \subset \sigma(\varrho)$,

(2) $M(\chi) = M(\varrho)$, and

(3) $\forall \pi \in \Gamma_{\sigma(\chi)}\, \forall v \in \sigma(\chi) : v^{[\pi]} \in \sigma(\chi)$.

Then $\chi \in \widetilde{\mathcal{R}}_k^{\max}$.

Proof. Because of (1) the relation χ is non-trivial. Because of (3) we only have to show, that for every $\emptyset \subset \sigma' \subseteq \sigma(\chi)$ there is a relational homomorphism $\varphi : E_k \to E_h$ from σ' to $M(\chi)$, such that $\varphi(r) = \eta_h$ for some $r \in \sigma'$, i.e., $(\varphi(r_0), \dots, \varphi(r_{h-1})) = (0, \dots, h-1)$ for some $r := (r_0, \dots, r_{h-1}) \in \sigma'$. But this follows from $\sigma' \subseteq \sigma(\chi) \subset \sigma(\varrho)$, (2) and ϱ coherent. Thus χ is coherent. □

Lemma 4.3 for a fixed $M(\varrho)$ implies that all $\chi' \notin \widetilde{\mathcal{R}}_k^{\max}$ with $\chi \notin \widetilde{\mathcal{R}}_k^{\max}$, $\chi' \supset \chi$ and $M(\chi') = M(\chi)$, i.e., we can cut off the subset tree early.

Lemma 4.4. *Let $\varrho, \chi \in \widetilde{\mathcal{R}}_k^{\max}$. Let $\chi \subset \varrho$ and $\chi \prec \varrho$ with*

(1) $\emptyset \subset \sigma(\chi) \subset \sigma(\varrho)$,

(2) $M(\chi) = M(\varrho)$, and

(3) ϱ is a quasi-minimal relation.

Then χ is a quasi-minimal relation.

Proof. Let $\varrho =: \{v_1, \dots, v_{|\varrho|}\}$ with $v_i \prec v_j$ for all $i < j$. Then $\chi = \{v_1, \dots, v_{|\chi|}\}$ because $\chi \prec \varrho$ and $\chi \subset \varrho$.

Assume χ is not a quasi-minimal relation. Then there is some $\chi' \in \mathrm{qclass}(\chi)$ with $\chi' \prec \chi$, i.e., there is some $f \in S_k$ with $f(\chi) \prec \chi$. Thus there are $a \in \{1, \dots, |\chi| - 1\}$ and $w \in f(\chi)$ with $v_{a-1} \prec w \prec v_a$ and $\{v_1, \dots, v_{a-1}\} \subseteq f(\chi)$. Then $\{v_1, \dots v_{a-1}, w\} \subseteq f(\varrho)$ implying $f(\varrho) \prec \varrho$, i.e., ϱ is not a quasi-minimal relation, in contradiction to the assumption. □

Lemma 4.4 for a fixed $M(\varrho)$ implies that all χ' are not quasi-minimal and thus not minimal if χ is not quasi-minimal, $\chi' \supset \chi$ and $M(\chi') = M(\chi)$. Thus we can cut off the subset tree early, if we encounter a coherent relation which is not quasi-minimal.

We can not cut off the subtree if we encounter a relation which is not a minimal relation. For example

$$\varrho := \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 2 \end{pmatrix}$$

is quasi-minimal, but not minimal because

$$\begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 2 & 1 \end{pmatrix} = \min_{\prec} \operatorname{class}(\varrho).$$

On the other hand

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 3 \end{pmatrix}$$

is minimal and is found in a subtree starting with ϱ.

5 CONCLUSION

The lists of all coherent relations for $k \in \{2, 3, 4, 5, 6\}$ as given by the program and the source code of the program can be found at
`http://www.math.uni-rostock.de/~schoelzel/papers/numbers/`
The program is written in Haskell, is single-threaded and needed about 52 hours on a SunFire V490 to compute all coherent relations for $k = 6$. The number of coherent relations is given in the following tables. The numbers agree with the ones given without computer in [1] for $k = 2$, [8] and [10] (independent from each other) for $k = 3$, and [7] with corrections in [15] for $k = 4$. As the number of maximal partial clones grows fast it does not seem feasible nor useful to find all coherent relations for any $k > 6$.

k	$\lvert p\mathscr{M}_k \rvert$	$\lvert\mathcal{S}\rvert$				$\lvert\mathcal{L}\rvert$		$P_k \cup C_\emptyset$
		3	4	5	6			
2	8					1	1	1
3	58	1				1	1	1
4	1102	15	1			4	2	1
5	325722	1023	31	1		46	16	1
6	5242621816	1048575	32767	63	1	4141	786	1

k	$\lvert\mathcal{Q} \cup \mathcal{A}\rvert$					
	1	2	3	4	5	6
2	2	3				
3	6	30	18			
4	14	416	505	144		
5	30	16457	295080	11945	1092	
6	62	1934514	5008589703	230676900	319722	14581

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