

Clones of Partial Functions on Finite Sets

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Karsten Schölzel, geb. am 08.06.1982 in Grevesmühlen

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Chapter 1

Introduction

This thesis covers parts of the theory of multiple-valued logics which belongs to discrete mathematics as well as Universal Algebra. Multiple-valued logics deals with functions with finite domains and finite images, and relations on finite sets.

These two topics are strongly connected and it has proven fruitful to use the theory on one of them to proceed on the other one. We have to state more accurately what are the objects we deal with and what are the questions connected to them we want to answer.

Functions and Clones

First consider a finite set E . An n -ary *function* (or *total function*) on E is the map $f : E^n \rightarrow E$ of the n -th cartesian power E^n into the set E with $n \in \{1, 2, \dots\}$. If we do not know the definition of the function at every point of E^n we call it a *partial function* on E . These can be either considered as functions from some set $A \subseteq E^n$ into E or as functions from E^n into \tilde{E} where \tilde{E} has exactly one extra element with respect to E . We choose $\infty \in \tilde{E} \setminus E$ as the extra element, but also any other symbol could be taken.

Functions on arbitrary sets especially finite sets started to be used in mathematics first in the second half of the 19th century; prior to that the word function was only used for real or complex valued functions.

One of the first examples of functions on finite sets were the *Functions of the Algebra of Logic*, which appeared in works of *G. Boole* (1815–1864), *A. de Morgan* (1801–1871), *G. Frege* (1848–1925), and others in the construction of propositional logic and which are known as *Boolean functions* (or *functions of 2-valued logic*).

Boolean Functions

Boolean functions are n -ary functions on a 2-element set with $n \geq 1$ whose elements are often called truth values denoted by 0 (false) and 1 (true), i.e., $E = \{0, 1\}$.

In propositional logic we take variables, e.g. x_1, \dots, x_n , which can take truth values (i.e., 0 or 1) for some concrete statements. From them we build up some compound statements with the help of elementary logic functions which correspond to words like *and*, *or*, *if ... then ...*, *... if and only if ...* or *not ...*: $x_1 \wedge x_2$, $x_1 \vee x_2$, $x_1 \Rightarrow x_2$, $x_1 \Leftrightarrow x_2$, $\neg x_1$. By iterative application one obtains more complex statements, e.g. $\neg(x_1 \wedge x_2)$, $(\neg(x_1 \vee x_2)) \Rightarrow (\neg x_1)$,

or $((x_1 \wedge x_2) \vee (x_2 \wedge x_3)) \Rightarrow (x_1 \wedge x_3)$. These statements are nothing else than Boolean functions which can be characterized by their truth tables:

x_1	x_2	$\neg x_1$	$x_1 \wedge x_2$	$x_1 \vee x_2$	$\neg(x_1 \wedge x_2)$	$\neg(x_1 \vee x_2) \Rightarrow (\neg x_1)$
0	0	1	0	0	1	1
0	1	1	0	1	1	1
1	0	0	0	1	1	1
1	1	0	1	1	0	1

Every Boolean function $f : \{0, 1\}^n \rightarrow \{0, 1\}$ can be given by a truth table, e.g. if $n = 3$:

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	1

One problem arises here: can we express every Boolean function in terms of the elementary logic functions? The answer is long known and yes. For example the f above can be written as $((x_1 \wedge x_2) \vee (x_2 \wedge x_3)) \Rightarrow (x_1 \wedge x_3)$.

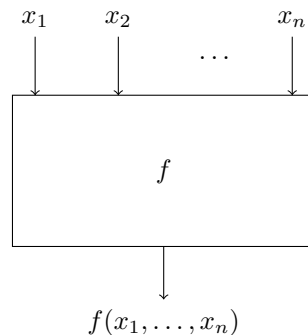
In general this translates to the *problem of functional completeness*:

Let X be a set of Boolean functions. Which conditions have to be fulfilled such that we can express all Boolean functions only in terms of functions from X . This problem was first solved by E.L. Post [47, 48]. Given a solution to this problem does not imply any construction how to express a function in terms of the basic functions. This is the *problem of synthesis*.

Other problems include the search for some

- *normal form*, i.e., given a function find a unique expression in terms of the basic functions which is in a particular form;
- *optimal form*, i.e., an expression with the least possible number of basic functions.

An important application of Boolean functions consists of the description of technical systems, especially electronic digital circuits. Such a circuit can be seen as a simple information processing system which gets values from $E = \{0, 1\}$ on its inputs and emits a value from $\{0, 1\}$ as output. These systems are modeled to have no time delay and no memory.



In the case of partial functions as considered in this thesis systems are modeled in such a way

- that there might be no signals at some inputs resulting in no output signal; and
- that for some combinations of the inputs there is no output signal.

We note explicitly that the first point states that the circuit f depends on all its inputs x_1, \dots, x_n if f represents a partial function, since it depends on the presence of a signal at all inputs although it might not depend on the actual value of some of its inputs.

Clones

Now we lift the restriction of only two values and consider an arbitrary finite set E . This might be useful to model combinatorial systems, information transmission systems or biological systems, under the assumption they can be described as a simple information processing system.

Let P_E be the set of all functions on the set E with finitely many inputs. We want to know which functions are expressible by some basic functions. If we have two functions we are allowed to do the following

- change the order of inputs (*exchange of variables*);
- connect inputs such that they always get the same signal (*identification of variables*);
- connect the output of one function with some input of another function (*composition*);
- add a fictitious input to a function (*addition fictitious variables*).

If a subset $C \subseteq P_E$ is closed with respect to the operations above and contains the projections (some special functions) it is called a *clone*. A more detailed and exact definition follows in Chapter 2.

Total Functions and Total Clones

As stated before E.L. Post solved the completeness problem for the Boolean functions [47, 48] by obtaining the maximal clones in the lattice of clones of Boolean functions, also known as lattice of Boolean clones. In fact he determined the complete lattice of Boolean clones and showed that there are countably many Boolean clones. See Figure 8.1 on page 79 for an impression of how the lattice looks like.

In 1958 Jablonski determined the maximal clones in three-valued logic and thus solving the completeness problem for three-valued functions [32]. Less than a decade later Rosenberg solved the general case for k -valued logic for every $k \geq 2$ by his completeness theorem [55, 56].

The books of Pöschel and Kaluzhnin [49] and Lau [40] cover a wide area of clone theory including some related topics. They both work very well as introduction to clone theory and the second book even covers partial clones in some detail. The papers by Rosenberg [60] and Hurst [31] present a good overview on clones and their applications.

Partial Functions and Partial Clones

One of the first results on partial functions was given by Freivald in 1966 [13] who solved the completeness problem for the Boolean partial functions by determining the maximal partial

Boolean clones. Independently Lau [37] and Romov [51] solved the problem for three-valued functions. In 1988 [14, 22, 24] Haddad and Rosenberg solved the completeness problem for partial clones in k -valued logic and described the maximal partial clones. This description is one of the foundations this thesis is build upon.

We note that there are continuum many partial clones in k -valued logic for all $k \geq 2$. Especially the Boolean case shows that the partial logic is inherently more difficult since there are only countably many total Boolean clones.

Since every chapter contains its own introduction to the topic of the chapter we do not write about these topics here. Some topics which are not covered in this thesis are e.g.

- minimal partial clones; see e.g. [7];
- what is the join or intersection of some clones? See e.g. [10, 20];
- let two clones and their corresponding relations be given (see Chapter 3). What is the connection between the two clones on one side and the clone described by the concatenation of the two relations on the other side? See e.g. [25];
- commutation theory; see e.g. [21].

An Application: Constraint Satisfaction Problems

There are several applications of partial clones especially in computer science, some already mentioned in the introduction of functions and clones. We show one further application to complexity theory here.

A *constraint satisfaction problem (CSP)* is a certain kind of combinatorial decision problem (see e.g. [3] for a definition). A well-known example is the *Satisfiability*, especially in the Boolean case:

Given a set of variables x_1, \dots, x_q and a term, e.g. $((x_1 \wedge x_2) \vee (x_2 \wedge x_3)) \Rightarrow (x_1 \wedge x_3)$, is it possible to assign truth-values to the variables such that the whole statement is true?

Many problems in computer-science can be expressed as CSPs. The general CSP is known to be NP-complete but some restrictions are tractable and it is interesting to know when this is the case. For a given set of relations Γ which define a specific CSP called $\text{CSP}(\Gamma)$ we can assign a total clone $\text{Pol}(\Gamma)$ and a partial clone $\text{pPol}(\Gamma)$; see Chapter 3.

If Γ_1 and Γ_2 are sets of relations, Γ_1 is finite and $\text{Pol}(\Gamma_2) \subseteq \text{Pol}(\Gamma_1)$, then $\text{CSP}(\Gamma_1)$ is polynomial-time reducible to $\text{CSP}(\Gamma_2)$ [35, 33].

The power of this is e.g. that the complexity classification of CSPs over the Boolean domain (seperating the cases in P from the NP-complete cases; due to Schaefer [63]) follows trivially with the knowledge of the Boolean clones due to Post [47, 48].

If we use partial clones instead we obtain a stronger result:

If Γ_1 and Γ_2 are sets of relations, Γ_1 is finite and $\text{pPol}(\Gamma_2) \subseteq \text{pPol}(\Gamma_1)$, then $\text{CSP}(\Gamma_1)$ is solvable in $O(f(n))$ if $\text{CSP}(\Gamma_2)$ is solvable in $O(f(n))$ (where n is the number of variables); see [45].

Unfortunately this is more difficult to apply since the lattice of partial clones is more complicated; even in the Boolean case. But there is research to determine the lattice better similar to the things we present in Chapter 8.

This topic is still young and an active field of research as many recently published papers show; e.g. [34, 3, 64, 9, 44, 36].

Chapter Overview

In Chapter 2 we introduce partial functions and the closed sets called *clones* preparing for all the later chapters.

In Chapter 3 a Galois-connection is developed between the partial function algebras and some newly introduced relation algebras. In comparison to older work ([52, 59, 61]) this yields a characterization of the Galois closed sets without using the Galois connection. The structure is taken from the chapter “The Galois-Connection Between Function- and Relation-Algebras” in [40] and is only changed to accommodate the extra difficulties due to the extension to partial functions.

In Chapter 4 we identify more properties of the Galois-connection. This ultimately yields a new, more elaborate proof of a statement of Romov [52].

In Chapter 5 we show that one special partial clone is maximal and then state the characterization of the maximal partial clones due to Haddad and Rosenberg [14, 22, 24]. This is extensively used in the following two chapters.

In Chapter 6 we determine the maximal partial clones for k -valued logics with $k \leq 6$.

In Chapter 7 we show that there is a unique minimal covering of the maximal partial clones. This is closely related to partial Sheffer functions [68] and lifts an important part of the result of Schofield [65] and Rousseau [62] to the case of partial clones.

Chapter 8 concludes this thesis with the study of an interval in partial two-valued logic by determining part of the lattice of partial Boolean clones.

Chapter 2

Basics

In this chapter two different notions for partial functions are introduced. These can be used interchangeably as the last section shows. Thus these notions are used in the rest of this thesis as we see them fit best.

2.1 Partial Functions

Let $E_k := \{0, 1, \dots, k-1\}$ and $\tilde{E}_k := E_k \cup \{\infty\}$ for any $k \in \mathbb{N}$ with $k \geq 2$ where ∞ is just some symbol which is not in E_k .

Then $f^{(n)} : E_k^n \rightarrow \tilde{E}_k$ is called an n -ary partial function. If the arity n of the function $f^{(n)}$ is known we just write f instead of $f^{(n)}$. Let $\tilde{P}_k^{(n)}$ be the set of all n -ary partial functions, i.e.,

$$\tilde{P}_k^{(n)} := \left\{ f^{(n)} \mid f^{(n)} : E_k^n \rightarrow \tilde{E}_k \right\}$$

and

$$\tilde{P}_k := \bigcup_{n \geq 1} \tilde{P}_k^{(n)}.$$

The set of all total functions $P_k \subset \tilde{P}_k$ is defined by

$$P_k := \{ f^{(n)} \in \tilde{P}_k \mid \forall x \in E_k^n : f^{(n)}(x) \in E_k \}.$$

In the study of partial functions it is useful to specify the domain of a function $f^{(n)}$ which is defined by

$$\text{dom } f := \{ x \in E_k^n \mid f(x) \neq \infty \}.$$

The n -ary function with empty domain is denoted by $c_\infty^{(n)} \in \tilde{P}_k$, i.e.,

$$\forall x \in E_k^n : c_\infty^{(n)}(x) := \infty.$$

Let

$$C_\infty := \{ c_\infty^{(n)} \mid n \geq 1 \}.$$

Example 2.1.1. Let $k = 2$ and $f_1^{(2)}, f_2^{(2)} \in \tilde{P}_k$ defined by

$$f_1 \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} := \begin{pmatrix} f_1(0,0) \\ f_1(0,1) \\ f_1(1,0) \\ f_1(1,1) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \infty \\ 0 \end{pmatrix} \quad \text{and} \quad f_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} := \begin{pmatrix} f_2(0,0) \\ f_2(0,1) \\ f_2(1,0) \\ f_2(1,1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then $f_1 \in \tilde{P}_k \setminus P_k$ and $f_2 \in P_k$.

The following holds

$$\text{dom } f_1 = \{(0, 0), (0, 1), (1, 1)\}, \quad \text{dom } f_2 = E_2^2, \quad \text{dom } c_\infty^{(n)} = \emptyset.$$

Sometimes we want to consider the functions of \tilde{P}_k as functions from \tilde{E}_k^n to \tilde{E}_k which can be done in a natural way.

Definition 2.1.2. Let $f^{(n)} \in \tilde{P}_k$. Then we define the n -ary function $(f^\infty)^{(n)} : \tilde{E}_k^n \rightarrow \tilde{E}_k$ by

$$(f^\infty)(x_1, \dots, x_n) := \begin{cases} f(x_1, \dots, x_n) & \text{if } (x_1, \dots, x_n) \in E_k^n, \\ \infty & \text{otherwise.} \end{cases}$$

Since it is clear from the context when we use the extended function (f^∞) we also write f for it. Moreover we can assume f to be (f^∞) since the restriction of (f^∞) to tuples from E_k^n is the function f .

2.2 Operations on \tilde{P}_k

We consider the following five *elementary operations* (or *superposition operations*; or *Mal'tsev operations*; see [42] for the original definition for the total case) ζ , τ , Δ , ∇ , \star on \tilde{P}_k , with which we can form more complex operations.

For arbitrary $f^{(n)}, g^{(m)} \in \tilde{P}_k$ one can define these operations by

$$\zeta f \in \tilde{P}_k^{(n)}, \tau f \in \tilde{P}_k^{(n)}, \Delta f \in \tilde{P}_k^{(\max(1, n-1))}, \nabla f \in \tilde{P}_k^{(n+1)}, f \star g \in \tilde{P}_k^{(m+n-1)}$$

and

$$\begin{aligned} (\zeta f)(x_1, x_2, \dots, x_n) &:= f(x_2, x_3, \dots, x_n, x_1), \\ (\tau f)(x_1, x_2, \dots, x_n) &:= f(x_2, x_1, x_3, \dots, x_n), \\ (\Delta f)(x_1, x_2, \dots, x_{n-1}) &:= f(x_1, x_1, x_2, \dots, x_{n-1}) \text{ for } n \geq 2, \\ \zeta f := \tau f := \Delta f := f &\text{ for } n = 1, \\ (\nabla f)(x_1, x_2, \dots, x_{n+1}) &:= f(x_2, x_3, \dots, x_{n+1}) \text{ and} \\ (f \star g)(x_1, \dots, x_{m+n-1}) &:= (f^\infty)(g(x_1, \dots, x_m), x_{m+1}, \dots, x_{m+n-1}) \end{aligned}$$

We can define a *general composition* $f(g_1, \dots, g_n) \in \tilde{P}_k^{(m)}$ for arbitrary $f^{(n)}, g_1^{(m)}, \dots, g_n^{(m)} \in \tilde{P}_k$ by

$$f(g_1, \dots, g_n)(x_1, \dots, x_m) := (f^\infty)(g_1(x_1, \dots, x_m), \dots, g_n(x_1, \dots, x_m)).$$

Furthermore we define n -ary projections $e_i^{(n)} \in \tilde{P}_k^{(n)}$ for $i \in \{1, \dots, n\}$ by

$$\forall x_1, \dots, x_n \in E_k : e_i^{(n)}(x_1, \dots, x_n) := x_i.$$

Let J_k be the set of all projections on E_k , i.e.,

$$J_k := \{e_i^{(n)} \mid n, i \in \mathbb{N}, 1 \leq i \leq n\}.$$

Let α be an unary operation. Then α^l is defined by $\alpha^0 := \text{id}$ (the identity function), $\alpha^1 := \alpha$ and $\alpha^l := \alpha \alpha^{l-1}$ for $l \geq 2$.

2.3 Function Algebras and Clones

The algebra

$$(\tilde{P}_k; \zeta, \tau, \Delta, \nabla, \star, e_1^{(2)})$$

is called the *full partial function algebra* on E_k . Every subalgebra A of \tilde{P}_k (written $A \leq \tilde{P}_k$) is a *partial function algebra* on E_k .

Let $A \subseteq \tilde{P}_k$. Then $[A]_P$ is the universe of the smallest partial function algebra, which contains A . If $[A]_P = A \subseteq \tilde{P}_k$ we say that A is *closed*, or A is a *clone* of \tilde{P}_k , or A is a *partial clone*. If A is a partial clone and $A \subseteq P_k$ then A is called a *total clone*.

Lemma 2.3.1. *Let $A \subseteq \tilde{P}_k$. Then A is a clone if and only if A is closed with respect to general composition and $J_k \subseteq A$.*

Proof. First let A be a clone. Then A is closed with respect to ζ , τ , Δ , ∇ and \star , and contains $e_1^{(2)}$.

Since $e_i^{(n)} = \zeta^{n-i} \nabla^{n-2} \tau e_1^{(2)}$ for $n \geq 2$ and $e_1^{(1)} = \Delta e_1^{(2)}$ we have $J_k \subseteq A$. Furthermore $f(g_1, \dots, g_n)$ for functions $f^{(n)}, g_1^{(m)}, \dots, g_n^{(m)}$ can be expressed by \star , Δ and permuting variables. Thus A is closed with respect to general composition and $J_k \subseteq A$.

Now let A be closed with respect to general composition and $J_k \subseteq A$. Let $f^{(n)}, g^{(m)} \in A$. Since

$$\begin{aligned} \zeta f &= f(e_2^{(n)}, e_3^{(n)}, \dots, e_n^{(n)}, e_1^{(n)}), \\ \tau f &= f(e_2^{(n)}, e_1^{(n)}, e_3^{(n)}, \dots, e_n^{(n)}), \\ \Delta f &= f(e_1^{(n-1)}, e_1^{(n-1)}, e_2^{(n-2)}, \dots, e_{n-1}^{(n-1)}) \text{ if } n \geq 2, \\ \Delta f &= f \text{ if } n = 1, \\ \nabla f &= f(e_2^{(n+1)}, e_3^{(n+1)}, \dots, e_{n+1}^{(n+1)}), \text{ and} \\ f \star g &= f(g(e_1^{(m+n-1)}, \dots, e_m^{(m+n-1)}), e_{m+1}^{(m+n-1)}, \dots, e_{m+n-1}^{(m+n-1)}) \end{aligned}$$

and $e_1^{(2)} \in J_k$ we have that A is a clone. □

2.4 Partial Functions via Domains

The set of *partial functions* can also be described with the help of the domain. Let $f^{(n)} : A \rightarrow E_k$ with $A \subseteq E_k^n$. Then f is called an n -ary partial function f on E_k . The set A is called the domain of f and we write $\text{dom } f := A$. For the specification of a partial function in this style we consequently need to specify the domain of the function explicitly.

Let $(\tilde{P}_k^{\text{dom}})^{(n)}$ be the set of all n -ary partial functions in the sense defined here, i.e.,

$$(\tilde{P}_k^{\text{dom}})^{(n)} := \left\{ f^{(n)} : A \rightarrow E_k \mid A \subseteq E_k^n \right\}$$

and

$$\tilde{P}_k^{\text{dom}} := \bigcup_{n \geq 1} (\tilde{P}_k^{\text{dom}})^{(n)}.$$

2.5 One-Point Extension

We can translate this definition into the language of the previous chapters. Let $f^{(n)} \in \tilde{P}_k^{\text{dom}}$. Then define $f_+^{(n)} \in \tilde{P}_k$ by

$$f_+(x_1, \dots, x_n) := \begin{cases} f(x_1, \dots, x_n) & \text{if } (x_1, \dots, x_n) \in \text{dom } f, \\ \infty & \text{otherwise.} \end{cases}$$

For $F \subseteq \tilde{P}_k^{\text{dom}}$ let $F_+ := \{f_+ \mid f \in F\}$.

Similar let $g^{(m)} \in \tilde{P}_k$. Then define $g_-^{(m)} : B \rightarrow E_k$ by

$$B := \{(x_1, \dots, x_m) \in E_k^m \mid g(x_1, \dots, x_m) \neq \infty\}$$

and

$$g_-(x_1, \dots, x_m) := g(x_1, \dots, x_m)$$

for all $(x_1, \dots, x_m) \in B$. For $G \subseteq \tilde{P}_k$ let $G_- := \{g_- \mid g \in G\}$.

We consider \tilde{P}_k as a subset of P_{k+1} . Let $H := [(J_k)_+] \subseteq P_{k+1}$ and $U := [(\tilde{P}_k^{\text{dom}})_+] \subseteq P_{k+1}$. Then the following theorem states that we can use either convention to denote partial functions.

Theorem 2.5.1 ([61], [8], [4]; also in [40]; (without proof)).

- (a) For every partial clone $F \subseteq \tilde{P}_k^{\text{dom}}$ we have $F = ([F_+]_P)_-$.
- (b) For every clone $G \subseteq P_{k+1}$ with $H \subseteq G$ the set G_- is a partial clone of \tilde{P}_k^{dom} with the property $G = ([G_-]_P)_+$.
- (c) The mapping $\varphi : \mathbb{L}(H; U) \longrightarrow \mathbb{L}(J_k; \tilde{P}_k^{\text{dom}}), G \mapsto G_-$ is a lattice isomorphism between the lattices $\mathbb{L}(H; U)$ and $\mathbb{L}(J_k; \tilde{P}_k^{\text{dom}})$ where $\varphi^{-1}(F) = [F_+]$ holds for every $F \in \mathbb{L}(J_k; \tilde{P}_k^{\text{dom}})$.

Chapter 3

Galois Theory

Clones are infinite objects since the set J_k is infinite and is a subset of a clone. Furthermore to every function f in a clone C the set of functions $\{\nabla^l f, \Delta^l f \mid l \in \mathbb{N}_0\}$ is a subset of C and contains functions of all arities. Since it is often more complicated to deal with infinite objects compared to finite objects, it is reasonable to look for a characterization of the functions in a clone by other means, either finite or at most countably infinite.

Several different approaches are used to characterize clones. For example

- by Galois connections to relation algebras;
- by functional terms [12];
- by forbidden subfunctions [74]; and
- by hyperidentities [11].

We use the first approach using an antitone Galois connection to relation algebras. Especially this helps with big clones, e.g., the maximal clones, since the corresponding relation algebras are small and can often be described by only one relation.

For the total clones, i.e., clones in P_k , this was shown by Bodnarchuk, Kaluzhnin, Kotov and Romov in 1969 [2]. An improved proof can be found in [49] or Lau's book [40]. We use it as main guidance and sometimes take it verbatim for this chapter. An interesting paper connected to the task of this chapter is by Börner [5].

Galois connections for partial clones have been developed for example by Romov [52] and Rosenberg [59]. They proved very fruitful in the study of partial clones. Yet these studies lack a concrete characterization of the Galois closed sets on the relational side without using the Galois connection. This chapter is devoted to this task and we give a characterization similar to the one for total clones with the help of elementary operations on relations.

3.1 Relations

Relations are useful to describe the partial clones of \tilde{P}_k . We often write the elements of relations as columns and a relation can then be given as a matrix. For example the relation $\varrho = \{(0, 1, 2), (1, 2, 0), (3, 4, 5), (2, 3, 1)\}$ can also be written as

$$\varrho = \begin{pmatrix} 0 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 2 & 0 & 5 & 1 \end{pmatrix}.$$

Let $\tilde{E}_k^{a \times b}$ be the set of all $(a \times b)$ -matrices on \tilde{E}_k .

Let a matrix be given by $C = (c_{ij})_{h,n} \in \tilde{E}_k^{h \times n}$. Then c_{i*} are the rows of the matrix with $i \in \{1, \dots, h\}$, i.e., $c_{i*} = (c_{i1}, c_{i2}, \dots, c_{in})$, and c_{*j} are the columns of the matrix with $j \in \{1, \dots, n\}$, i.e., $c_{*j} = (c_{1j}, c_{2j}, \dots, c_{hj})^T$.

Denote $\widehat{\mathcal{R}}_k^{(h)}$ the set of all h -ary relations on \tilde{E}_k , i.e.,

$$\widehat{\mathcal{R}}_k^{(h)} := \{A \mid A \subseteq E_k^h\} \text{ for } h \geq 1,$$

and let

$$\widehat{\mathcal{R}}_k := \bigcup_{h \geq 1} \widehat{\mathcal{R}}_k^{(h)}.$$

For a relation $\varrho \in \widehat{\mathcal{R}}_k$ we write $\varrho^{(h)}$ to indicate that $\varrho \in \widehat{\mathcal{R}}_k^{(h)}$, i.e., that ϱ is an h -ary relation.

Similarly, $\mathcal{R}_k^{(h)}$ denotes the set of all h -ary relations on E_k and let

$$\mathcal{R}_k := \bigcup_{h \geq 1} \mathcal{R}_k^{(h)}.$$

Definition 3.1.1. Let $f^{(n)} \in \tilde{P}_k$ and $r_{*1}, \dots, r_{*n} \in \tilde{E}_k^h$ with $h \geq 1$ and $r_{*i} = (r_{1i}, \dots, r_{hi})$ for all $i \in \{1, \dots, n\}$. Then

$$f(r_{*1}, \dots, r_{*n}) := f \begin{pmatrix} r_{11} & \dots & r_{1n} \\ r_{21} & \dots & r_{2n} \\ \dots & \dots & \dots \\ r_{h1} & \dots & r_{hn} \end{pmatrix} := \begin{pmatrix} f(r_{11}, \dots, r_{1n}) \\ f(r_{21}, \dots, r_{2n}) \\ \dots \\ f(r_{h1}, \dots, r_{hn}) \end{pmatrix}$$

Remember that we use f in place of (f^∞) in this context (see Definition 2.1.2) since the elements r_{ij} belong to \tilde{E}_k , i.e., $r_{ij} = \infty$ is possible for some i and j .

3.2 Preserving of Relations; pPol, pInv

We say a function $f^{(n)} \in \tilde{P}_k$ *preserves the relation* $\varrho \in \widehat{\mathcal{R}}_k^{(h)}$ (or ϱ is an *invariant of* f), if

$$\forall r_1, \dots, r_n \in \varrho : f(r_1, \dots, r_n) \in \varrho.$$

By

$$\text{pPol}_k \varrho$$

or, briefly, $\text{pPol} \varrho$ we denote the set of all functions $f \in \tilde{P}_k$ which preserve the relation ϱ . For $Q \subseteq \widehat{\mathcal{R}}_k$ we put

$$\text{pPol}_k Q := \bigcap_{\varrho \in Q} \text{pPol}_k \varrho.$$

$\text{pPol}_k \varrho$ and $\text{pPol}_k Q$ are short-cuts for *polymorphisms* of ϱ and Q , respectively. The set of all relations $\varrho \in \widehat{\mathcal{R}}_k$ that are preserved by the function $f \in \tilde{P}_k$ is called

$$\text{pInv}_k f.$$

The set of all invariants of $A \subseteq \tilde{P}_k$ is called

$$\text{pInv}_k A := \bigcap_{f \in A} \text{pInv}_k f.$$

For the study of total clones one defines Pol_k and Inv_k by

$$\text{Pol}_k Q := (\text{pPol}_k Q) \cap P_k \text{ and } \text{Inv}_k A := (\text{pInv}_k A) \cap \mathcal{R}_k$$

for $Q \subseteq \widehat{\mathcal{R}}_k$ and $A \subseteq \widetilde{P}_k$. Then the following identities hold

$$\text{Pol}_k Q = \text{Pol}_k(Q \cap \mathcal{R}_k) \text{ and } \text{Inv}_k A = \text{Inv}_k(A \cap P_k).$$

Definition 3.2.1. Let $\varrho^{(h)} \subseteq \widetilde{E}_k^h$. Then ϱ is called ∞ -strict if

$$e_1^{(2)} \in \text{pPol}_k \varrho,$$

or equivalently

$$\forall r, s \in \varrho : e_1^{(2)}(r, s) \in \varrho.$$

Since we want to describe clones with the help of relations and the function $e_1^{(2)}$ belongs to every partial clone, we can restrict to ∞ -strict relations.

Denote $\widetilde{\mathcal{R}}_k^{(h)}$ the set of all h -ary ∞ -strict relations on \widetilde{E}_k and let

$$\widetilde{\mathcal{R}}_k := \bigcup_{h \geq 1} \widetilde{\mathcal{R}}_k^{(h)}.$$

If $Q \subseteq \widetilde{\mathcal{R}}_k$ then let $Q^{(h)} := Q \cap \widetilde{\mathcal{R}}_k^{(h)}$.

3.3 Diagonal Relations

Definition 3.3.1. Let ε be an arbitrary equivalence relation on E_h . Define

$$\widetilde{\delta}_{k;\varepsilon}^{(h)} := \left\{ (a_0, \dots, a_{h-1}) \in \widetilde{E}_k^h \mid (i, j) \in \varepsilon \implies a_i = a_j \right\}.$$

If h or k can be deduced from the context we just write $\widetilde{\delta}_\varepsilon$ or $\widetilde{\delta}_\varepsilon^{(h)}$ or $\widetilde{\delta}_{k;\varepsilon}$. If the relation ε is given by the non-singular equivalence classes $\varepsilon_1, \dots, \varepsilon_r$ then we write $\widetilde{\delta}_{k;\varepsilon_1, \dots, \varepsilon_r}^{(h)}$ or $\widetilde{\delta}_{\varepsilon_1, \dots, \varepsilon_r}^{(h)}$ or $\widetilde{\delta}_{\varepsilon_1, \dots, \varepsilon_r}$ instead of $\widetilde{\delta}_{k;\varepsilon}^{(h)}$. For example $\widetilde{\delta}_{k;E_h}^{(h)} = \left\{ (x, x, \dots, x) \in \widetilde{E}_k^h \mid x \in \widetilde{E}_k \right\}$.

These relations are called *diagonal relations*. Especially \widetilde{E}_k^h for any h is a diagonal relation.

Additionally define $\delta_{k;\varepsilon}^{(h)} := \widetilde{\delta}_{k;\varepsilon}^{(h)} \cap E_k^h$ and use similar notations as above.

3.4 Elementary Relations on $\widehat{\mathcal{R}}_k$

In this section we define some operations ζ , τ , pr , \times , \wedge and $\hat{\kappa}$ on $\widehat{\mathcal{R}}_k$ which can be used to define more complex operations later. We call the operations ζ , τ , pr , \times , \wedge and $\hat{\kappa}$ the *elementary operations* on $\widehat{\mathcal{R}}_k$.

Let $\varrho^{(h)}, \sigma^{(\mu)} \in \widehat{\mathcal{R}}_k$ with $\varrho \neq \emptyset$ and $\sigma \neq \emptyset$. Then let

- $\zeta \varrho, \tau \varrho \in \widehat{\mathcal{R}}_k^h$,
- $\text{pr} \varrho \in \widehat{\mathcal{R}}_k^{h-1}$ for $h \geq 2$, and $\text{pr} \varrho = \emptyset$ for $h = 1$,
- $\varrho \times \sigma \in \widehat{\mathcal{R}}_k^{h+\mu}$,

- $\varrho \wedge \sigma \in \widehat{\mathcal{R}}_k^h$ (only for $h = \mu$),
- $\hat{\kappa}\varrho \in \widehat{\mathcal{R}}_k^h$

defined by

$$\begin{aligned}
\zeta\varrho &:= \{(a_2, a_3, \dots, a_h, a_1) \mid (a_1, a_2, \dots, a_h) \in \varrho\} \\
&\quad (\text{cyclical exchange of the rows}), \\
\tau\varrho &:= \{(a_2, a_1, a_3, \dots, a_h) \mid (a_1, a_2, \dots, a_h) \in \varrho\} \\
&\quad (\text{exchange of the first two rows}), \\
\text{pr } \varrho &:= \{(a_2, \dots, a_h) \mid \exists a_1 \in \widetilde{E}_k : (a_1, a_2, \dots, a_h) \in \varrho\} \\
&\quad (\text{projection onto the rows } 2, \dots, h), \\
\varrho \times \sigma &:= \{(a_1, \dots, a_h, b_1, \dots, b_\mu) \mid (a_1, \dots, a_h) \in \varrho \text{ and } (b_1, \dots, b_\mu) \in \sigma\} \\
&\quad (\text{cartesian product of } \varrho \text{ and } \sigma), \\
\varrho \wedge \sigma &:= \varrho \cap \sigma, \\
\hat{\kappa}\varrho &:= \varrho \cup \{\infty\} \\
&\quad \text{for } h = 1 \text{ or } \varrho = \emptyset, \\
\hat{\kappa}\varrho &:= \varrho \cup \{(a_1, a_2, a_3, \dots, a_h) \in \widetilde{E}_k^h \mid a_1 = a_2 = \infty\} \\
&\quad \text{for } h \geq 2.
\end{aligned}$$

Lemma 3.4.1. *Let $\varrho^{(h)}, \sigma^{(\mu)} \in \widetilde{\mathcal{R}}_k$ then $\zeta\varrho, \tau\varrho, \text{pr } \varrho, \varrho \times \sigma, \varrho \wedge \sigma, \hat{\kappa}\varrho \in \widetilde{\mathcal{R}}_k$, i.e., the elementary operations on $\widehat{\mathcal{R}}_k$ preserve the ∞ -strictness.*

Proof. Let $\alpha \in \{\zeta, \tau\}$ and $r, s \in \alpha\varrho$ be arbitrary.

Then $\alpha^{-1}r, \alpha^{-1}s \in \varrho$ and $e_1^{(2)}(\alpha^{-1}r, \alpha^{-1}s) \in \varrho$ because ϱ is ∞ -strict. This implies

$$e_1^{(2)}(r, s) = \alpha \left(e_1^{(2)}(\alpha^{-1}r, \alpha^{-1}s) \right) \in \alpha\varrho$$

and thus $\alpha\varrho$ is ∞ -strict.

Let $r, s \in \text{pr } \varrho$ be arbitrary. Then there are $r', s' \in \varrho$ with $\text{pr } r' = r$ and $\text{pr } s' = s$. Because ϱ is ∞ -strict we have $e_1^{(2)}(r', s') \in \varrho$. This implies

$$e_1^{(2)}(r, s) = e_1^{(2)}(\text{pr } r', \text{pr } s') = \text{pr} \left(e_1^{(2)}(r', s') \right) \in \text{pr } \varrho$$

and thus $\text{pr } \varrho$ is ∞ -strict.

Let $r, s \in \varrho \times \sigma$. Then there are $r', s' \in \varrho$ and $r'', s'' \in \sigma$ with

$$r = \begin{pmatrix} r' \\ r'' \end{pmatrix} \text{ and } s = \begin{pmatrix} s' \\ s'' \end{pmatrix}.$$

Because ϱ and σ are ∞ -strict we have $e_1^{(2)}(r', s') \in \varrho$ and $e_1^{(2)}(r'', s'') \in \sigma$. This implies

$$e_1^{(2)}(r, s) = e_1^{(2)} \begin{pmatrix} r' & s' \\ r'' & s'' \end{pmatrix} = \begin{pmatrix} e_1^{(2)}(r', s') \\ e_1^{(2)}(r'', s'') \end{pmatrix} \in \varrho \times \sigma$$

and thus $\varrho \times \sigma$ is ∞ -strict.

Let $h = \mu$ and $r, s \in \varrho \wedge \sigma$ be arbitrary. Because ϱ and σ are ∞ -strict we have $e_1^{(2)}(r, s) \in \varrho$ and $e_1^{(2)}(r, s) \in \sigma$. This implies $e_1^{(2)}(r, s) \in \varrho \wedge \sigma$ and thus $\varrho \wedge \sigma$ is ∞ -strict.

Let $r, s \in \hat{\kappa}\varrho$. If $r, s \in \varrho$ then $e_1^{(2)}(r, s) \in \varrho \subseteq \hat{\kappa}\varrho$ because ϱ is ∞ -strict. If $r \notin \varrho$ then $r = (\infty, \infty, r_3, \dots, r_h)$ and thus

$$e_1^{(2)}(r, s) = e_1^{(2)} \begin{pmatrix} \infty & s_1 \\ \infty & s_2 \\ r_3 & s_3 \\ \dots & \dots \\ r_h & s_h \end{pmatrix} = \begin{pmatrix} \infty \\ \infty \\ t_3 \\ \dots \\ t_h \end{pmatrix} \in \{(a_1, \dots, a_h) \in \tilde{E}_k^h \mid a_1 = a_2 = \infty\} \subseteq \hat{\kappa}\varrho$$

for some t_i . Similarly if $s \notin \varrho$. And thus $\hat{\kappa}\varrho$ is ∞ -strict. \square

Theorem 3.4.2. *The set $\tilde{\mathcal{R}}_k$ is closed with respect to the application of ζ , τ , pr , \times , \wedge and $\hat{\kappa}$.* \square

For any set A denote by $\mathcal{P}(A)$ the powerset of A , i.e.,

$$\mathcal{P}(A) := \{X \mid X \subseteq A\}.$$

Definition 3.4.3. Let $\varrho^{(h)} \in \tilde{\mathcal{R}}_k$. Then define $I_\infty(\varrho) \subseteq \mathcal{P}(\{1, \dots, h\})$ by

$$I_\infty(\varrho) := \{I \in \mathcal{P}(\{1, \dots, h\}) \setminus \{\emptyset\} \mid \exists (a_1, \dots, a_h) \in \varrho \forall i \in \{1, \dots, h\} (a_i = \infty \iff i \in I)\}.$$

3.5 Some Operations on $\tilde{\mathcal{R}}_k$ Derivable from the Elementary Operations

A list of derivable operations follows. Let $\varrho^{(h)}, \sigma^{(\mu)} \in \tilde{\mathcal{R}}_k$.

(A) *Permutation of coordinates (or permutation of rows) $\varrho^{[\pi]}$*

Let S_h be the full symmetric group on $\{1, \dots, h\}$ and let $\pi \in S_h$ be an arbitrary permutation. Let $r = (r_1, r_2, \dots, r_h) \in \tilde{E}_k^h$ be arbitrary. Then define

$$r^{[\pi]} := (r_{\pi(1)}, r_{\pi(2)}, \dots, r_{\pi(h)})$$

and

$$\begin{aligned} \varrho^{[\pi]} &:= \{r^{[\pi]} \mid r \in \varrho\} \\ &= \{(r_{\pi(1)}, r_{\pi(2)}, \dots, r_{\pi(h)}) \mid (r_1, r_2, \dots, r_h) \in \varrho\}. \end{aligned}$$

This operation is derivable from ζ and τ since the permutations $(12 \dots h)$ and (12) (written with cycle notation) constitute a generating set of S_h .

(B) *Projection onto the α_1 -th, \dots , α_t -th coordinates (or deleting of rows)*

Let

$$\text{pr}_{\alpha_1, \dots, \alpha_t} \varrho := \{(a_{\alpha_1}, \dots, a_{\alpha_t}) \mid \exists a_1, \dots, a_h \in \tilde{E}_k : (a_1, \dots, a_h) \in \varrho\}.$$

Additionally we define a short notation: if $\alpha_1 < \alpha_2 < \dots < \alpha_t$ and $A = \{\alpha_1, \dots, \alpha_t\}$ then let $\text{pr}_A := \text{pr}_{\alpha_1, \dots, \alpha_t}$.

(C) *Identification of coordinates Δ_I*

Let $I := \{i_1, \dots, i_l\} \subseteq \{1, \dots, h\}$ with $i_1 < i_2 < \dots < i_l$. Let

$\{j_1, \dots, j_m\} := (\{1, \dots, h\} \setminus I) \cup \{i_1\}$ with $j_1 < j_2 < \dots < j_m$. Now we can define

$$\Delta_I \varrho := \{(a_{j_1}, a_{j_2}, \dots, a_{j_m}) \mid \exists a_1, \dots, a_h \in \tilde{E}_k : (a_1, \dots, a_h) \in \varrho\}.$$

If $|I| \leq 1$ then $\Delta_I \varrho = \varrho$. We define one special identification operator $\Delta \varrho := \Delta_{\{1,2\}} \varrho$.

- (D) *Doubling of coordinates (rows) ν_i*
Let $i \in \{1, \dots, h\}$. Define ν_i by

$$\nu_i(\varrho) := \{(a_1, \dots, a_{i-1}, a_i, a_i, a_{i+1}, \dots, a_h) \mid (a_1, \dots, a_h) \in \varrho\}.$$

- (E) *Adding of fictitious coordinates ∇ and unpr_J^h*
Let

$$\nabla\varrho := \{(a_1, a_2, \dots, a_{h+1}) \in \tilde{E}_k^{h+1} \mid (a_2, \dots, a_{h+1}) \in \varrho\} = \tilde{E}_k \times \varrho.$$

Similar one can add a fictitious variable at the i -th coordinate with ∇_i defined by

$$\nabla_i\varrho := (\nabla\varrho)^{[\pi_i]}$$

with the permutation $\pi_i = (i \ i-1 \ \dots \ 2 \ 1)$.

Let $\{j_1, \dots, j_\mu\} := J \subseteq \{1, \dots, h\}$ and $|J| = \mu$. Define unpr_J^h by

$$\text{unpr}_J^h(\sigma) := \{(a_1, \dots, a_h) \in \tilde{E}_k^h \mid (a_{j_1}, a_{j_2}, \dots, a_{j_\mu}) \in \sigma\}.$$

Then $\text{pr}_J \text{unpr}_J^h \sigma = \sigma$ and $\text{unpr}_J^h \text{pr}_J \varrho \supseteq \varrho$.

- (F) *General composition (relation product) \circ_t*
Let $1 \leq t < \min(h, \mu)$. Define \circ_t by

$$\varrho \circ_t \sigma := \left\{ \begin{array}{l} (a_1, \dots, a_{h+\mu-t}) \in \tilde{E}_k^{h+\mu-t} \mid \\ (a_1, \dots, a_h) \in \varrho \text{ and } (a_{h+1-t}, \dots, a_{h+\mu-t}) \in \sigma \end{array} \right\}$$

and let $\varrho \circ \sigma := \varrho \circ_1 \sigma$.

- (G) *General ∞ -extension κ_I*
Let $I \subseteq \{1, \dots, h\}$ and $I \neq \emptyset$. Then define

$$\kappa_I\varrho := \varrho \cup \{(a_1, \dots, a_h) \in \tilde{E}_k^h \mid \forall i \in I : a_i = \infty\}.$$

For $I = \{1, 2\}$ and $h \geq 2$ we have $\kappa_I = \hat{\kappa}$. With permutations of coordinates one gets κ_I for all I with $|I| = 2$. Then for $I = \{i\}$ we have $\kappa_I\varrho = \text{pr}_{1, \dots, h} \kappa_{\{i, h+1\}}(\varrho \times \tilde{E}_k)$ and for $|I| \geq 3$ we have

$$\kappa_I\varrho = \bigcap_{\substack{J \subseteq I \\ |J|=2}} \kappa_J\varrho.$$

- (H) *Full ∞ -extension κ*
Define κ by

$$\kappa\varrho := \varrho \cup \{(a_1, \dots, a_h) \in \tilde{E}_k^h \mid \exists i \in \{1, \dots, h\} : a_i = \infty\}.$$

Then $\kappa\varrho = \kappa_{\{1\}}\kappa_{\{2\}} \dots \kappa_{\{h\}}\varrho$.

- (I) *Partial ∞ -extension $\bar{\kappa}$*
Define $\bar{\kappa}$ by

$$\bar{\kappa}\varrho := \varrho \cup \{(a_1, \dots, a_h) \in \tilde{E}_k^h \mid \exists I \in I_\infty(\varrho) \forall i \in I : a_i = \infty\}.$$

If $I_\infty(\varrho) = \{I_1, \dots, I_l\}$ for some $l \geq 1$ then $\bar{\kappa}\varrho = \kappa_{I_1}\kappa_{I_2} \dots \kappa_{I_l}\varrho$.

- (J) *General ∞ -subextension κ_I^**

holds.

The strong partial clones play a very important role and for these it is good to define pPOL_k for any relation $\varrho \in E_k^h$ by

$$\text{pPOL}_k \varrho := \text{pPol}_k(\kappa\varrho).$$

3.7 Relation Algebras and Co-Clones

The algebra

$$(\tilde{\mathcal{R}}_k; \tilde{\delta}_{k;\{1,2\}}^{(3)}, \zeta, \tau, \text{pr}, \wedge, \times, \hat{\kappa})$$

is called the *full partial relation algebra* on E_k . Every subalgebra Q of $\tilde{\mathcal{R}}_k$ (written $Q \leq \tilde{\mathcal{R}}_k$) is a *partial relation algebra* on E_k .

Let $Q \subseteq \tilde{\mathcal{R}}_k$. Then $[Q]_{\text{P}}$ is the universe of the smallest partial relation algebra, which contains Q .

If $[Q]_{\text{P}} = Q \subseteq \tilde{\mathcal{R}}_k$ we say that Q is *closed* or Q is a *co-clone* of $\tilde{\mathcal{R}}_k$.

The algebra

$$(\mathcal{R}_k; \delta_{k;\{1,2\}}^{(3)}, \zeta, \tau, \text{pr}, \wedge, \times)$$

is called the *full total relation algebra* on E_k .

3.8 Galois Connection

The following theorem describes certain properties of a Galois connection which can already be proven in the setting of general Galois connections between two powersets.

Theorem 3.8.1. *For arbitrary $A, B \subseteq \tilde{P}_k$ and arbitrary $S, T \subseteq \tilde{\mathcal{R}}_k$, it holds:*

- (a) $A \subseteq B \implies \text{pInv } B \subseteq \text{pInv } A,$
 $S \subseteq T \implies \text{pPol } T \subseteq \text{pPol } S;$
- (b) $A \subseteq \text{pPol pInv } A,$
 $S \subseteq \text{pInv pPol } S;$
- (c) $\text{pInv pPol pInv } A = \text{pInv } A,$
 $\text{pPol pInv pPol } S = \text{pPol } S;$
- (d) $A \subseteq \text{pPol } S \iff S \subseteq \text{pInv } A;$
- (e) $\text{pPol}(S \cup T) = \text{pPol } S \cap \text{pPol } T,$
 $\text{pInv}(A \cup B) = \text{pPol } A \cap \text{pPol } B.$

Proof. The statements (a), (b), (d), and (e) follow directly from the definitions of pPol and pInv .

Now we show (c). Let $S := \text{pInv } A$. From (b) follows $\text{pInv } A = S \subseteq \text{pInv pPol } S = \text{pInv pPol pInv } A$. On the other hand $A \subseteq \text{pPol pInv } A$ by (b) and this implies together with (a): $\text{pInv pPol pInv } A \subseteq \text{pInv } A$. Therefore $\text{pInv pPol pInv } A = \text{pInv } A$. Similarly we can show $\text{pPol pInv pPol } S = \text{pPol } S$. \square

Theorem 3.8.2. For every $A \subseteq \tilde{P}_k$ and every $Q \subseteq \tilde{\mathcal{R}}_k$ the sets $\text{pInv } A$ and $\text{pPol } Q$ are closed (in respect to the operations defined above, respectively); i.e., $\text{pPol } Q$ is a partial clone of \tilde{P}_k and $\text{pInv } A$ is a partial co-clone of $\tilde{\mathcal{R}}_k$.

Furthermore, it holds:

$$\text{pInv } [A]_{\text{P}} = \text{pInv } A \text{ and } \text{pPol } [Q]_{\text{P}} = \text{pPol } Q.$$

Proof. Let $\varrho^{(h)}, \sigma^{(\mu)} \in \text{pInv } A$ and $f^{(n)} \in A$ be arbitrary.

- Let $\alpha \in \{\zeta, \tau\}$. Because $f(r_1, \dots, r_n) = \alpha f(\alpha^{-1}r_1, \dots, \alpha^{-1}r_n) = \alpha r$ and $\alpha^{-1}r_i \in \varrho$ for all $r_1, \dots, r_n \in \alpha\varrho$, we have $r \in \varrho$ and thus $f(r_1, \dots, r_n) = \alpha r \in \alpha\varrho$, i.e., f preserves $\alpha\varrho$.

- Let $r_1, \dots, r_n \in \text{pr } \varrho$. Then there are $r'_1, \dots, r'_n \in \varrho$ with $r_i = \text{pr } r'_i$ for all $i \in \{1, \dots, n\}$. Then

$$f(r_1, \dots, r_n) = f(\text{pr } r'_1, \dots, \text{pr } r'_n) = \text{pr } f(r'_1, \dots, r'_n) \in \text{pr } \varrho,$$

i.e., f preserves $\text{pr } \varrho$.

- Let $\mu = h$ and $r_1, \dots, r_n \in \varrho \wedge \sigma$. Then $f(r_1, \dots, r_n) \in \varrho$ because f preserves ϱ and $f(r_1, \dots, r_n) \in \sigma$ because f preserves σ . Thus $f(r_1, \dots, r_n) \in \varrho \wedge \sigma$, i.e., f preserves $\varrho \wedge \sigma$.

- Let $r_1, \dots, r_n \in \varrho \times \sigma$. Then there are $r'_i \in \varrho$ and $r''_i \in \sigma$ for all $i \in \{1, \dots, h\}$ with

$$r_i = \begin{pmatrix} r'_i \\ r''_i \end{pmatrix}.$$

Then

$$f(r_1, \dots, r_n) = f \begin{pmatrix} r'_1 & \dots & r'_n \\ r''_1 & \dots & r''_n \end{pmatrix} = \begin{pmatrix} f(r'_1, \dots, r'_n) \\ f(r''_1, \dots, r''_n) \end{pmatrix} \in \varrho \times \sigma,$$

i.e., f preserves $\varrho \times \sigma$.

- Let $r_1, \dots, r_n \in \hat{\kappa}\varrho$. If $r_1, \dots, r_n \in \varrho$ then $f(r_1, \dots, r_n) \in \varrho \subseteq \hat{\kappa}\varrho$. Otherwise there is some r_i with

$$r_i = \begin{pmatrix} \infty \\ \infty \\ r_{3i} \\ \dots \\ r_{hi} \end{pmatrix}$$

and then there are $b_3, \dots, b_n \in \tilde{E}_k$ such that

$$\begin{aligned} f(r_1, \dots, r_i, \dots, r_n) &= \begin{pmatrix} f(\dots, \infty, \dots) \\ f(\dots, \infty, \dots) \\ f(\dots, r_{3i}, \dots) \\ \dots \\ f(\dots, r_{hi}, \dots) \end{pmatrix} = \begin{pmatrix} \infty \\ \infty \\ b_3 \\ \dots \\ b_h \end{pmatrix} \\ &\in \{(a_1, \dots, a_h) \in \tilde{E}_k^h \mid a_1 = a_2 = \infty\} \subseteq \hat{\kappa}\varrho. \end{aligned}$$

Thus f preserves $\hat{\kappa}\varrho$.

This implies $\text{pInv } A$ is closed.

Now let $f^{(n)}, g^{(m)} \in \text{pPol } Q$ and $\varrho \in Q$ be arbitrary. It is clear that the functions ζf , τf , Δf and ∇f also preserve ϱ . Furthermore the relation ϱ is also an invariant of $f \star g$, because

$(f \star g)(r_1, \dots, r_{m+n-1}) = f(g(r_1, \dots, r_m), r_{m+1}, \dots, r_{m+n-1})$ holds for all r_1, \dots, r_{m+n-1} since $g(r_1, \dots, r_m) \in \varrho$. Thus $\text{pPol } Q$ is closed.

Now we show $\text{pInv } [A]_{\mathbb{P}} = \text{pInv } A$.

- Because $A \subseteq [A]_{\mathbb{P}}$ and Theorem 3.8.1 (a), we have $\text{pInv } [A]_{\mathbb{P}} \subseteq \text{pInv } A$.
- By Theorem 3.8.1 (b), we have $A \subseteq \text{pPol } \text{pInv } A$. Since $\text{pPol } \text{pInv } A$ is closed, this implies $[A]_{\mathbb{P}} \subseteq \text{pPol } \text{pInv } A$. By Theorem 3.8.1 (c) and (a) we obtain the other direction $\text{pInv } A = \text{pInv } \text{pPol } \text{pInv } A \subseteq \text{pInv } [A]_{\mathbb{P}}$.

Thus $\text{pInv } [A]_{\mathbb{P}} = \text{pInv } A$.

Analogously, one can prove $\text{pPol } [Q]_{\mathbb{P}} = \text{pPol } Q$. □

3.9 The Relations χ_n and $G_n(A)$

For arbitrary $n \in \mathbb{N}$ and $k \in \mathbb{N}$, $k \geq 2$ denote $\chi_{k;n}$ (or short χ_n) the k^n -ary relation, whose rows are just all $(x_1, \dots, x_n) \in E_k^n$ that are arranged *lexicographically*.

For example

$$\chi_{2;3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Obviously, there is exactly one function $f_r \in \tilde{P}_k^{(n)}$ with $f_r(\chi_n) = r$ for every column $r \in \tilde{E}_k^{k^n}$.

The relation

$$G_n(A) := \{r \in E_k^{k^n} \mid f_r \in A^{(n)}\},$$

is called the *n-th graphic* of $A \subseteq \tilde{P}_k$.

The following theorem summarizes elementary properties of the relation $G_n(A)$.

Theorem 3.9.1. *Let $A \subseteq \tilde{P}_k$ be an arbitrary partial clone. Then*

- (a) $\forall n \in \mathbb{N} : G_n(A) \in \text{pInv } A$;
- (b) $f^{(n)} \in A^{(n)} \iff f^{(n)} \in \text{pPol } G_n(A)$;
- (c) $A \subseteq \dots \subseteq \text{pPol } G_n(A) \subseteq \text{pPol } G_{n-1}(A) \subseteq \dots \subseteq \text{pPol } G_2(A) \subseteq \text{pPol } G_1(A)$;
- (d) $A = \bigcap_{n \geq 1} \text{pPol } G_n(A)$;
- (e) $\{\emptyset, \{\infty\}, \tilde{E}_k\} \subseteq [\{G_n(A) \mid n \geq 1\}]_{\mathbb{P}}$;
- (f) $\forall \varrho \in \text{pInv } A : \varrho \in [\{G_n(A) \mid n \geq 1\}]_{\mathbb{P}}$;
- (g) $\text{pInv } A = [\{G_n(A) \mid n \geq 1\}]_{\mathbb{P}}$.

Proof. (a): Let $g^{(m)} \in A$ and $r_1, \dots, r_m \in G_n(A)$ be arbitrary. Then

$$g(r_1, \dots, r_m) = g(f_{r_1}(\chi_n), \dots, f_{r_m}(\chi_n)) = g(f_{r_1}, \dots, f_{r_m})(\chi_n) = h(\chi_n)$$

with $h = g(f_{r_1}, \dots, f_{r_m})$. Since A is partial clone we have $h \in A$ and h is n -ary. Thus $h(\chi_n) \in G_n(A)$ by definition of $G_n(A)$, and $G_n(A) \in \text{pInv } A$.

(b): If $f^{(n)} \in A$ then we have $f \in \text{pPol } \varrho$ for every $\varrho \in \text{pInv } A$. Consequently, $f \in \text{pPol } G_n(A)$ by (a).

On the other hand, $f^{(n)} \in \text{pPol } G_n(A)$ implies the existence of a certain $r \in G_n(A)$ with $f(\chi_n) = r$; thus $f = f_r \in A^{(n)}$ is valid.

(c): Let $n \in \mathbb{N}$ be arbitrary. We have $\{G_n(A)\} \subseteq \text{pInv } A$ by (a) and we get $A \subseteq \text{pPol pInv } A \subseteq \text{pPol } \{G_n(A)\} = \text{pPol } G_n(A)$ by Theorem 3.8.1 (b) and (a).

Now let $n \geq 2$, $f^{(m)} \in \text{pPol } G_n(A)$ and $r_1, \dots, r_m \in G_{n-1}(A)$.

Define $\theta : \tilde{E}_k^{k^n} \rightarrow \tilde{E}_k^{k^{n-1}}$ by

$$\theta t = (\Delta f_t)(\chi_{n-1})$$

with $t \in \tilde{E}_k^{k^n}$.

Let $t \in G_n(A)$ be arbitrary. Then $\theta t = (\Delta f_t)(\chi_{n-1}) \in G_{n-1}(A)$ because $f_t \in A^{(n)}$ and $\Delta f_t \in A^{(n-1)}$. Thus $\theta(G_n(A)) \subseteq G_{n-1}(A)$.

Then

$$r_i = f_{r_i}(\chi_{n-1}) = \theta((\nabla f_{r_i})(\chi_n))$$

for all $i \in \{1, \dots, m\}$. Since $\nabla f_{r_i} \in A^{(n)}$ we have $(\nabla f_{r_i})(\chi_n) \in G_n(A)$. Thus

$$f(r_1, \dots, r_m) = \theta f(\nabla f_{r_1}, \dots, \nabla f_{r_m})(\chi_n) = \theta(g(\chi_n)) \in \theta(G_n(A)) \subseteq G_{n-1}(A)$$

with $g^{(n)} \in \text{pPol } G_n(A)$. This means that $f \in \text{pPol } G_{n-1}(A)$ and therefore $\text{pPol } G_n(A) \subseteq \text{pPol } G_{n-1}(A)$.

(d): We have $A \subseteq \bigcap_{n \geq 1} \text{pPol } G_n(A)$ by (c) and $A \supseteq \bigcap_{n \geq 1} \text{pPol } G_n(A)$ by (b), i.e.,

$$f^{(m)} \in \bigcap_{n \geq 1} \text{pPol } G_n(A) \implies f^{(m)} \in \text{pPol } G_m(A) \implies f^{(m)} \in A^{(m)}.$$

(e): We have $\emptyset = \text{pr pr}_1 G_1(A)$, $\{\infty\} = \hat{\kappa} \emptyset$ and $\tilde{E}_k = \hat{\kappa} \text{pr}_i G_k(A)$ for some i ; choose i such that the i -th row of χ_k equals $(0, 1, 2, \dots, k-1)$.

(f): We show a construction of ϱ from the relations $G_n(A)$ by induction over the arity h . Because of (e) we can assume $\varrho \notin \{\emptyset, \{\infty\}\}$.

We start with arity $h = 0$. Then $\varrho = \emptyset$ is constructable.

Thus let $h \geq 1$ be arbitrary and assume that $\sigma \in \{G_n(A) \mid n \geq 1\}_{\text{P}}$ for every relation $\sigma^{(\mu)} \in \text{pInv } A$ with $\mu < h$.

Let $\varrho^{(h)} \in \text{pInv } A$ be arbitrary. Let $\bar{\varrho} := \varrho \cap E_k^h$, i.e., the columns of ϱ without any ∞ , and let $t := |\bar{\varrho}|$.

There are two cases to consider

- $\bar{\varrho} \neq \emptyset$. Then we first construct

$$\hat{\varrho} := \{f(r_1, \dots, r_t) \mid r_1, \dots, r_t \in \bar{\varrho}, f \in A^{(t)}\}.$$

As $\bar{\varrho} \subseteq \varrho$ and $\varrho \in \text{pInv } A$ we get $\hat{\varrho} \subseteq \varrho$. Moreover

$$\hat{\varrho} \cap E_k^h = \varrho \cap E_k^h$$

because $J_k^{(t)} \subseteq A^{(t)}$ and $\bar{\varrho} = \varrho \cap E_k^h$.

Since $J_k \subseteq A$ we have $\chi_t \subseteq G_t(A)$ and for every row i of $\bar{\varrho}$ there is some row α_i of χ_t , i.e.,

$$\bar{\varrho} = \text{pr}_{\alpha_1, \dots, \alpha_h} \chi_t.$$

Then

$$\begin{aligned} \hat{\varrho} &= \{f(r_1, \dots, r_t) \mid r_1, \dots, r_t \in \text{pr}_{\alpha_1, \dots, \alpha_h} \chi_t, f \in A^{(t)}\} \\ &= \text{pr}_{\alpha_1, \dots, \alpha_h} \{f(r'_1, \dots, r'_t) \mid r'_1, \dots, r'_t \in \chi_t, f \in A^{(t)}\} \\ &= \text{pr}_{\alpha_1, \dots, \alpha_h} G_t(A) \end{aligned}$$

Thus $\hat{\varrho}$ is constructable.

- $\bar{\varrho} = \emptyset$. Then there is some $r := (r_1, \dots, r_h) \in \varrho$ and $I := \{i \in \{1, \dots, h\} \mid r_i = \infty\}$ with $I \neq \emptyset$. Without restriction we can assume $I = \{1, \dots, |I|\}$.

Then $\hat{\varrho} := \{\infty\}^{|I|} \times \text{pr}_{|I|+1, \dots, h} \varrho$ is constructable by induction because $\text{pr}_{|I|+1, \dots, h} \varrho \in (\text{pInv } A)^{(\mu)}$ for some $\mu < h$. Furthermore, $\hat{\varrho} \subseteq \varrho$ because ϱ is ∞ -strict, i.e., for all $s \in \varrho$ we have $e_1^{(2)}(s, r) \in \varrho$, and $\hat{\varrho} \cap E_k^h = \emptyset = \varrho \cap E_k^h$.

Let $H := \{1, \dots, h\}$. Let $\varrho_1 := \hat{\varrho}$, $\{I_1, \dots, I_{l-1}\} := I_\infty(\varrho)$ and construct for $1 \leq j \leq l-1$ the relations ϱ_{j+1} recursively.

Let $I := I_j$ and $\varrho_{j+1} := (\kappa_I \varrho_j) \cap (\tilde{E}_k^{|I|} \times \text{pr}_{H \setminus I} \varrho)$ where we assume w.l.o.g. $I = \{1, \dots, |I|\}$ (otherwise we can permute the rows of ϱ_j , ϱ and I first, and then permuting back the rows of ϱ_{j+1}). Otherwise let $\varrho_{j+1} := \varrho_j$.

Then $\varrho_j \subseteq \varrho_{j+1}$, $\varrho_{j+1} \cap E_k^h = \varrho \cap E_k^h$,

$$\forall s = (s_1, \dots, s_n) \in \varrho \ ((s_i = \infty \iff i \in I) \implies s \in \varrho_{j+1})$$

and $\varrho_{j+1} \subseteq \varrho$.

Then $\varrho = \varrho_l \in [\{G_n(A) \mid n \geq 1\}]_{\mathbb{P}}$.

(g): By (f) we have $\text{Inv } A \subseteq [\{G_n(A) \mid n \geq 1\}]_{\mathbb{P}}$. Furthermore, because of (a) and Theorem 3.8.2, we have $\text{Inv } A \supseteq [\{G_n(A) \mid n \geq 1\}]_{\mathbb{P}}$. \square

3.10 The Operator Γ_A

For arbitrary $A \subseteq \tilde{P}_k$ denote Γ_A a mapping from $\tilde{\mathcal{R}}_k$ into $\tilde{\mathcal{R}}_k$, which is defined for $\sigma^{(h)} \in \tilde{\mathcal{R}}_k^{(h)}$ by

$$\Gamma_A(\sigma) := \bigcap \left\{ \varrho \in \tilde{\mathcal{R}}_k \mid \varrho \in \text{pInv } A \wedge \sigma \subseteq \varrho \right\}. \quad (3.1)$$

Theorem 3.10.1. *Let $A \subseteq \tilde{P}_k$ be an arbitrary partial clone and $n \in \mathbb{N}$ arbitrary. Then*

- $\Gamma_A(\chi_n) \in \text{pInv } A$;
- $\Gamma_A(\chi_n) = G_n(A)$;
- $A^{(n)} = \{f_r \mid r \in \Gamma_A(\chi_n)\}$.

Proof. (a): follows from $[\text{pInv } A]_{\text{P}} = \text{pInv } A$ by Theorem 3.8.2 and the definition of $\Gamma_A(\chi_n)$.

(b): Since every projection $e_i^{(n)}$ belongs to $A^{(n)}$, we have $\chi_n \subseteq G_n(A)$.

Let ϱ be an arbitrary k^n -ary relation of $\tilde{\mathcal{R}}_k$ with $\chi_n \subseteq \varrho$. If $\varrho \in \text{pInv } A$, then $f(\chi_n) \in \varrho$ for every $f \in A^{(n)}$, i.e., $G_n(A) \subseteq \varrho$. Consequently, we have shown that $G_n(A) \subseteq \Gamma_A(\chi_n)$ holds. $\Gamma_A(\chi_n) \subseteq G_n(A)$ follows from $G_n(A) \in \text{pInv } A$ (see Theorem 3.9.1 (a)).

(c) follows from (b) and the definition of $G_n(A)$. \square

3.11 Galois Theory for Partial Function- and Relation-Algebras

Theorem 3.11.1. *Let A be a partial clone of \tilde{P}_k . Then*

$$A = \text{pPol pInv } A.$$

Proof. By Theorem 3.8.1 (b) we have $A \subseteq \text{pPol pInv } A$.

To prove $\text{pPol pInv } A \subseteq A$ let $f^{(n)} \in \text{pPol pInv } A$ be arbitrary. Because $G_n(A) \in \text{pInv } A$ by Theorem 3.9.1 (a) we have $f \in \text{pPol } G_n(A)$ and then we have $f \in A^{(n)}$ by Theorem 3.9.1 (b).

Thus $A = \text{pPol pInv } A$. \square

Lemma 3.11.2. *Let $\beta^{(m)} \in \tilde{\mathcal{R}}_k$, $f^{(t)} \in \tilde{P}_k$ and $r_1, \dots, r_t \in \beta$ with $f(r_1, \dots, r_t) \notin \beta$.*

Then there is a relation $\bar{\beta}^{(\bar{m})} \in [\{\beta\}]_{\text{P}} \subseteq \tilde{\mathcal{R}}_k$ and $\bar{r}_1, \dots, \bar{r}_t \in \bar{\beta} \cap E_k^{\bar{m}}$ with $f(\bar{r}_1, \dots, \bar{r}_t) \notin \bar{\beta}$.

Proof. Let $M := \{1, \dots, m\}$. Let $(r_j^1, \dots, r_j^m) := r_j$ and $I_j := \{i \in M \mid r_j^i = \infty\}$ for all $j \in \{1, \dots, t\}$. Moreover let $s := (s^1, \dots, s^m) := f(r_1, \dots, r_t)$ and $I_s := \{i \in M \mid s^i = \infty\}$. By definition of function application we know $I := \bigcup_{j=1}^t I_j \subseteq I_s \subseteq M$.

If $I = M$ then $I_s = M$, and thus $s = (\infty, \dots, \infty) \in \beta$ because $r_1, \dots, r_t \in \beta$ and β is ∞ -strict. But this contradicts $s = f(r_1, \dots, r_t) \notin \beta$.

Thus $M \setminus I \neq \emptyset$. Let $\bar{m} := |M \setminus I|$, $\bar{\beta}^{(\bar{m})} := \text{pr}_{M \setminus I} \beta \in [\{\beta\}]_{\text{P}}$ and $\bar{r}_j := \text{pr}_{M \setminus I} r_j \in \bar{\beta}$ for all $j \in \{1, \dots, t\}$. Since $I_j \subseteq I$ we have $\bar{r}_j \in E_k^{\bar{m}}$.

We now assume w.l.o.g. that $I = \{1, \dots, |I|\}$. Suppose $\text{pr}_{M \setminus I} s = f(\bar{r}_1, \dots, \bar{r}_t) \in \bar{\beta} = \text{pr}_{M \setminus I} \beta$. Then $s \in \{\infty\}^{|I|} \times \text{pr}_{M \setminus I} \beta \subseteq \beta$ because β is ∞ -strict in contradiction to the assumption. Thus $f(\bar{r}_1, \dots, \bar{r}_t) \notin \bar{\beta}$. \square

Theorem 3.11.3. *Let Q be a partial co-clone of $\tilde{\mathcal{R}}_k$. Then*

$$Q = \text{pInv pPol } Q.$$

Proof. Let $A := \text{pPol } Q$. By Theorem 3.8.1 (b) we have $Q \subseteq \text{pInv } A$. To prove that $\text{pInv } A \subseteq Q$ it is sufficient to show that $\Gamma_A(\chi_t) \in Q$ for all $t \in \mathbb{N}$, since $\left[\bigcup_{t \geq 1} \{\Gamma_A(\chi_t)\} \right]_{\text{P}} = \text{pInv } A$ by Theorem 3.9.1 (g) and Theorem 3.10.1 (b).

Now let $t \in \mathbb{N}$ with $t \geq 1$ be arbitrary and set

$$\gamma := \bigcap \{ \varrho \in Q \mid \chi_t \subseteq \varrho \}.$$

The intersection is not empty because $\tilde{E}_k^{k^t} \in Q$. Since Q is closed with respect to $\wedge = \cap$, we have $\chi_t \subseteq \gamma$, $\gamma \in Q$ and γ is the smallest relation in Q which contains χ_t , with respect to cardinality. Because $\gamma \in Q \subseteq \text{pInv } A$ we have $\Gamma_A(\chi_t) \subseteq \gamma$. Consequently, our theorem is proven, if we can show $\Gamma_A(\chi_t) = \gamma$.

Suppose $\Gamma_A(\chi_t) \subset \gamma$. Then there is a column $r \in \gamma \setminus \Gamma_A(\chi_t)$. Since $A^{(t)} = \{f_s \mid s \in \Gamma_A(\chi_t)\}$ by Theorem 3.10.1 (c) we have $f_r \notin A^{(t)} = \text{pPol}^{(t)} Q$. Thus there is some m -ary relation $\beta \in Q$ with $f_r \notin \text{pPol} \beta$ and certain columns $r_1, \dots, r_t \in \beta$ with $f(r_1, \dots, r_t) \notin \beta$. By Lemma 3.11.2 we can assume $r_1, \dots, r_t \in E_k^m$ because $\tilde{\beta} \in [\{\beta\}]_P \subseteq Q$, i.e., $\tilde{\beta} \in Q$. Thus every row of the matrix (r_1, \dots, r_m) is also a row of the matrix χ_t . Let i_j be the number of the row of χ_t which agrees with the j -th row of (r_1, \dots, r_t) for all $j \in \{1, \dots, m\}$. Let now

$$\gamma' := \text{pr}_{1,2,\dots,k^t}(\gamma \times \beta) \cap \delta_{\{i_1, k^t+1\}, \{i_2, k^t+2\}, \dots, \{i_m, k^t+m\}}^{(k^t+m)}$$

Since Q is closed, γ' belongs to Q , and by construction of γ' we have $\chi_t \subseteq \gamma' \subseteq \gamma$. Furthermore, we have $r \in \gamma \setminus \gamma'$, since $r_1, \dots, r_t \in \beta$, $f_r(r_1, \dots, r_t) \notin \beta$ and $f_r(\chi_t) = r \in \gamma$. But with γ' we received a contradiction to the choice of γ . Thus $\gamma = \Gamma_A(\chi_t)$ and therefore $\text{pInv } A \subseteq Q$. \square

Theorem 3.11.4. *Let $\mathbb{L}(\tilde{P}_k)$ be the set of all partial clones of \tilde{P}_k and let $\mathbb{L}(\tilde{\mathcal{R}}_k)$ be the set of all partial co-clones of $\tilde{\mathcal{R}}_k$. Then the mappings*

$$\text{pInv} : \mathbb{L}(\tilde{P}_k) \longrightarrow \mathbb{L}(\tilde{\mathcal{R}}_k), A \mapsto \text{pInv } A$$

and

$$\text{pPol} : \mathbb{L}(\tilde{\mathcal{R}}_k) \longrightarrow \mathbb{L}(\tilde{P}_k), Q \mapsto \text{pPol } Q$$

are bijective mappings, which reverse the partial order \subseteq , i.e., it holds

$$\forall A, B \in \mathbb{L}(\tilde{P}_k) : A \subseteq B \implies \text{pInv } B \subseteq \text{pInv } A$$

and

$$\forall S, T \in \mathbb{L}(\tilde{\mathcal{R}}_k) : S \subseteq T \implies \text{pPol } T \subseteq \text{pPol } S.$$

In other words:

The lattices $(\mathbb{L}(\tilde{P}_k), \subseteq)$ and $(\mathbb{L}(\tilde{\mathcal{R}}_k), \subseteq)$ are antiisomorphic.

Proof. By Theorem 3.8.2 the mappings pInv and pPol are mappings from $\mathbb{L}(\tilde{P}_k)$ (or $\mathbb{L}(\tilde{\mathcal{R}}_k)$) to $\mathbb{L}(\tilde{\mathcal{R}}_k)$ (or $\mathbb{L}(\tilde{P}_k)$), respectively.

Let $A \in \mathbb{L}(\tilde{P}_k)$. Then $A = \text{pPol } \text{pInv } A$ by Theorem 3.11.1 and $\text{pInv } A \in \mathbb{L}(\tilde{\mathcal{R}}_k)$ by Theorem 3.8.2. Thus pPol is surjective. Let $S, T \in \mathbb{L}(\tilde{\mathcal{R}}_k)$ with $\text{pPol } S = \text{pPol } T$. Then $S = \text{pInv } \text{pPol } S = \text{pInv } \text{pPol } T = T$ by Theorem 3.11.3. Thus pPol is injective. Therefore pPol is a bijective mapping. Analogously, pInv is a bijective mapping.

In Theorem 3.8.1 (a) it was shown that pPol and pInv reverse the partial order \subseteq . \square

Theorem 3.11.4 implies the long known result for total clones and co-clones:

Theorem 3.11.5 (Theorem of V.G. Bodnarchuk, L.A. Kaluzhnin, V.A. Kotov and B.A. Romov; [2]). *Let $\mathbb{L}(P_k)$ be the set of all clones of P_k and let $\mathbb{L}(\mathcal{R}_k)$ be the set of all co-clones of \mathcal{R}_k . Then the mappings*

$$\text{Inv} : \mathbb{L}(P_k) \longrightarrow \mathbb{L}(\mathcal{R}_k), A \mapsto \text{Inv } A$$

and

$$\text{Pol} : \mathbb{L}(\mathcal{R}_k) \longrightarrow \mathbb{L}(P_k), Q \mapsto \text{Pol } Q$$

are bijective mappings, which reverse the partial order \subseteq , i.e., it holds

$$\forall A, B \in \mathbb{L}(P_k) : A \subseteq B \implies \text{Inv } B \subseteq \text{Inv } A$$

and

$$\forall S, T \in \mathbb{L}(\mathcal{R}_k) : S \subseteq T \implies \text{Pol } T \subseteq \text{Pol } S.$$

In other words:

The lattices $(\mathbb{L}(P_k), \subseteq)$ and $(\mathbb{L}(\mathcal{R}_k), \subseteq)$ are antiisomorphic.

Chapter 4

Representation Lemma of Romov

In the last chapter we have shown the Galois theory for partial clones. Now we extend this topic and give some further results. These allow us to describe all partial clones with a subset of the relations used before. These relations have some useful properties needed later on.

Furthermore if we only consider strong partial clones then we obtain even better results and a new proof of the Representation Lemma of Romov (Theorem 4.5.9 in this chapter; see also [52]). This characterizes the subset relation between two strong partial clones by means of homomorphisms between the relations describing those clones. The Representation Lemma of Romov was a substantial basis of the characterization of all maximal partial clones by Haddad and Rosenberg [14, 22, 24].

4.1 Definitions and Properties

Definition 4.1.1. Define $\tilde{\mathcal{R}}_k^* \subseteq \tilde{\mathcal{R}}_k$ by

$$\tilde{\mathcal{R}}_k^* := \{\varrho \in \tilde{\mathcal{R}}_k \mid \bar{\kappa}\varrho = \varrho\}. \quad (\text{for } \bar{\kappa} \text{ see p. 15})$$

The following relations belong to $\tilde{\mathcal{R}}_k^*$:

- all relations in $\tilde{\mathcal{R}}_k$ since $\bar{\kappa}$ can not change a relation if there is no ∞ in it;
- $\chi_1 := \begin{pmatrix} 0 & 0 & 1 & \infty \\ 0 & \infty & \infty & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix} \in \tilde{\mathcal{R}}_2^*$

The relation

$$\chi_2 := \begin{pmatrix} 0 & 0 & \infty \\ 0 & \infty & \infty \\ 1 & \infty & \infty \end{pmatrix} \in \tilde{\mathcal{R}}$$

does not belong to $\tilde{\mathcal{R}}_2^*$ since $\bar{\kappa}\chi_2 = \chi_1 \neq \chi_2$.

Lemma 4.1.2. Let $f^{(n)} \in \tilde{P}_k$ and $\varrho^{(h)} \in \tilde{\mathcal{R}}_k^*$. Then $f \in \text{pPol}_k \varrho$ if and only if

$$\forall r_{*1}, \dots, r_{*n} \in \varrho \cap E_k^h : f(r_{*1}, \dots, r_{*n}) \in \varrho.$$

Proof. Let $f^{(n)} \in \text{pPol}_k \varrho$ and $r_{*1}, \dots, r_{*n} \in \varrho \cap E_k^h$ be arbitrary. Then $f(r_{*1}, \dots, r_{*n}) \in \varrho$ because of the definition of pPol_k and $\varrho \cap E_k^h \subseteq \varrho$.

Now let $f^{(n)}$ be arbitrary such that

$$\forall r_{*1}, \dots, r_{*n} \in \varrho \cap E_k^h : f(r_{*1}, \dots, r_{*n}) \in \varrho.$$

We want to show $f \in \text{pPol}_k \varrho$, i.e.,

$$\forall r_{*1}, \dots, r_{*n} \in \varrho : f(r_{*1}, \dots, r_{*n}) \in \varrho.$$

Let $r_{*1}, \dots, r_{*n} \in \varrho$ be arbitrary. If $\{r_{*1}, \dots, r_{*n}\} \subseteq \varrho \cap E_k^h$ then we have $f(r_{*1}, \dots, r_{*n}) \in \varrho$ by choice of f . Thus we can assume $\{r_{*1}, \dots, r_{*n}\} \not\subseteq \varrho \cap E_k^h$, w.l.o.g. $r_{*1} \notin E_k^h$. Let $s := (s_1, \dots, s_h) := f(r_{*1}, \dots, r_{*n})$. Then $r_{i1} = \infty$ implies $s_i = \infty$ and thus $s \in \bar{\kappa}\varrho = \varrho$. \square

Lemma 4.1.3. Let $\varrho^{(h)} \in \tilde{\mathcal{R}}_k^*$, $I \in I_\infty(\varrho)$ and $J \subseteq \{1, \dots, h\}$ with $I \subseteq J$. Then $J \in I_\infty(\varrho)$.

Proof. Since $I \in I_\infty(\varrho)$ there is some $(a_1, \dots, a_h) \in \varrho$ with $a_i = \infty$ if and only if $i \in I$. By definition of $\bar{\kappa}\varrho$ all tuples $(b_1, \dots, b_h) \in \tilde{E}_k^h$ with $b_i = \infty$ for all $i \in I$ belong to $\bar{\kappa}\varrho$, especially the tuple (b_1, \dots, b_h) with $b_j = \infty$ if $j \in J$ and $b_j = 0$ if $j \notin J$. Thus $J \in I_\infty(\varrho)$. \square

Definition 4.1.4. Let $\varrho \in \tilde{\mathcal{R}}_k^*$. Then define $I_\infty^{\min}(\varrho)$ by

$$I_\infty^{\min}(\varrho) := \{I \in I_\infty(\varrho) \mid \forall J \in I_\infty(\varrho) \setminus \{I\} : J \not\subseteq I\}.$$

For every relation $\varrho \in \tilde{\mathcal{R}}_k^*$ the set $I_\infty(\varrho)$ is fully determined by $I_\infty^{\min}(\varrho)$ because of Lemma 4.1.3.

Lemma 4.1.5. Let $\varrho^{(h)} \in \tilde{\mathcal{R}}_k^*$. Then there is some $\chi^{(h)} \in \tilde{\mathcal{R}}_k^*$ with $\text{pPol}_k \varrho = \text{pPol}_k \chi$ and

$$\begin{aligned} \forall I \in I_\infty^{\min}(\chi) \exists f^{(n)} \in \text{pPol}_k \chi \exists r_{*0} \in \chi \exists r_{*1}, \dots, r_{*n} \in \chi \cap E_k^h : \\ r_{*0} = f(r_{*1}, \dots, r_{*n}) \wedge (r_{i0} = \infty \iff i \in I). \end{aligned}$$

Proof. Let $\chi' := (\varrho \cap E_k^h) \cup \{f(r_{*1}, \dots, r_{*n}) \mid r_{*1}, \dots, r_{*n} \in \varrho \cap E_k^h\}$ and $\chi := \bar{\kappa}\chi' \in \tilde{\mathcal{R}}_k^*$. Then $\text{pPol}_k \varrho = \text{pPol}_k \chi$ follows from $\chi \cap E_k^h = \varrho \cap E_k^h$ and Lemma 4.1.2. The rest follows from the construction of χ . \square

Example 4.1.6. Define $\varrho^{(3)}, \chi^{(3)} \in \tilde{\mathcal{R}}_2^*$ by

$$\varrho := \chi_1 \text{ and } \chi := \chi_3 := \begin{pmatrix} 0 & \infty \\ 0 & \infty \\ 1 & \infty \end{pmatrix}.$$

Then $\text{pPol}_k \varrho = \text{pPol}_k \chi$ and Lemma 4.1.5 holds for them.

Definition 4.1.7. The relation $\varrho^{(h)} \in \mathcal{R}_k$ is called *irredundant*, iff it fulfills the following two conditions:

- (a) ϱ has no *duplicate rows*, i.e., for all i, j with $1 \leq i < j \leq h$, there is a tuple $(a_1, \dots, a_h) \in \varrho$ with $a_i \neq a_j$;
- (b) ϱ has no *fictitious coordinates*, i.e., there is no $i \in \{1, \dots, h\}$, such that $(a_1, \dots, a_h) \in \varrho$ implies $(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_h) \in \varrho$ for all $x \in E_k$.

4.2 Forcing Relations

Definition 4.2.1. Let $A, B \subseteq E_k$ and $A, B \neq \emptyset$. Then define the ∞ -forcing relation $\lambda_{A \Rightarrow B}^{(2)}$ by

$$\lambda_{A \Rightarrow B} := (A \times B) \cup (\tilde{E}_k \times \{\infty\}).$$

Let $\chi^{(\mu)} \in \mathcal{R}_k$. Then define the *special* ∞ -forcing relation $\lambda_\chi^{*(\mu+1)}$ by

$$\lambda_\chi^* := \kappa_2 \kappa_3 \dots \kappa_{\mu+1}(E_k \times \chi).$$

Let Λ_k be the set of all ∞ -forcing relations in $\tilde{\mathcal{R}}_k$, i.e.,

$$\Lambda_k := \{\lambda_{A \Rightarrow B} \mid A, B \subseteq E_k, A, B \neq \emptyset\} \cup \{\lambda_\chi^* \mid \chi \in \mathcal{R}_k\}.$$

Let Λ_k^* be the set of all irredundant ∞ -forcing relations in $\tilde{\mathcal{R}}_k$, i.e.,

$$\Lambda_k^* := \{\lambda_{A \Rightarrow B} \mid A, B \subseteq E_k, A, B \neq \emptyset\} \cup \{\lambda_\chi^* \mid \chi \in \mathcal{R}_k \text{ and } \chi \text{ is irredundant}\}.$$

We see that $\lambda_{A \Rightarrow B} = \bar{\kappa} \lambda_{A \Rightarrow B}$ and $\lambda_\chi^* = \bar{\kappa} \lambda_\chi^*$ by construction, i.e., $\lambda_{A \Rightarrow B}, \lambda_\chi^* \in \tilde{\mathcal{R}}_k^*$. Thus $\Lambda_k \subseteq \tilde{\mathcal{R}}_k^*$.

Lemma 4.2.2. Let $A, B \subseteq E_k$ and $A, B \neq \emptyset$. Then $f^{(n)} \in \text{pPol}_k \lambda_{A \Rightarrow B}$ if and only if

- (a) whenever there are some $a_1, \dots, a_n \in A$ with $f(a_1, \dots, a_n) \notin A$ then $f(b_1, \dots, b_n) = \infty$ for all $b_1, \dots, b_n \in B$; and
- (b) $f \in \text{pPol}_k \kappa B$

hold.

Proof. Let $f^{(n)} \in \text{pPol}_k \lambda_{A \Rightarrow B}$ and $a_1, \dots, a_n \in A$ with $f(a_1, \dots, a_n) \notin A$. Let $b_1, \dots, b_n \in B$ be arbitrary. Then $(a_i, b_i) \in \lambda_{A \Rightarrow B}$ for all $i \in \{1, \dots, n\}$ and

$$f \begin{pmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{pmatrix} \in \lambda_{A \Rightarrow B} \cap ((\tilde{E}_k \setminus A) \times \tilde{E}_k) = (\tilde{E}_k \setminus A) \times \{\infty\}.$$

Thus $f(b_1, \dots, b_n) = \infty$ and the implication holds. Since $\kappa B = \text{pr}_2 \lambda_{A \Rightarrow B}$ we have $f \in \text{pPol}_k \kappa B$.

Let $f^{(n)} \in \text{pPol}_k \kappa B$ and $(a_1, b_1), \dots, (a_n, b_n) \in \lambda_{A \Rightarrow B}$ be arbitrary. Because of Lemma 4.1.2 we can assume $(a_i, b_i) \in \lambda_{A \Rightarrow B} \cap E_k^2 = A \times B$. Since $f \in \text{pPol}_k \kappa B$ we have $b_0 := f(b_1, \dots, b_n) \in B \cup \{\infty\} = \kappa B$. If $a_0 := f(a_1, \dots, a_n) \in A$ then $(a_0, b_0) \in A \times (B \cup \{\infty\}) \subseteq \lambda_{A \Rightarrow B}$. If $a_0 \notin A$ then $b_0 = \infty$ by (a) and thus $(a_0, b_0) \in \lambda_{A \Rightarrow B}$. Therefore $f \in \text{pPol}_k \lambda_{A \Rightarrow B}$ holds. \square

Lemma 4.2.3. Let $\chi^{(\mu)} \in \mathcal{R}_k$. Then $f^{(n)} \in \text{pPol}_k \lambda_\chi^*$ if and only if

- (a) whenever there are some $a_1, \dots, a_n \in E_k$ with $f(a_1, \dots, a_n) = \infty$ then $f(b_1, \dots, b_n) \in \tilde{E}_k^\mu \setminus E_k^\mu$ for all $b_1, \dots, b_n \in \chi$; and
- (b) $f \in \text{pPol}_k \kappa \chi$

hold.

Proof. Let $f^{(n)} \in \text{pPol}_k \lambda_\chi^*$ and $a_1, \dots, a_n \in E_k$ with $f(a_1, \dots, a_n) = \infty$. Let $b_1, \dots, b_n \in \chi$ be arbitrary. Then $(a_i, b_i) \in \lambda_\chi^*$ for all $i \in \{1, \dots, n\}$ and

$$f \begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{pmatrix} \in \lambda_\chi^* \cap (\{\infty\} \times (\tilde{E}_k^\mu \setminus E_k^\mu)) = \{\infty\} \times (\tilde{E}_k^\mu \setminus E_k^\mu).$$

Thus $f(b_1, \dots, b_n) \in \tilde{E}_k^\mu \setminus E_k^\mu$ and the implication holds. Since $\kappa\chi = \text{pr}_{2, \dots, \mu+1} \lambda_\chi^*$ we have $f \in \text{pPol}_k \kappa\chi$.

Let $f^{(n)} \in \text{pPol}_k \kappa\chi$ and $(a_1, b_1), \dots, (a_n, b_n) \in \lambda_\chi^*$ be arbitrary. Because of Lemma 4.1.2 we can assume $(a_i, b_i) \in \lambda_\chi^* \cap E_k^{\mu+1} = E_k \times \chi$. Since $f \in \text{pPol}_k \kappa\chi$ we have $b_0 := f(b_1, \dots, b_n) \in \chi \cup (\tilde{E}_k^\mu \setminus E_k^\mu) = \kappa\chi$. If $a_0 := f(a_1, \dots, a_n) \neq \infty$ then $(a_0, b_0) \in E_k \times (\kappa\chi) \subseteq \lambda_\chi^*$. If $a_0 = \infty$ then $b_0 \in \tilde{E}_k^\mu \setminus E_k^\mu$ by (a) and thus $(a_0, b_0) \in \lambda_\chi^*$. Therefore $f \in \text{pPol}_k \lambda_\chi^*$ holds. \square

Lemma 4.2.4. *Let $A, B, C, D \subseteq E_k$ with $B \cap C \neq \emptyset$. Then*

$$\text{pPol}_k \{\lambda_{A \Rightarrow B}, \lambda_{C \Rightarrow D}\} \subseteq \text{pPol}_k \lambda_{A \Rightarrow D}.$$

Proof. Let $f^{(n)} \in \text{pPol}_k \{\lambda_{A \Rightarrow B}, \lambda_{C \Rightarrow D}\}$ be arbitrary.

By Lemma 4.2.2 we have $f \in \text{pPol}_k \kappa D$.

Let $a_1, \dots, a_n \in A$ with $f(a_1, \dots, a_n) \notin A$ and $c_1, \dots, c_n \in B \cap C$. Then $f(c_1, \dots, c_n) = \infty$ by Lemma 4.2.2 and thus we have $f(d_1, \dots, d_n) = \infty$ for all $d_1, \dots, d_n \in D$ by Lemma 4.2.2 again.

Applying Lemma 4.2.2 once more we get $f \in \text{pPol}_k \lambda_{A \Rightarrow D}$. \square

Example 4.2.5. Let $k = 3$, $A = \{0, 1\}$ and $B = \{1, 2\}$. Then

$$\lambda_{A \Rightarrow B} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 1 & 2 & \infty \\ 1 & 2 & 1 & 2 & \infty & \infty & \infty & \infty \end{pmatrix}.$$

Let $f^{(n)} \in \text{pPol}_3 \lambda_{A \Rightarrow B}$ be arbitrary such that there are $a_1, \dots, a_n \in A$ with $f(a_1, \dots, a_n) = \infty$. Let $b_1, \dots, b_n \in B$ be arbitrary. Then $(a_i, b_i) \in \lambda_{A \Rightarrow B}$ and thus

$$\begin{pmatrix} a_0 \\ b_0 \end{pmatrix} := f \begin{pmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \end{pmatrix} \in \lambda_{A \Rightarrow B}$$

implies $b_0 = \infty$. That means

$$\forall b_1, \dots, b_n \in B : f(b_1, \dots, b_n) = \infty.$$

The forcing relations $\lambda_{E_k \Rightarrow B}$ and $\lambda_{A \Rightarrow E_k}$ play an important role in subsequent proofs. For instance

$$\text{pPol}_k \lambda_{E_k \Rightarrow E_k} = P_k \cup C_\infty.$$

Lemma 4.2.6. *Let $A, B \subseteq E_k$ and $A, B \neq \emptyset$. Then*

$$\text{pPol}_k \kappa_{\{1,2\}}(A \times B) = \text{pPol}_k \{\lambda_{A \Rightarrow B}, \lambda_{B \Rightarrow A}\}.$$

Proof. Let $\chi := \kappa_{\{1,2\}}(A \times B)$. Then $\chi = (A \times B) \cup \{(\infty, \infty)\}$.

Since $\chi = \lambda_{A \Rightarrow B} \cap \zeta \lambda_{B \Rightarrow A}$, $\lambda_{A \Rightarrow B} = \kappa_{\{2\}} \chi$ and $\lambda_{B \Rightarrow A} = \kappa_{\{2\}} \tau \chi$ we obtain the equality. \square

4.3 Representations

The following theorem shows that we do not need all relations in $\widetilde{\mathcal{R}}_k$ but only the relations in $\widetilde{\mathcal{R}}_k^*$ to classify all partial clones. These have a simpler structure and therefore they are easier to reason about.

Theorem 4.3.1. *Let $\varrho^{(h)} \in \widetilde{\mathcal{R}}_k$ and $R := \{\bar{\kappa} \text{pr}_J \varrho \mid J \subseteq \{1, \dots, h\}\} \subseteq \widetilde{\mathcal{R}}_k^*$. Then*

$$\text{pPol}_k \varrho = \text{pPol}_k R.$$

Proof. Since $R \subseteq [\varrho]_{\text{P}}$ we have $\text{pPol}_k \varrho \subseteq \text{pPol}_k R$. We now show $\varrho \in [R]_{\text{P}}$.

For $h \in \{0, 1\}$ we get $R = \{\varrho\}$ since $\bar{\kappa}\varrho = \varrho$ and thus $\varrho \in [R]_{\text{P}}$.

Let $h \geq 2$. We show that $\varrho = \tilde{\varrho}$ where

$$\tilde{\varrho} := (\bar{\kappa}\varrho) \cap \bigcap_{\substack{I \in I_{\infty}(\varrho) \\ J = \{1, \dots, h\} \setminus I}} (\text{unpr}_J^h \bar{\kappa} \text{pr}_J \varrho). \quad (4.1)$$

We first show $\varrho \subseteq \tilde{\varrho}$. Let $r \in \varrho$ be arbitrary. Then $r \in \bar{\kappa}\varrho$ since $\varrho \subseteq \bar{\kappa}\varrho$. Let $I \in I_{\infty}(\varrho)$ be arbitrary and $J := \{1, \dots, h\} \setminus I$, w.l.o.g. $I = \{1, \dots, l\}$. Then $r \in \tilde{E}_k^l \times (\text{pr}_J \varrho) \subseteq \tilde{E}_k^l \times (\bar{\kappa} \text{pr}_J \varrho) = \text{unpr}_J^h \bar{\kappa} \text{pr}_J \varrho$ and thus $\varrho \subseteq \tilde{\varrho}$.

Now we show $\varrho \supseteq \tilde{\varrho}$. Let $r := (r_1, \dots, r_h) \in \tilde{\varrho}$ be arbitrary and set $I_r := \{i \in \{1, \dots, h\} \mid r_i = \infty\}$ and $J_r := \{1, \dots, h\} \setminus I_r$.

(a) $I_r = \emptyset$.

Then $r \in E_k^h$ and thus $r \in \varrho$ since $(\bar{\kappa}\varrho) \cap E_k^h = \varrho \cap E_k^h$.

(b) $I_r \neq \emptyset$ and $I_r \in I_{\infty}(\varrho)$.

Then $\text{pr}_{J_r} r \in (\bar{\kappa} \text{pr}_{J_r} \varrho) \cap E_k^{h-l}$ and this implies $\text{pr}_{J_r} r \in \text{pr}_{J_r} \varrho$ as in the previous case. Because there is some $s \in \varrho$ with $I_s = I_r$ and ϱ is ∞ -strict we know that

$$\varrho \cap (\{\infty\}^l \times \tilde{E}_k^{h-l}) = \{\infty\}^l \times (\text{pr}_{I_{r+1}, \dots, h} \varrho) = \text{unpr}_J^h \text{pr}_J \varrho.$$

Thus $r \in \varrho$.

(c) $I_r \neq \emptyset$ and $I_r \notin I_{\infty}(\varrho)$.

(i) $\forall I \in I_{\infty}(\varrho) : I \not\subseteq I_r$.

Then $r \notin \bar{\kappa}\varrho$ since

$$\bar{\kappa}\varrho = (\varrho \cap E_k^h) \cup \bigcup_{I \in I_{\infty}(\varrho)} \{(a_1, \dots, a_h) \in \tilde{E}_k^h \mid \forall i \in I : a_i = \infty\}.$$

But this contradicts $r \in \tilde{\varrho}$.

(ii) $\exists I \in I_{\infty}(\varrho) : I \subset I_r$.

Let $H := \{1, \dots, h\}$ and $r' := \text{pr}_{H \setminus I} r$ and $\varrho_1^{(h')} := \text{pr}_{H \setminus I} \varrho$. Then $r \in \varrho$ if and only if $r' \in \varrho'$.

Since $I \neq \emptyset$ the relation ϱ' has smaller arity than ϱ , i.e. $h' < h$, and since $I \not\subseteq I_r \subseteq \{1, \dots, h\}$ we have $h' \geq 1$.

Then

$$I_{\infty}(\varrho') = \left\{ \left\{ i - l \mid i \in \tilde{I}, i > l \right\} \mid \tilde{I} \in I_{\infty}(\varrho), I \not\subseteq \tilde{I} \right\}$$

holds, and since $I_{r'} = \{i - l \mid i \in I_r, i > l\}$ we have $I_{r'} \neq \emptyset$ and $I_{r'} \notin I_\infty(\varrho')$. But by induction over the arity we know that $r' \in \varrho'$ if and only if $I_{r'} = \emptyset$ or $I_{r'} \in I_\infty(\varrho')$. Thus $r' \notin \varrho'$ and consequently $r \notin \varrho \subseteq \tilde{\varrho}$ in contradiction to the assumption.

So $r \in \varrho$ and thus $\varrho = \tilde{\varrho}$ implying $\varrho \in [R]_P$. \square

Lemma 4.3.2. *Let $\varrho \in \tilde{\mathcal{R}}_k^*$ with $C := \text{pPol}_k \varrho \subseteq P_k$. Then there is some $\varrho' \in \tilde{\mathcal{R}}_k^*$ with $\text{pPol}_k \varrho' \not\subseteq P_k$ and $C = \text{pPol}_k \{\varrho', E_k\} = (\text{pPol}_k \varrho') \cap P_k$.*

Proof. Let $\varrho' := \kappa \varrho$. The statement follows from $(\text{pPol}_k \kappa \varrho) \cap P_k = (\text{pPol}_k \varrho) \cap P_k$ and $C_\infty \subseteq (\text{pPol}_k \varrho') \setminus P_k$. \square

Lemma 4.3.3. *Let $\varrho^{(h)} \in \tilde{\mathcal{R}}_k^*$ and $C := \text{pPol}_k \varrho$ such that the h -th row of $\varrho \cap E_k^h$ is fictitious. Then there are $(\varrho')^{(h-1)} \in \tilde{\mathcal{R}}_k^*$ and $L \subseteq \{\varrho', E_k\} \cup \Lambda_k$ with*

$$C = \text{pPol}_k L.$$

Proof. There are the following cases

- $h = 1$.
Then $E_k \subseteq \varrho$. If $\varrho = E_k$ then we set $L = \{E_k\}$. Otherwise $\varrho = \tilde{E}_k$ and we set $L = \emptyset$. We get $\text{pPol}_k \varrho = \text{pPol}_k L$.
- $h = 2$.
Then $\varrho \cap E_k^2 = A \times E_k =: \chi$ for some $A \subseteq E_k$. Since $\bar{\kappa} \varrho = \varrho$ there are four cases
 - If $\varrho = \chi$ then we get $\text{pPol}_k \varrho \subseteq \text{pPol}_k(\text{pr}_2 \varrho) = \text{pPol}_k E_k = P_k$ and $\text{pPol}_k \varrho \subseteq \text{pPol}_k(\text{pr}_1 \varrho) = \text{pPol}_k A$. Let $L := \{A, E_k\}$. Since $\varrho = A \times E_k$ we have $\text{pPol}_k L \subseteq \text{pPol}_k \varrho$.
 - If $\varrho = \kappa_{\{2\}} \chi = \lambda_{A \Rightarrow E_k}$ then set $L = \{\lambda_{A \Rightarrow E_k}\}$.
 - If $\varrho = \kappa_{\{1\}} \chi = \tau \kappa_{\{2\}} \tau \chi = \tau \lambda_{E_k \Rightarrow A}$ then set $L = \{\lambda_{E_k \Rightarrow A}\}$.
 - If $\varrho = \kappa_{\{1,2\}} \chi$ then set $L = \{\lambda_{A \Rightarrow E_k}, \lambda_{E_k \Rightarrow A}\}$. By Lemma 4.2.6 we have $\text{pPol}_k \varrho = \text{pPol}_k L$.

We obtain $\text{pPol}_k \varrho = \text{pPol}_k L$.

- $h \geq 3$.
If $I_\infty(\varrho) = \emptyset$ then $\varrho = \varrho \cap E_k^h$ and thus $C \subseteq \text{pPol}_k \text{pr}_h \varrho = \text{pPol}_k E_k = P_k$.
If $C \subseteq P_k$ then let $\varrho' := \text{pr}_{1,\dots,h-1}(\varrho \cap E_k^h)$. Since $C = \text{pPol}_k \{\varrho, E_k\} = \text{pPol}_k \{\varrho \cap E_k^h, E_k\} \subseteq \text{pPol}_k \{\varrho', E_k\}$ and $\varrho \cap E_k^h = \varrho' \times E_k$ we can set $L := \{\varrho', E_k\}$ and obtain $C = \text{pPol}_k L$.

We can now assume $C \setminus P_k \neq \emptyset$ and consequently $I_\infty(\varrho) \neq \emptyset$.

Let $\varrho' := (\kappa \text{pr}_{1,\dots,h-1}(\varrho \cap E_k^h)) \cap (\text{pr}_{1,\dots,h-1} \varrho) \in \tilde{\mathcal{R}}_k^*$.

We first show $C \subseteq \text{pPol}_k \varrho'$. Let $f^{(n)} \in C$ and $r_{*1}, \dots, r_{*n} \in \varrho' \cap E_k^{h-1}$ be arbitrary. Let $r_{*0} := f(r_{*1}, \dots, r_{*n})$.

We have $r_{*0} \in \text{pr}_{1,\dots,h-1} \varrho$ since $r_{*i} \in (\kappa \text{pr}_{1,\dots,h-1}(\varrho \cap E_k^h)) \cap E_k^{h-1} = \text{pr}_{1,\dots,h-1}(\varrho \cap E_k^h) \subseteq \text{pr}_{1,\dots,h-1} \varrho$ and $f \in C \subseteq \text{pPol}_k(\text{pr}_{1,\dots,h-1} \varrho)$.

If $r_{*0} \notin E_k^{h-1}$ then $r_{*0} \in \kappa \text{pr}_{1,\dots,h-1}(\varrho \cap E_k^h)$ since $(\kappa \text{pr}_{1,\dots,h-1}(\varrho \cap E_k^h)) \setminus E_k^{h-1} = \tilde{E}_k^{h-1} \setminus E_k^{h-1}$ by definition of κ . Thus $r_{*0} \in \varrho'$.

If $r_{*0} \in E_k^{h-1}$ then $r_{1i} \in E_k$ for all $i \in \{1, \dots, n\}$ and

$$f \begin{pmatrix} r_{*1} & \cdots & r_{*n} \\ r_{11} & \cdots & r_{1n} \end{pmatrix} \in \varrho \cap E_k^h$$

since $f \in C$. Thus $r_{*0} \in \text{pr}_{1, \dots, h-1}(\varrho \cap E_k^h) \subseteq \kappa \text{pr}_{1, \dots, h-1}(\varrho \cap E_k^h)$ implying $r_{*0} \in \varrho'$.

Thus $C \subseteq \text{pPol}_k \varrho'$.

We want to show the following property of ϱ :

$$\forall I \in I_\infty^{\min}(\varrho) : (h \notin I \text{ or } \{h\} = I \text{ or } I = \{1, \dots, h\}). \quad (4.2)$$

We assume otherwise. Then there is some $I \in I_\infty^{\min}(\varrho)$ with $h \in I$, $I \neq \{h\}$ and $I \neq \{1, \dots, h\}$. By Lemma 4.1.5 we can assume that there are $f^{(n)} \in C$, $r_{*0} \in \varrho$, $r_{*1}, \dots, r_{*n} \in \varrho \cap E_k^h$ with $r_{*0} = f(r_{*1}, \dots, r_{*n})$ and $r_{i0} = \infty$ iff $i \in I$. Since $I \neq \{1, \dots, h\}$ there is some $j \in \{1, \dots, h\} \setminus I$ with $r_{j0} \neq \infty$. Then $j \neq h$ and

$$s := \begin{pmatrix} s_1 \\ \cdots \\ s_{h-1} \\ s_h \end{pmatrix} := f \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \cdots & \cdots & \cdots \\ r_{h-1,1} & \cdots & r_{h-1,n} \\ r_{j,1} & \cdots & r_{j,n} \end{pmatrix} \in \varrho.$$

Since $s_h = s_j \neq \infty$ and $s_i = r_{i0}$ for all $i \leq h-1$ we have $s_i = \infty$ iff $i \in I \setminus \{h\}$. But then $I \setminus \{h\} \in I_\infty(\varrho)$ in contradiction to $I \in I_\infty^{\min}(\varrho)$. Thus (4.2) holds.

Now there are the following cases.

$$- I := \{1, \dots, h\} \in I_\infty^{\min}(\varrho).$$

Then $I_\infty(\varrho) = \{I\}$ and

$$\varrho = (\varrho \cap E_k^h) \cup \{\{\infty\}^h\}.$$

With Lemmas 4.2.6 and 4.2.4 we get

$$\begin{aligned} C &\subseteq \text{pPol}_k(\text{pr}_{1,h} \varrho) = \text{pPol}_k\{\lambda_{\text{pr}_1 \varrho \Rightarrow E_k}, \lambda_{E_k \Rightarrow \text{pr}_1 \varrho}\} \\ &\subseteq \text{pPol}_k \lambda_{E_k \Rightarrow E_k} = P_k \cup C_\infty. \end{aligned}$$

Furthermore we have $\varrho = \varrho' \circ (\kappa_{\{1,2\}} E_k^2)$ and $\kappa_{\{1,2\}} E_k^2 = \lambda_{E_k \Rightarrow E_k} \cap \tau \lambda_{E_k \Rightarrow E_k}$. Let $L := \{\varrho', \lambda_{E_k \Rightarrow E_k}\}$. Then follows $C = \text{pPol}_k L$.

$$- \{h\} \in I_\infty^{\min}(\varrho) \text{ and } \{1, \dots, h\} \notin I_\infty^{\min}(\varrho).$$

Then $\varrho = \varrho' \times E_k^h$ since for every $I \in I_\infty^{\min}(\varrho)$ with $I \neq \{h\}$ we have $h \notin I$. Thus $\text{pPol}_k \varrho' \subseteq \text{pPol}_k \varrho = C$ and this implies $C = \text{pPol}_k L$ with $L := \{\varrho'\}$.

$$- \{h\}, \{1, \dots, h\} \notin I_\infty^{\min}(\varrho) \text{ and } I^* := \bigcap I_\infty^{\min}(\varrho) \neq \emptyset.$$

Let $i \in I^*$ be arbitrary, w.l.o.g. $i = h-1$. Then $C \subseteq \text{pPol}_k \lambda_{E_k \Rightarrow \text{pr}_i \varrho}$ and $\varrho = \varrho' \circ (\tau \lambda_{E_k \Rightarrow \text{pr}_i \varrho})$. Thus $C = \text{pPol}_k L$ with $L := \{\varrho', \lambda_{E_k \Rightarrow \text{pr}_i \varrho}\}$.

$$- \text{Let } \{h\}, \{1, \dots, h\} \notin I_\infty^{\min}(\varrho) \text{ and } \bigcap I_\infty^{\min}(\varrho) = \emptyset. \text{ Then let } \chi := \varrho' \cap E_k^{h-1} \text{ and } L := \{\varrho', \lambda_\chi^*\}. \text{ Since } \lambda_\chi^* \cap E_k^h = \varrho \cap E_k^h \text{ and } \varrho \subseteq \lambda_\chi^* \text{ we get } C \subseteq \text{pPol}_k \lambda_\chi^* \text{ by Lemma 4.1.2 and therefore } C \subseteq \text{pPol}_k L. \text{ Furthermore } \varrho = (\varrho' \times \tilde{E}_k) \cap \lambda_\chi^* \text{ implies } \text{pPol}_k L \subseteq C \text{ and we get } C = \text{pPol}_k L. \quad \square$$

Lemma 4.3.4. *Let $\chi^{(\mu)} \in \mathcal{R}_k$ and the μ -th row is fictious. Then*

- $\text{pPol}_k \lambda_\chi^* = \text{pPol}_k \lambda_{\text{pr}_{1, \dots, \mu-1} \chi}^*$, or
- $\mu = 1$ and $\lambda_\chi^* = \lambda_{E_k \Rightarrow E_k}$.

Proof. If $\mu = 1$ then $\chi = E_k$ since the first row is fictious. Thus $\lambda_\chi^* = \lambda_{E_k \Rightarrow E_k}$.

Let $\mu \geq 2$. Then the $(\mu+1)$ -th row of $\lambda_\chi^* \cap E_k^{\mu+1}$ is fictious and $\{\mu+1\} \in I_\infty^{\min}(\lambda_\chi^*)$. Applying Lemma 4.3.3 and the case distinction in that proof yields $L = \{\varrho'\} = \{(\kappa \text{pr}_{1,\dots,\mu}(\lambda_\chi^* \cap E_k^{\mu+1})) \cap (\text{pr}_{1,\dots,\mu} \lambda_\chi^*)\} = \{\lambda_{\text{pr}_{1,\dots,\mu-1} \chi}^*\}$ and $\text{pPol}_k \lambda_\chi^* = \text{pPol}_k L = \text{pPol}_k \lambda_{\text{pr}_{1,\dots,\mu-1} \chi}^*$. \square

Lemma 4.3.5. *Let $\varrho^{(h)} \in \widetilde{\mathcal{R}}_k^*$ such that $\varrho \cap E_k^h$ has two duplicate rows i and j with $i < j$.*

Then

$$\text{pPol}_k \varrho = \text{pPol}_k(\Delta_{i,j} \varrho).$$

Proof. Let $\chi^{(h-1)} = \Delta_{i,j} \varrho$ and $\varrho' := (\nabla_j \chi) \cap \widetilde{\delta}_{\{i,j\}}^{(h)}$. We have $\text{pPol}_k \varrho \subseteq \text{pPol}_k \chi \subseteq \text{pPol}_k \varrho'$ and $\varrho' = \varrho \cap \widetilde{\delta}_{\{i,j\}}$ by construction. Since the rows i and j are duplicate we have $\text{pPol}_k \varrho = \text{pPol}_k(\varrho \cap \widetilde{\delta}_{\{i,j\}})$ by Lemma 4.1.2. Thus $\text{pPol}_k \chi = \text{pPol}_k \varrho$. \square

Lemma 4.3.6. *Let $\lambda_\chi^* \in \Lambda_k \setminus \Lambda_k^*$ for some $\chi^{(\mu)} \in \mathcal{R}_k$. Then there is some $\chi' \in \mathcal{R}_k$ with*

$$\text{pPol}_k \lambda_{\chi'}^* = \text{pPol}_k \lambda_\chi^*$$

and $\lambda_{\chi'}^ \in \Lambda_k^*$.*

Proof. If $\mu = 1$ and χ has a fictious row then $\chi = E_k$ and $\lambda_\chi^* = \lambda_{E_k \Rightarrow E_k} \in \Lambda_k^*$ in contradiction to the assumption. Thus we can assume $\mu \geq 2$.

If χ has duplicate rows then Lemma 4.3.5 applies to λ_χ^* and $\text{pPol}_k \lambda_\chi^* = \text{pPol}_k \lambda_{\Delta_{i,j} \chi}^*$ for some i and j . Let $\chi' := \Delta_{i,j} \chi$.

If $\mu \geq 2$ and χ has a fictious row, w.l.o.g. the μ -th row is fictious, then Lemma 4.3.4 applies to λ_χ^* and $\text{pPol}_k \lambda_\chi^* = \text{pPol}_k \lambda_{\text{pr}_{1,\dots,\mu-1} \chi}^*$. Let $\chi' := \text{pr}_{1,\dots,\mu-1} \chi$.

If χ is not irredundant then $\text{pPol}_k \lambda_{\chi'}^* = \text{pPol}_k \lambda_\chi^*$, and χ' has smaller arity than χ .

Iterating these steps yields some χ' with χ' irredundant or $\chi' = E_k$. Then $\lambda_{\chi'}^* \in \Lambda_k^*$ and $\text{pPol}_k \lambda_{\chi'}^* = \text{pPol}_k \lambda_\chi^*$. \square

Theorem 4.3.7. *Let $\varrho^{(h)} \in \widetilde{\mathcal{R}}_k^*$ and $C := \text{pPol}_k \varrho$.*

Then there is some $\chi^{(\mu)} \in \widetilde{\mathcal{R}}_k^$ and some*

$$L \subseteq \{\chi, E_k\} \cup \Lambda_k^*$$

with $\mu \leq h$, such that $\chi \cap E_k^\mu$ is irredundant and

$$C = \text{pPol}_k L.$$

Proof. Let $\varrho_0 := \varrho$ and $L_0 := \{\varrho_0\}$. Then $C = \text{pPol}_k L_0$.

Let $L_i \subseteq \{\varrho_i, E_k\} \cup \Lambda_k^*$ for some $i \geq 0$ be given. We show that L_i is the L we search for or there is some L_{i+1} with $\text{pPol}_k L_{i+1} = \text{pPol}_k L_i$ and some $\varrho_{i+1} \in \widetilde{\mathcal{R}}_k^*$ such that $L_{i+1} \subseteq \{\varrho_{i+1}, E_k\} \cup \Lambda_k^*$.

- (a) $\varrho_i^{(h_i)} \cap E_k^{h_i}$ is irredundant. Then let $L := L_i$, $\chi := \varrho_i$ and we are done.
- (b) $\varrho_i^{(h_i)} \cap E_k^{h_i}$ has a fictious row. Then Lemma 4.3.3 applies and we get some $L' \subseteq \{\varrho', E_k\} \cup \Lambda_k$ with $\text{pPol}_k \varrho_i = \text{pPol}_k L'$, $\varrho' = \bar{\kappa} \varrho'$ and the arity of ϱ' is less than the arity of ϱ . Because of Lemma 4.3.6 we can assume $L' \subseteq \{\varrho', E_k\} \cup \Lambda_k^*$.

Let $\varrho_{i+1} := \varrho'$ and $L_{i+1} := L_i \setminus \{\varrho_i\} \cup L' \subseteq \{\varrho_{i+1}, E_k\} \cup \Lambda_k^*$. Then $\text{pPol}_k L_{i+1} = \text{pPol}_k L_i$.

(c) Otherwise $\varrho_i^{(h_i)} \cap E_k^{h_i}$ has duplicate rows. Then Lemma 4.3.5 applies and we get with $\text{pPol}_k \varrho_i = \text{pPol}_k \Delta_{a,b} \varrho_i$ for some a, b .

Let $\varrho_{i+1} := \Delta_{a,b} \varrho_i$ and $L_{i+1} := (L_i \setminus \{\varrho_i\}) \cup \{\varrho_{i+1}\}$. Then $\text{pPol}_k L_{i+1} = \text{pPol}_k L_i$.

Since $h_{i+1} < h_i$ as long as $\varrho_i \cap E_k^{h_i}$ is not irredundant, and h is finite, there is some finite l with $\varrho_l \cap E_k^{h_l}$ irredundant. Let $\chi := \varrho_l$ and $L := L_l$. Then $\text{pPol}_k \varrho = \text{pPol}_k L$ and χ has all required properties by construction. \square

4.4 Implications

Lemma 4.4.1. *Let $\varrho^{(h)} \in \mathcal{R}_k$. Then there is some irredundant relation $\chi^{(\mu)} \in \mathcal{R}_k$ with $\text{pPOL}_k \varrho = \text{pPOL}_k \chi$ or $\text{pPOL}_k \varrho = \tilde{P}_k$.*

Proof. If ϱ has two duplicate rows i and j then we set $\chi^{(h-1)} = \Delta_{i,j} \varrho$. Then $\kappa\chi = \Delta_{i,j} \kappa\varrho \in [\{\kappa\varrho\}]_{\text{P}}$ and $\kappa\varrho = \kappa\sigma_{(j-1 \dots i+1)} \nu_i(\kappa\chi) \in [\{\kappa\chi\}]_{\text{P}}$. Thus $\text{pPOL}_k \chi = \text{pPOL}_k \varrho$ and χ has smaller arity than ϱ .

If $h = 1$ and ϱ has a fictious coordinate then $\varrho = E_k$ and thus $\text{pPOL}_k \varrho = \text{pPol}_k \kappa E_k = \text{pPol}_k \tilde{E}_k = \tilde{P}_k$. Let $h \geq 2$. If ϱ has a fictious coordinate i then we set $\chi^{(h-1)} = \Delta_{i,j} \varrho$ for some $j \neq i$. Then $\kappa\chi = \Delta_{i,j} \kappa\varrho \in [\{\kappa\varrho\}]_{\text{P}}$ and $\kappa\varrho = \kappa\nabla_i(\kappa\chi) \in [\{\kappa\chi\}]_{\text{P}}$. Thus $\text{pPOL}_k \chi = \text{pPOL}_k \varrho$ and χ has smaller arity than ϱ .

Iterating these steps yields an irredundant relation χ or $\text{pPOL}_k \varrho = \tilde{P}_k$ since h is finite. \square

The following result by Romov can now be proven

Lemma 4.4.2 ([52]). *Let $C \subset \tilde{P}_k$ be a strong partial clone. Then there is a certain non-empty set $M \subseteq \mathcal{R}_k$ of irredundant relations with $C = \bigcap_{\varrho \in M} \text{pPOL}_k \varrho$.*

Proof. By Theorem 3.9.1 (d) we have $C = \bigcap_{n \geq 1} \text{pPol}_k G_n(C)$.

Let $n \geq 1$ and $J \subseteq \{1, \dots, k^n\}$, $I \subseteq \{1, \dots, |J|\}$ be arbitrary. Since $G_n(C) \cap E_k^{k^n} \neq \emptyset$, $e_1^{(1)} = \text{id} \in C$ and C is strong, there is some $(r_1, \dots, r_{|J|}) \in \text{pr}_J G_n(C)$ with $r_i = \infty$ iff $i \in I$. Thus

$$\bar{\kappa} \text{pr}_J G_n(C) = \kappa \text{pr}_J G_n(C) \quad (4.3)$$

Let $T_n := \{\text{pr}_J G_n(C) \mid \emptyset \subset J \subseteq \{1, \dots, k^n\}\}$, $N_n := \{\chi \cap E_k^\mu \mid \chi^{(\mu)} \in T_n\}$ and $M_n = \{\chi \in \mathcal{R}_k \mid \chi \text{ irredundant, } \varrho \in N_n, \text{pPOL}_k \chi = \text{pPOL}_k \varrho \neq \tilde{P}_k\}$

Then

$$\begin{aligned} \text{pPol}_k G_n(C) &= \text{pPol}_k \{\bar{\kappa}\chi \mid \chi \in T_n\} \text{ (by Theorem 4.3.1)} \\ &= \text{pPol}_k \{\kappa\chi \mid \chi \in T_n\} \text{ (by (4.3))} \\ &= \text{pPOL}_k N_n \text{ (by definition of pPOL}_k \text{ and } \kappa) \\ &= \text{pPOL}_k M_n \text{ (by Lemma 4.4.1)} \end{aligned}$$

Let $M = \bigcup_{n \geq 1} M_n$. Then

$$C = \bigcap_{n \geq 1} \bigcap_{\varrho \in M_n} \text{pPOL}_k \varrho = \bigcap_{\varrho \in M} \text{pPOL}_k \varrho. \quad \square$$

4.5 Representation Lemma of Romov

Let

$$\Delta' \varrho := \{(x_1, x_2, \dots, x_h) \in \varrho \mid x_1 = x_2\}$$

and

$$\Delta'_I \varrho := \{(x_1, x_2, \dots, x_h) \in \varrho \mid \forall \{i, j\} \subseteq I : x_i = x_j\}$$

for $\varrho \in \widetilde{\mathcal{R}}_k^{(h)}$ and $I \subseteq \{1, \dots, h\}$ with $|I| \geq 2$. Obviously $\Delta' = \Delta'_{1,2} := \Delta'_{\{1,2\}}$ and Δ'_I are derivable from the elementary operations. For completeness $\Delta_{\{i\}} \varrho := \varrho$ for every i .

Furthermore we define a special projection $\widehat{\text{pr}}_i$ which removes the i -th row by

$$\widehat{\text{pr}}_i \varrho^{(h)} := \text{pr}_{1, \dots, i-1, i+1, \dots, h} \varrho = \zeta^{h-i} \text{pr} \zeta^{i-1} \varrho.$$

In the following lemma the variables $i, j \in \mathbb{N}$ and $I, J \subset \mathbb{N}$ are arbitrary for each expression. If there is some I' in an expression then this means it can be chosen depending on I such that equality holds. The roles of I and I' can also be exchanged, i.e., I' freely chosen, and I depending on I' .

Lemma 4.5.1 (without proof). *Let $\varrho \in \widetilde{\mathcal{R}}_k^{(h)}$ and $\varrho' \in \widetilde{\mathcal{R}}_k^{(h')}$. Then the following holds*

- (a) $\varrho \times \varrho' = (\zeta^{h'} \nabla^{h'} \varrho) \wedge (\nabla^{h'} \varrho')$;
- (b) $\emptyset = \text{pr}^h \varrho$,
 $\delta_{k; \{1,2\}}^{(3)} = \Delta' \nabla^3 \emptyset$;
- (c) $\zeta(\text{pr} \varrho) = \text{pr}(\zeta(\tau(\varrho)))$,
 $\tau(\text{pr} \varrho) = \text{pr}(\zeta^{h-1}(\tau(\zeta(\varrho))))$,
 $\Delta'(\text{pr} \varrho) = \text{pr}(\zeta^{h-1}(\Delta'(\zeta(\varrho))))$,
 $\nabla(\text{pr} \varrho) = \text{pr}(\tau(\nabla(\varrho)))$;
- (d) if $h = h'$ then $\alpha(\varrho \wedge \varrho') = (\alpha\varrho) \wedge (\alpha\varrho')$ for all $\alpha \in \{\zeta, \tau, \Delta', \nabla, \text{pr}, \widehat{\kappa}, \kappa\}$;
- (e) $\Delta'(\widehat{\kappa}\varrho) = \widehat{\kappa}(\Delta'\varrho)$,
 $\widehat{\kappa}(\nabla(\widehat{\kappa}\varrho)) = \widehat{\kappa}(\nabla\varrho)$;
- (f) $\alpha(\kappa\varrho) = \kappa(\alpha\varrho)$ for $\alpha \in \{\zeta, \tau\}$,
 $\kappa(\Delta'(\kappa\varrho)) = \kappa(\Delta'\varrho)$,
 $\kappa(\nabla(\kappa\varrho)) = \kappa(\nabla\varrho)$;
- (g) $\text{pr}(\kappa\varrho) = \text{pr}(\kappa_{\{1\}}\varrho) = \widetilde{E}_k^{h-1} = \nabla^{h-1}(\text{pr}^h \varrho)$;
- (h) $\nabla_i \nabla_j \varrho = \nabla_{j-1} \nabla_i \varrho$ for $i < j$,
 $\nabla_i \nabla_{i+1} \varrho = \nabla_i^2 \varrho$,
 $\nabla_i \Delta'_I \varrho = \Delta'_{I'} \nabla_i \varrho$ if $i \notin I'$,
 $\Delta'_I \Delta'_J \varrho = \Delta'_J \Delta'_I \varrho$,
 $\Delta'_I \Delta'_J \varrho = \Delta'_{I \cup J} \varrho$ if $I \cap J \neq \emptyset$;
- (i) $\widehat{\text{pr}}_g \nabla_i \varrho = \begin{cases} \nabla_{i-1} \widehat{\text{pr}}_g \varrho & \text{if } g < i, \\ \varrho & \text{if } g = i, \\ \nabla_i \widehat{\text{pr}}_g \varrho & \text{if } g > i; \end{cases}$
- (j) $\widehat{\text{pr}}_i \Delta'_I \nabla_j \varrho = (\Delta'_{I'} \varrho)^{[(j-1 \dots i)]}$ if $\{i, j\} \subseteq I$,
 $\widehat{\text{pr}}_i \Delta'_I \varrho = \Delta'_{I'} \widehat{\text{pr}}_i \varrho$ if $i \notin I$;

- (k) $\widehat{\text{pr}}_i \zeta \varrho = \zeta \widehat{\text{pr}}_{i+1} \varrho,$
 $\widehat{\text{pr}}_1 \tau \varrho = \tau \widehat{\text{pr}}_2 \varrho,$
 $\widehat{\text{pr}}_2 \tau \varrho = \tau \widehat{\text{pr}}_1 \varrho,$
 $\widehat{\text{pr}}_i \tau \varrho = \tau \widehat{\text{pr}}_i \varrho$ for $i > 2;$
- (l) $\Delta'_I \zeta \varrho = \zeta \Delta'_I \varrho,$
 $\Delta'_I \tau \varrho = \tau \Delta'_I \varrho;$
- (m) $\nabla \zeta \varrho = \zeta \tau \nabla \varrho,$
 $\nabla \tau \varrho \in [\nabla \varrho]_{\zeta, \tau}.$

Theorem 4.5.2 (without proof). *Let $Q \subseteq \widetilde{\mathcal{R}}_k$. Then the following holds*

- (a) $[Q]_{\text{P}} = [Q]_{\zeta, \tau, \Delta', \nabla, \text{pr}, \widehat{\kappa}, \wedge};$
- (b) $[Q]_{\zeta, \tau, \Delta', \nabla, \text{pr}, \widehat{\kappa}, \wedge} = \left[[Q]_{\zeta, \tau, \Delta', \nabla, \text{pr}, \widehat{\kappa}} \right]_{\wedge}.$

Lemma 4.5.3. *Let $\alpha_i \in \mathcal{R}_k^{(h_i)}$ for $i = 1, \dots, n$ and $\beta \in \mathcal{R}_k^t$. Then*

$$\bigcap_{i=1}^n \text{pPOL}_k \alpha_i \subseteq \text{pPOL}_k \beta$$

if and only if there are $\gamma_1, \dots, \gamma_l \in \bigcup_{i=1}^n [\kappa \alpha_i]_{\zeta, \tau, \Delta', \nabla, \text{pr}, \widehat{\kappa}}$ for which

$$\kappa \beta = \bigcap_{j=1}^l \kappa \gamma_j.$$

Proof. Let $Q := \{\kappa \alpha_i \mid i \in \{1, \dots, n\}\}$. Since $\text{pPOL}_k \varrho = \text{pPol}_k(\kappa \varrho)$ for all ϱ we have

$$\bigcap_{i=1}^n \text{pPOL}_k \alpha_i = \bigcap_{i=1}^n \text{pPol}_k(\kappa \alpha_i) = \text{pPol}_k Q$$

and

$$\text{pPOL}_k \beta = \text{pPol}_k(\kappa \beta).$$

Thus

$$\begin{aligned} & \bigcap_{i=1}^n \text{pPOL}_k \alpha_i \subseteq \text{pPOL}_k \beta \\ \iff & \kappa \beta \in [Q]_{\text{P}} \\ \iff & \kappa \beta \in \left[[Q]_{\zeta, \tau, \Delta', \nabla, \text{pr}, \widehat{\kappa}} \right]_{\wedge} \\ \iff & \exists \gamma_1, \dots, \gamma_l \in [Q]_{\zeta, \tau, \Delta', \nabla, \text{pr}, \widehat{\kappa}} : \kappa \beta = \bigcap_{j=1}^l \gamma_j \\ \iff & \exists \gamma_1, \dots, \gamma_l \in \bigcup_{i=1}^n [\kappa \alpha_i]_{\zeta, \tau, \Delta', \nabla, \text{pr}, \widehat{\kappa}} : \kappa \beta = \bigcap_{j=1}^l \gamma_j \end{aligned}$$

The statement follows from $\kappa \beta = \kappa \kappa \beta = \kappa \bigcap_{j=1}^l \gamma_j = \bigcap_{j=1}^l \kappa \gamma_j$. \square

Let $h \geq 1$ and $I \subseteq \{1, \dots, h\}$ be arbitrary. Then we define

$$L_I^{(h)} := \left\{ (a_1, \dots, a_h) \in \widetilde{E}_k^h \mid \forall i \in I : a_i = \infty \right\}.$$

Then $\kappa_I \varrho^{(h)} = \varrho^{(h)} \cup L_I^{(h)}$. If $I = \emptyset$ then $L_I^{(h)} = \tilde{E}_k^h$.

Lemma 4.5.4. *Let $\varrho^{(h)} \in \tilde{\mathcal{R}}_k$, $\chi^{(\mu)} \in \tilde{\mathcal{R}}_k^*$, $\omega_1, \dots, \omega_l \in \{\zeta, \tau, \Delta', \nabla, \text{pr}\}$ and*

$$\chi = \omega_1 \dots \omega_l \hat{\kappa} \varrho.$$

Then there is some $I \subseteq \{1, \dots, \mu\}$ such that $\chi \in \text{pInv } \tilde{P}_k$ or

$$\chi = \kappa_I \omega_1 \dots \omega_l \varrho.$$

Proof.

$$\begin{aligned} \chi &= \omega_1 \dots \omega_l \hat{\kappa} \varrho \\ &= \omega_1 \dots \omega_l (\varrho \cup L_{\{1,2\}}^{(h)}) \\ &= (\omega_1 \dots \omega_l \varrho) \cup (\omega_1 \dots \omega_l L_{\{1,2\}}^{(h)}) \\ &= (\omega_1 \dots \omega_l \varrho) \cup L' \end{aligned}$$

with $L' \subseteq \tilde{E}_k^\mu$. If $L'' := L' \cap E_k^\mu \neq \emptyset$ then $L'' = \delta_\varepsilon^{(\mu)}$ for some equivalence relation ε and $(\omega_1 \dots \omega_l \varrho) \cap E_k^\mu \subseteq L''$ by construction since $\omega_1, \dots, \omega_l$ are applied in the same way to ϱ and $L_{\{1,2\}}^{(h)}$, i.e., if two rows in L'' are duplicate rows, then these rows are also duplicate rows in $(\omega_1 \dots \omega_l \varrho) \cap E_k^\mu$. Thus $\chi = L' = \tilde{\delta}_\varepsilon^{(\mu)} \in \text{pInv } \tilde{P}_k$ since $\chi \in \tilde{\mathcal{R}}_k^*$.

If $L'' = \emptyset$ then $L' = L_I^{(\mu)}$ for some $I \subseteq \{1, \dots, \mu\}$ and $|I| \neq \emptyset$ since $\chi \in \tilde{\mathcal{R}}_k^*$. Thus $\chi = \kappa_I \omega_1 \dots \omega_l \varrho$. \square

Lemma 4.5.5. *Let $\varrho^{(h)} \in \tilde{\mathcal{R}}_k$, $\chi^{(\mu)} \in \tilde{\mathcal{R}}_k^*$, and*

$$\chi \in [\zeta, \tau, \Delta', \nabla, \text{pr}, \hat{\kappa}] (\varrho)$$

Then $\chi \in \text{pInv } \tilde{P}_k$ or there are

$$\chi' \in [\zeta, \tau, \Delta', \nabla, \text{pr}] (\varrho)$$

and $I_1, \dots, I_m \subseteq \{1, \dots, \mu\}$ such that

$$\chi = \kappa_{I_1} \dots \kappa_{I_m} \chi'.$$

Proof. We can write

$$\chi = \theta_1 \hat{\kappa} \theta_2 \hat{\kappa} \dots \theta_m \hat{\kappa} \theta_{m+1} \varrho$$

with $m \geq 0$, $\theta_i = \omega_{i,1} \dots \omega_{i,l(i)}$ and $\{\omega_{i,1}, \dots, \omega_{i,l(i)}\} \subseteq \{\zeta, \tau, \Delta', \nabla, \text{pr}\}$. Then

$$\begin{aligned} \chi &= (\theta_1 \theta_2 \dots \theta_{m+1} \varrho) \cup \bigcup_{j=1}^m (\theta_1 \theta_2 \dots \theta_j L_{\{1,2\}}^{(h_j)}) \\ &= (\theta_1 \theta_2 \dots \theta_{m+1} \varrho) \cup \bigcup_{j=1}^m L'_j \end{aligned}$$

for some $h_j \geq 1$ and L'_j . Exactly like in the proof of the previous lemma we can conclude that either $\chi \in \text{pInv } \tilde{P}_k$ or for all j there is some $I_j \subseteq \{1, \dots, \mu\}$ with $I_j \neq \emptyset$ and $L'_j = L_{I_j}^{(\mu)}$.

Thus we have $\chi \in \text{pInv } \tilde{P}_k$ or

$$\chi = \kappa_{I_1} \dots \kappa_{I_m} \chi'$$

with $\chi' = \theta_1 \theta_2 \dots \theta_{m+1} \varrho \in [\zeta, \tau, \Delta', \nabla, \text{pr}] (\varrho)$. \square

Theorem 4.5.6. *Let $Q \subseteq \tilde{\mathcal{R}}_k$ and $\varrho^{(h)} \in \tilde{\mathcal{R}}_k^*$. Then*

$$\varrho \in [Q]_{\mathbb{P}}$$

if and only if there are $\gamma_1, \dots, \gamma_l \in [Q]_{\zeta, \tau, \Delta', \nabla, \text{pr}}$ and $I_1, \dots, I_m \subseteq \{1, \dots, h\}$ with $m \geq 0$ such that

$$\varrho = \kappa_{I_1} \dots \kappa_{I_m} \bigcap_{j=1}^l \gamma_j$$

holds.

Proof. \Leftarrow : Since κ_I is derived from the elementary operations we know that this direction holds.

\Rightarrow : Let $\varrho \in [Q]_{\mathbb{P}}$. Then

$$\varrho = \bigcap_{j=1}^l \gamma'_j$$

for some $\gamma'_1, \dots, \gamma'_l \in [Q]_{\zeta, \tau, \Delta', \nabla, \text{pr}, \hat{\kappa}}$ by Theorem 4.5.2.

Therefore Lemma 4.5.5 applies and either $\gamma'_j = \kappa_{I_j^1} \dots \kappa_{I_j^{i(j)}} \gamma_j$ or $\gamma'_j = \gamma_j$ holds for some $\gamma_j \in [Q]_{\zeta, \tau, \Delta', \nabla, \text{pr}}$, $i(j) \geq 0$ and $I_j^i \subseteq \{1, \dots, h\}$. Thus

$$\varrho = \bigcap_{j=1}^l \kappa_{I_j^1} \dots \kappa_{I_j^{i(j)}} \gamma_j = \bigcap_{j=1}^l \left(\gamma_j \cup \bigcup_{i=1}^{i(j)} L_i \right) = \left(\bigcap_{j=1}^l \gamma_j \right) \cup L'$$

for some $L' := \subseteq \tilde{E}_k^h \setminus E_k^h$. Since $\varrho \in \tilde{\mathcal{R}}_k^*$ we have $L_I \subseteq \varrho$ for every $I \subseteq \{1, \dots, h\}$ where there is $(a_1, \dots, a_h) \in L'$ with $a_i = \infty$ iff $i \in I$.

Thus there are some I_1, \dots, I_m with

$$\varrho = \bigcap_{j=1}^l \kappa_{I_j} \gamma_j = \left(\bigcap_{j=1}^l \gamma_j \right) \cup L_{I_1} \cup \dots \cup L_{I_m} = \kappa_{I_1} \dots \kappa_{I_m} \bigcap_{j=1}^l \gamma_j. \quad \square$$

Lemma 4.5.7. *Let $\varrho^{(h)} \in \tilde{\mathcal{R}}_k$, $\omega_1, \dots, \omega_l \in \{\zeta, \tau, \Delta', \nabla, \text{pr}\}$ and $\chi^{(\mu)} := \kappa_{\omega_1} \dots \omega_l \hat{\kappa} \varrho$.*

Then $\chi \in \text{pInv}_k \tilde{P}_k$ or $\chi = \kappa_{\omega_1} \dots \omega_l \varrho$.

Proof. Assume $\chi \notin \text{pInv}_k \tilde{P}_k$. Then Lemma 4.5.4 applies and

$$\chi = \kappa_{\omega_1} \dots \omega_l \hat{\kappa} \varrho = \kappa_{\omega_1} \dots \omega_l \varrho. \quad \square$$

Lemma 4.5.8. *Let $\alpha \in \mathcal{R}_k^{(h)}$, $\gamma \in \mathcal{R}_k^{(t)} \setminus (\text{Inv}_k P_k)$ and γ has no duplicate rows. Then*

$$\kappa \gamma \in \left[[\kappa \alpha]_{\zeta, \tau, \Delta', \nabla, \text{pr}, \hat{\kappa}} \right]_{\kappa} \quad (4.4)$$

if and only if there is some $\varphi : \{1, \dots, h\} \rightarrow \{1, \dots, t\}$ with

$$\gamma = \{(g_1, \dots, g_t) \in E_k^t \mid (g_{\varphi(1)}, \dots, g_{\varphi(h)}) \in \alpha\}. \quad (4.5)$$

Proof. First let there be some $\varphi : \{1, \dots, h\} \rightarrow \{1, \dots, t\}$ such that (4.5) is true.

For every $\varrho \in \widetilde{\mathcal{R}}_k$ and $I := \{i_1, \dots, i_l\}$ with $i_1 < i_2 < \dots < i_l$ we have $\Delta_I \varrho = \widehat{\text{pr}}_{i_2} \dots \widehat{\text{pr}}_{i_l} \Delta'_I \varrho$. Thus

$$\kappa\gamma = \kappa \left(\nabla^{t-|\varphi(\{1, \dots, h\})|} \Delta_{\varphi^{-1}(1)} \Delta_{\varphi^{-1}(2)} \dots \Delta_{\varphi^{-1}(t)} \kappa\alpha \right)^{[\pi]} \in \left[[\kappa\alpha]_{\zeta, \tau, \Delta', \nabla, \text{pr}, \widehat{\kappa}} \right]_{\kappa}$$

for some $\pi \in S_t$, i.e., (4.4) is true.

Now let (4.4) be true. Then

$$\kappa\gamma = \kappa \widehat{\omega}_1 \widehat{\omega}_2 \dots \widehat{\omega}_l \kappa\alpha$$

with $\widehat{\omega}_i \in \{\zeta, \tau, \Delta', \nabla, \text{pr}, \widehat{\kappa}\}$ for all $i \in \{1, \dots, l\}$.

By Lemma 4.5.7 we can assume $\widehat{\omega}_i \neq \widehat{\kappa}$ for all $i \in \{1, \dots, l\}$ since $\kappa\gamma \in \text{pInv}_k \widetilde{P}_k$ implies $\gamma \in \text{Inv}_k P_k$. By Lemma 4.5.1 (f), (k) – (m) we can move ζ and τ to the front and obtain

$$\kappa\gamma = \omega'_1 \dots \omega'_l \kappa\omega_1 \dots \omega_l \kappa\alpha$$

with $\omega'_1, \dots, \omega'_l \in \{\zeta, \tau\}$ and $\omega_1, \dots, \omega_l \in \{\Delta'_I, \nabla_i, \widehat{\text{pr}}_i \mid i \in \mathbb{N}, I \subseteq \mathbb{N}, |I| \geq 2\}$. That means

$$\kappa\gamma \in [\kappa\gamma']_{\zeta, \tau}$$

with $\kappa\gamma' \notin \text{pInv}_k \widetilde{P}_k$ and

$$\gamma' \in [\kappa\alpha]_{\{\Delta'_I, \nabla_i, \widehat{\text{pr}}_i \mid i \in \mathbb{N}, I \subseteq \mathbb{N}, |I| \geq 2\}},$$

i.e., $\kappa\gamma' = \kappa\omega_1 \dots \omega_l \kappa\alpha$.

From $\text{pr}_i \kappa\alpha = \widetilde{E}_k^{h-1} \in \text{pInv} \widetilde{P}_k$ we conclude $\omega_l \neq \widehat{\text{pr}}_i$ for all $i \in \mathbb{N}$. Now we consider the remaining cases for ω_l .

- $\omega_l = \Delta'_I$ for some $I \subseteq \mathbb{N}$. Let $\omega_a := \Delta'_I$ and look at ω_{a-1} . We show that we can move Δ'_I leftwards, i.e., switch ω_a and ω_{a-1} , or we have $\omega_{a-1} = \widehat{\text{pr}}_i$ with $i \in I$. Assume $\omega_{a-1} \neq \widehat{\text{pr}}_i$ for some $i \in I$.
 - $\omega_{a-1} = \nabla_j$ for some $j \in \mathbb{N}$. Then $\omega_{a-1}\omega_a = \nabla_j \Delta'_I = \Delta'_{I'} \nabla_j$ with $I' = \{i \mid i \in I, i < j\} \cup \{i+1 \mid i \in \mathbb{N}, i \geq j\}$, i.e., replace $\omega_{a-1}\omega_a$ with $\Delta'_{I'} \nabla_j$ and call I' the new I . (Lemma 4.5.1 (h))
 - $\omega_{a-1} = \Delta'_J$ with $J \cap I = \emptyset$. Then $\omega_{a-1}\omega_a = \Delta'_J \Delta'_I = \Delta'_I \Delta'_J$, i.e., replace $\omega_{a-1}\omega_a$ with $\Delta'_I \Delta'_J$. (Lemma 4.5.1 (h))
 - $\omega_{a-1} = \Delta'_J$ with $J \cap I \neq \emptyset$. Then $\omega_{a-1}\omega_a = \Delta'_J \Delta'_I = \Delta'_{I \cup J}$, i.e., replace $\omega_{a-1}\omega_a$ with $\Delta'_{I \cup J}$ and call $I \cup J$ the new I . (Lemma 4.5.1 (h))
 - $\omega_{a-1} = \widehat{\text{pr}}_j$ with $j \notin I$. Then $\omega_{a-1}\omega_a = \widehat{\text{pr}}_j \Delta'_I = \Delta'_{I'} \widehat{\text{pr}}_j$ with $I' = \{i \mid i \in I, i < j\} \cup \{i-1 \mid i \in I, i > j\}$, i.e., replace $\omega_{a-1}\omega_a$ with $\Delta'_{I'} \widehat{\text{pr}}_j$ and call I' the new I . (Lemma 4.5.1 (j))

The I never decreases in size in this process. If we get $a = 1$ at the end then γ has duplicate rows in contradiction to the assumption. Thus there is some $a > 1$ such that $\omega_{a-1} = \widehat{\text{pr}}_i$ for some $i \in I$.

If $|I| \geq 3$ then let $\{i_1, \dots, i_{|I|}\} := I$ and $i_1 = i$. Then

$$\widehat{\text{pr}}_i \Delta'_I = \widehat{\text{pr}}_i \Delta'_{I \setminus \{i\}} \Delta'_{\{i_1, i_2\}} = \Delta'_{I \setminus \{i\}} \widehat{\text{pr}}_i \Delta'_{\{i_1, i_2\}}$$

and we can deal with $\Delta'_{I \setminus \{i\}}$ the same way as before.

Thus we can conclude that if $\omega_a = \Delta'_I$ then $a \geq 2$, $|I| = 2$ and $\omega_{a-1} = \widehat{\text{pr}}_i$ for some $i \in I$, i.e., Δ'_I only occurs in combination with $\widehat{\text{pr}}_i$.

- $\omega_l = \nabla_i$ for some $i \in \mathbb{N}$. We show that we can either move the ∇_i to ω_1 or ∇_i collapses with some $\widehat{\text{pr}}_i$ or Δ'_I . Let $\omega_a := \nabla_i$ and look at ω_{a-1} . We show that

- we can move ω_i leftwards, i.e., switch ω_a and ω_{a-1} , or
- $\omega_{a-1} = \widehat{\text{pr}}_i$, or
- $\omega_{a-2}\omega_{a-1} = \widehat{\text{pr}}_j\Delta'_I$ with $\{i, j\} \in I$.

Now consider the following cases for ω_{a-1}

- $\omega_{a-1} = \nabla_j$. Then $\omega_{a-1}\omega_a = \nabla_j\nabla_i = \nabla_i\nabla_j$, i.e., replace $\omega_{a-1}\omega_a$ with $\nabla_i\nabla_j$.
- $\omega_{a-1} = \widehat{\text{pr}}_i$. Then $\omega_{a-1}\omega_a = \widehat{\text{pr}}_i\nabla_i = \text{id}$, i.e., we can remove $\omega_{a-1}\omega_a$.
- $\omega_{a-1} = \widehat{\text{pr}}_j$ for some $j \neq i$. Then $\omega_{a-1}\omega_a = \widehat{\text{pr}}_j\nabla_i = \nabla_{i'}\widehat{\text{pr}}_j$ with $i' = i - 1$ if $j < i$ and $i' = i$ if $j > i$, i.e., replace $\omega_{a-1}\omega_a$ with $\nabla_{i'}\widehat{\text{pr}}_j$ and call i' the new i .
- $\omega_{a-1} = \Delta'_J$ with $i \notin J$. Then $\omega_{a-1}\omega_a = \Delta'_J\nabla_i = \nabla_i\Delta'_{J'}$ with $J' = \{j \mid j \in J, j < i\} \cup \{j - 1 \mid j \in J, j > i\}$, i.e., replace $\omega_{a-1}\omega_a$ with $\nabla_i\Delta_J$.
- $\omega_{a-1} = \Delta'_J$ with $i \in J$. Then we can assume by the case discussion about $\omega_a = \Delta'_J$ that $J = 2$ and $\omega_{a-2} = \widehat{\text{pr}}_j$ for some $j \in J$. Let $\varrho^{(\mu)} := \omega_{a+1} \dots \omega_l \kappa \alpha$. Then $\omega_{a-2}\omega_{a-1}\omega_a\varrho = \widehat{\text{pr}}_j\Delta'_J\nabla_i\varrho = \varrho^{[\pi]}$ for some $\pi \in S_\mu$. Thus $\omega_{a-2}\omega_{a-1}\omega_a$ can be replaced by a combination of ζ and τ characterizing π . Since these can be moved leftwards we can assume that $\omega_{a-2}\omega_{a-1}\omega_a$ can be removed from the sequence.

Thus ∇_i either vanishes or can be moved leftwards. Because we permute the variables with $\omega'_1 \dots \omega'_l$ afterwards, we can assume that $\gamma' = \nabla^{t'}\omega_1 \dots \omega_l \kappa \alpha$ with $\omega_a \in \{\widehat{\text{pr}}_i, \Delta'_I \mid i \in I, I \subseteq \mathbb{N}, |I| \geq 2\}$ for all $a \in \{1, \dots, l\}$ and $t' \geq 0$. Furthermore we can assume $\omega_a \in \{\widehat{\text{pr}}_i, \Delta'_I \mid i \in I, I \subseteq \mathbb{N}, |I| = 2\}$ and if $\omega_a = \Delta'_I$ then $\omega_{a-1} = \widehat{\text{pr}}_i$ with $i \in I$. We have $\widehat{\text{pr}}_i\Delta'_I\varrho = (\Delta_I\varrho)^{[\pi]}$ for some permutation π and because we permute the variables with $\omega'_1 \dots \omega'_l$ afterwards we can assume $\pi = \text{id}$. Thus we can replace these pairs $\widehat{\text{pr}}_i\Delta'_I$ with Δ_I .

Therefore we can write $\gamma' = \nabla^{t'}\omega_1 \dots \omega_l \kappa \alpha$ with $\omega_a \in \{\widehat{\text{pr}}_i, \Delta_I \mid i \in \mathbb{N}, I \subseteq \mathbb{N}\}$.

Let $\omega_a = \widehat{\text{pr}}_i$ for some $a \in \{1, \dots, l\}$. Then we discuss the following cases

- $a = l$.
Then $\gamma' \in [\widehat{\text{pr}}_i \kappa \alpha]_{\text{P}} = \left[\widetilde{E}_k^{h-1} \right]_{\text{P}} = \text{pInv}_k \widetilde{P}_k$ in contradiction to $\gamma \notin \text{Inv}_k P_k$.
- $a < l$ and a is the maximal one.
Then $\omega_{a+1} = \Delta_J$.
 - $i \neq \min J$. Then $\omega_a\omega_{a+1} = \widehat{\text{pr}}_i\Delta_J = \Delta_J\widehat{\text{pr}}_{i'}$, i.e., $\widehat{\text{pr}}_i$ can be moved rightwards.
 - $i = \min J$. Then $\omega_a\omega_{a+1} = \widehat{\text{pr}}_i\Delta_J = \widehat{\text{pr}}_{j_1} \dots \widehat{\text{pr}}_{j_{|J|}}$ with $\{j_1, \dots, j_{|J|}\} := J$, i.e., Δ_J can be removed and more $\widehat{\text{pr}}$ added.

Since l is finite there are only finitely many Δ_J to the right which are moved to the left or removed. Therefore we finally get to the case $a = l$ and this yields a contradiction.

Thus $\omega_a \neq \widehat{\text{pr}}_i$ for all $a \in \{1, \dots, l\}$ and we can write

$$\kappa\gamma = \kappa \left(\nabla^{t'} \Delta_{I_1} \dots \Delta_{I_l} \kappa \alpha \right)^{[\pi]}$$

for some $\pi \in S_t$ and $I_1, \dots, I_l \subseteq \mathbb{N}$. That means we first identify rows, then add fictious rows and at last bring the rows in the right order. Thus every row of $\kappa\alpha$ is mapped to one $\kappa\gamma$. Since $(\kappa\alpha) \cap E_k^h = \alpha \cap E_k^h$ this mapping also works for α and γ . The mapping $\varphi : \{1, \dots, h\} \rightarrow \{1, \dots, t\}$ is obtained by looking at the rows. If the a -th row of α is mapped to the b -th row of γ then let $\varphi(a) = b$. We see that $\varphi(a_1) = \varphi(a_2)$ if and only if the a_1 -th row and the a_2 -th row of α are identified in the construction.

Thus we obtained φ and (4.5) is true. \square

Theorem 4.5.9 (Representation Lemma of B. A. Romov). *Let $\alpha_i \in \mathcal{R}_k^{(h_i)}$ for $i = 1, \dots, n$ and $\beta \in \mathcal{R}_k^t$ an irredundant relation. Then*

$$\bigcap_{i=1}^n \text{pPOL}_k \alpha_i \subseteq \text{pPOL}_k \beta \quad (4.6)$$

if and only if there are $\Phi_i \subseteq \{\varphi : \{1, \dots, h_i\}\}$ for all $i \in \{1, \dots, n\}$ such that

$$\beta = \{(b_1, \dots, b_t) \in E_k^t \mid \forall i \in \{1, \dots, n\} \forall \varphi \in \Phi_i : (b_{\varphi(1)}, \dots, b_{\varphi(h)}) \in \alpha_i\}$$

and

$$\{1, \dots, t\} \subseteq \bigcup_{i=1}^n \bigcup_{\varphi \in \Phi_i} \varphi(\{1, \dots, h\}).$$

Proof. By Lemma 4.5.3 we know that (4.6) holds if and only if there are $\gamma_1, \dots, \gamma_l \in \bigcup_{i=1}^n [\kappa\alpha_i]_{\zeta, \tau, \Delta', \nabla, \text{pr}, \hat{\kappa}}$ for which

$$\kappa\beta = \bigcap_{j=1}^l \kappa\gamma_j.$$

Because $\kappa\gamma_j = \kappa(\gamma_j \cap E_k^t)$ we can assume $\gamma_j \in \mathcal{R}_k$ for all j . Since β is irredundant the relations γ_j can not have duplicate rows. Furthermore $\gamma_j \notin \text{Inv}_k P_k$ because $\text{Inv}_k P_k$ consists only of diagonal relations, i.e. they have duplicate rows or $\gamma_j = E_k^t$ which can be ignored for the intersection because $\gamma_j \subseteq E_k^t$ by definition.

Then $\gamma_j \in [\kappa\alpha_i]_{\zeta, \tau, \Delta', \nabla, \text{pr}, \hat{\kappa}}$ for some i , and $\kappa\gamma \in \left[[\kappa\alpha]_{\zeta, \tau, \Delta', \nabla, \text{pr}, \hat{\kappa}} \right]_{\kappa}$. By Lemma 4.5.8 there is some $\varphi : \{1, \dots, h_i\} \rightarrow \{1, \dots, t\}$ with

$$\gamma_j = \{(g_1, \dots, g_t) \in E_k^t \mid (g_{\varphi(1)}, \dots, g_{\varphi(h)}) \in \alpha_i\}.$$

Combining these we have

$$\begin{aligned} \forall j \in \{1, \dots, l\} \exists i_j \in \{1, \dots, n\} \exists \varphi_j : \{1, \dots, h_{i_j}\} \rightarrow \{1, \dots, t\} : \\ \gamma_j = \{(g_1, \dots, g_t) \in E_k^t \mid (g_{\varphi_j(1)}, \dots, g_{\varphi_j(h)}) \in \alpha_{i_j}\}. \end{aligned}$$

Thus there are $\Phi_i \subseteq \{\varphi : \{1, \dots, h_i\}\}$ for all $i \in \{1, \dots, n\}$ such that

$$\begin{aligned} \beta &= \bigcap_{j=1}^l \kappa\gamma_j \\ &= \bigcap_{i=1}^n \bigcap_{\varphi \in \Phi_i} \{(g_1, \dots, g_t) \in E_k^t \mid (g_{\varphi(1)}, \dots, g_{\varphi(h)}) \in \alpha_i\} \\ &= \{(b_1, \dots, b_t) \in E_k^t \mid \forall i \in \{1, \dots, n\} \forall \varphi \in \Phi_i : (b_{\varphi(1)}, \dots, b_{\varphi(h)}) \in \alpha_i\}. \end{aligned}$$

Because β is irredundant it has no fictitious coordinates, i.e., for every coordinate a there is some γ_j such the a -th row is not fictitious. Then $a \in \varphi_j(\{1, \dots, h_{i_j}\})$ and thus

$$\{1, \dots, t\} \subseteq \bigcup_{i=1}^n \bigcup_{\varphi \in \Phi_i} \varphi(\{1, \dots, h\}). \quad \square$$

Chapter 5

Characterization of Maximal Partial Clones

An important question in the theory of clones concerns the completeness of a set A of (partial) functions, i.e., can we write all (partial) functions in terms of the functions of A together with the elementary operations and projections?

For the further studies we need to define maximal (partial) clones. Let $C \subseteq \tilde{P}_k$ be a partial clone. Then C is called a *maximal partial clone* iff

- there is no partial clone M that properly contains C and is properly contained in \tilde{P}_k , and
- C is not the full clone \tilde{P}_k .

That means $C \in \mathbb{L}(\tilde{P}_k)$ is maximal iff

$$C \neq \tilde{P}_k \text{ and } \forall M \in \mathbb{L}(\tilde{P}_k) \setminus \{C, \tilde{P}_k\} : C \not\subseteq M.$$

In a similar fashion one can define the *maximal clones* of P_k .

The first result in this direction was by E.L. Post in 1921 [47, 48] for the Boolean total functions. He showed that a set of Boolean function is complete if it is not contained in any of the five maximal clones of P_2 . Later Rosenberg generalized this result in 1965 [55, 56] to arbitrary finite k and described all maximal clones of P_k .

The partial case was solved around 1980 starting with papers by Lau [37], Romov [51, 52] and others, and concluded by Haddad and Rosenberg by the characterization of all maximal partial clones [14, 22, 24]. Independently Romov [54] and Zhukai [73] presented results about the completeness problem and the maximal partial clones.

We present the results of Haddad and Rosenberg but first we show that the special clone $P_k \cup C_\infty$ is maximal in \tilde{P}_k .

5.1 The Maximal Partial Clone $P_k \cup C_\infty$

Lemma 5.1.1. *Let $C \subset \tilde{P}_k$ be a partial clone.*

Then $C = \text{pPol}_k R$ for some $R \subseteq \tilde{\mathcal{R}}_k \setminus (\text{pInv}_k \tilde{P}_k)$.

Proof. The lemma holds for $R := (\text{pInv}_k C) \setminus (\text{pInv}_k \tilde{P}_k)$. □

Lemma 5.1.2. *Let $C \subset \tilde{P}_k$ be a partial clone.*

Then there is some partial clone M with $C \subseteq M \subset \tilde{P}_k$ and $M = \text{pPol}_k \varrho$ for some $\varrho \in \tilde{\mathcal{R}}_k \setminus (\text{pInv}_k \tilde{P}_k)$.

Proof. $C = \text{pPol}_k R \subseteq \text{pPol}_k \varrho =: M$ with $\varrho \in R \subseteq \tilde{\mathcal{R}}_k \setminus (\text{pInv}_k \tilde{P}_k)$ by Theorem 3.11.4. \square

Lemma 5.1.3. *Let $C = \text{pPol}_k \varrho^{(h)} \subset \tilde{P}_k$. Then*

- $P_k \subseteq C$, or
- $C \subseteq \text{pPOL}_k \varrho \subset \tilde{P}_k$.

Proof. Let $P_k \not\subseteq C$. Then there is some $f \in P_k$ with $f \notin C = \text{pPol}_k \varrho$. Because f is a total function and $\varrho \cap E_k^h = (\kappa \varrho) \cap E_k^h$ we have $f \notin \text{pPol}_k(\kappa \varrho) = \text{pPOL}_k \varrho$. Since $\kappa \varrho \in [\{\varrho\}]_P$ we have $C \subseteq \text{pPol}_k(\kappa \varrho) \subset \tilde{P}_k$. \square

Lemma 5.1.4. *Let $C \subset \tilde{P}_k$ be a partial clone with $P_k \subseteq C$. Then $C = P_k = \text{pPol}_k E_k$ or $C = P_k \cup C_\infty = \text{pPol}_k \kappa_{\{1,2\}} E_k^2$. Furthermore $P_k \cup C_\infty$ is a maximal partial clone of \tilde{P}_k .*

Proof. Let $P_k \subset C \subset \tilde{P}_k$.

Let $f^{(n)} \in \tilde{P}_k \setminus (P_k \cup C_\infty)$. Then there are $a, b \in E_k^n$ with $f(a) \in E_k$ and $f(b) = \infty$. Now let $g^{(m)} \in \tilde{P}_k$ be arbitrary. Then there is some function $g' \in P_k$ such that g is a subfunction of g' . Furthermore there are functions $h_1^{(m)}, \dots, h_n^{(m)} \in P_k$ with

$$(h_1(x), \dots, h_n(x)) = \begin{cases} a & \text{if } g(x) \in E_k \\ b & \text{if } g(x) = \infty \end{cases}$$

The function g can now be written as

$$g = e_1^{(2)}(g', f(h_1, \dots, h_n)) \in [P_k \cup \{f\}]_P$$

and thus $[P_k \cup \{f\}]_P = \tilde{P}_k$ and $f \notin C$.

Let $f^{(n)}, g_1^{(m)}, \dots, g_n^{(m)} \in \tilde{P}_k$ be arbitrary.

Since $c_\infty^{(n)}(g_1, \dots, g_n) = c_\infty^{(m)}$ and $f(c_\infty^{(m)}, g_2, \dots, g_n) = c_\infty^{(m)}$ we see that $P_k \cup C_\infty$ is closed and $C = P_k \cup C_\infty$. \square

5.2 Definitions

Definition 5.2.1. Let for all h with $1 \leq h \leq k$

$$\begin{aligned} \varrho_1 &:= \{(a, a, b, b), (a, b, a, b) \mid a, b \in E_k\}, \\ \varrho_2 &:= \{(a, a, b, b), (a, b, a, b), (a, b, b, a) \mid a, b \in E_k\}, \\ \iota_k^h &:= \{(x_1, \dots, x_h) \in E_k^h \mid |\{x_1, \dots, x_h\}| \leq h - 1\}. \end{aligned}$$

Definition 5.2.2. Let $\varrho^{(h)} \subseteq E_k^h$. Then we write $\sigma(\varrho) := \varrho \setminus \iota_k^h$ and $\delta(\varrho) := \varrho \cap \iota_k^h = \varrho \setminus \sigma(\varrho)$. If $\delta = \delta_{\varepsilon'}$ for some equivalence relation ε' then we write $\varepsilon(\varrho) := \varepsilon'$.

Definition 5.2.3. Let $\varrho^{(h)} \subseteq E_k^h$. Then ϱ is

- *areflexive*, if $h \geq 2$ and $\delta(\varrho) = \emptyset$, i.e., for each $(x_1, \dots, x_h) \in \varrho$ we have $x_i \neq x_j$ for all $1 \leq i < j \leq h$.

- *quasi-diagonal*, if $\sigma(\varrho)$ is a non-empty areflexive relation, $\delta(\varrho) = \delta_\varepsilon$ with $\varepsilon \neq \iota_h^2$ an equivalence relation on E_h .

For each $m \in \mathbb{N}$ let $\eta_m := (0, 1, \dots, m-1)^T$.

Definition 5.2.4. Let $\varrho^{(h)} \subseteq E_k^h$, $\sigma := \sigma(\varrho)$ and $\delta := \delta(\varrho)$.

Let

$$\Gamma_\sigma := \left\{ \pi \in S_h \mid \sigma \cap \sigma^{[\pi]} \neq \emptyset \right\}$$

be the symmetry group of the relation σ .

The *model* of ϱ is the h -ary relation $M(\varrho)$ on E_h defined by

$$M(\varrho) := \left\{ \eta_h^{[\pi]} \mid \pi \in \Gamma_\sigma \right\} \cup (\delta \cap E_h^h).$$

The relation ϱ is *coherent*, if the following conditions hold:

- $\varrho \neq E_k^h$, $\varrho \neq \emptyset$,
- ϱ is a unary relation, i.e., $h = 1$, or
 - ϱ is areflexive with $2 \leq h \leq k$, or
 - ϱ is quasi-diagonal with $2 \leq h \leq k$, or
 - $\delta = \iota_k^h$ with $3 \leq h \leq k$, or
 - $\delta = \varrho_i$ with $i \in \{1, 2\}$ (see Definition 5.2.1) and $h = 4$,
- if $\delta = \iota_k^h$ and $h \geq 3$ then $\Gamma_\sigma = S_h$,
 - if $\delta = \varrho_1$ then $\Gamma_\sigma = \langle (1342), (23) \rangle$, i.e., the permutation group generated by the cycles (1342) and (23),
 - if $\delta = \varrho_2$ then $\Gamma_\sigma = S_4$.
- $r^{[\pi]} \in \sigma$ for all $r \in \sigma$ and all $\pi \in \Gamma_\sigma$,
- for every σ' with $\emptyset \neq \sigma' \subseteq \sigma$ there is a relational homomorphism $\varphi : E_k \rightarrow E_h$ from σ' to $M(\varrho)$, such that $\varphi(r) = \eta_h$ for some $r \in \sigma'$, i.e., $(\varphi(r_0), \dots, \varphi(r_{h-1})) = (0, \dots, h-1)$ for some $r = (r_0, \dots, r_{h-1}) \in \sigma'$,

We remark that all non-empty non-diagonal totally reflexive, totally symmetric relations are coherent.

5.3 Characterization by Haddad and Rosenberg

Let $\tilde{\mathcal{R}}_k^{\max}$ be the set of all coherent relations with $\text{pPOL}_k \varrho \neq \text{pPOL}_k \chi$ for all $\varrho, \chi \in \tilde{\mathcal{R}}_k^{\max}$ and $\varrho \neq \chi$. Let

$$p\mathcal{M}_k := \{P_k \cup C_\infty\} \cup \left\{ \text{pPOL}_k \varrho \mid \varrho \in \tilde{\mathcal{R}}_k^{\max} \right\}.$$

Theorem 5.3.1 (of Haddad and Rosenberg; [22, 24]). *Let $k \geq 2$. For each $A \subset \tilde{P}_k$ with $A = [A]_P$ there is a maximal partial clone M_A with $A \subseteq M_A$. A clone M is a maximal partial clone of \tilde{P}_k if and only if $M \in p\mathcal{M}_k$, i.e., $p\mathcal{M}_k$ is the set of all maximal partial clones of \tilde{P}_k .*

Theorem 5.3.2 (Completeness criterion for \tilde{P}_k ; [24]). *Let $C \subseteq \tilde{P}_k$. Then $[C]_P = \tilde{P}_k$ if and only if $C \not\subseteq M$ for all $M \in p\mathcal{M}_k$.*

Example 5.3.3. The following relations $\varrho = \sigma(\varrho) \cup \delta(\varrho)$ are all coherent:

k	$\sigma(\varrho)$	$\delta(\varrho)$	$\Gamma_{\sigma(\varrho)}$	$M(\varrho)$
3	$\{0, 1\}$	\emptyset	$\{\text{id}\}$	$\{0\}$
2	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	\emptyset	$\{\text{id}\}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
4	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 3 & 3 \end{pmatrix}$	$\delta_{4;\{0,1\}}^{(3)}$	$\langle(12)\rangle$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 2 \end{pmatrix} \cup \delta_{3;\{0,1\}}^{(3)}$
5	\emptyset	ϱ_2	S_4	$\varrho_2 \cap E_4^4$
7	$\begin{pmatrix} 2 & 2 & 4 & 4 & 6 & 6 \\ 4 & 6 & 2 & 6 & 2 & 4 \\ 6 & 4 & 6 & 2 & 4 & 2 \end{pmatrix}$	ι_7^3	S_3	E_3^3

Example 5.3.4. The following relations are not coherent:

- $k = 2$, $\varrho = \{0, 1\}$. Then $\varrho = E_2$ is trivial.
- $k = 2$, $\varrho = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\delta(\varrho) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq \delta_\varepsilon^{(2)}$ for all ε .
- $k = 4$, $\varrho = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix}$. Then $\Gamma_{\sigma(\varrho)} = \langle(12)\rangle$ but $\begin{pmatrix} 2 \\ 3 \end{pmatrix}^{[(12)]} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \notin \varrho$.
- $k = 4$, $\varrho = \begin{pmatrix} 2 & 0 & 1 \\ 3 & 1 & 0 \\ 0 & 2 & 3 \end{pmatrix} \cup \delta_{4;\{0,1\}}^{(3)}$. Then there is no relational homomorphism $\varphi : E_k \rightarrow E_h$ from $\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 3 \end{pmatrix} \subseteq \sigma(\varrho)$ to $M(\varrho) = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \cup \delta_{3;\{0,1\}}^{(3)}$ as required. If $\varphi((0, 1, 2)) = (0, 1, 2) = \eta_3$ then $\varphi((1, 0, 3)) = (1, 0, x) \notin M(\varrho)$. On the other hand if $\varphi((1, 0, 3)) = (0, 1, 2) = \eta_3$ then $\varphi((0, 1, 2)) = (1, 0, x) \notin M(\varrho)$. Thus ϱ is not coherent.

Chapter 6

Number of Maximal Partial Clones

The maximal partial clones have been determined for $k = 2$ by Freivald [13], for $k = 3$ independently by Lau [37] (see also [39]) and Romov [51] (actually in the second paper three were missed), and for $k = 4$ by Haddad and Simons [26] (some clones were counted twice). Especially the last case inspired the author to write a computer program that lists all coherent relations and counts them effectively determining the maximal partial clones due to Theorem 5.3.2. Another motivation was the search for more examples of coherent relations to test hypotheses about members and non-members in minimal coverings (defined in the next chapter) since the cases $k \in \{2, 3, 4\}$ are quite restricted (nonetheless complicated).

So we want to determine all coherent relations for some k and count the maximal partial clones in this chapter. First we have to show that coherent relations which differ in more than just a permutation of the rows correspond to different maximal partial clones. Then we want to generate all coherent relations but only once for each maximal partial clone. Consider we start with small relations and use a backtracking algorithm to get every possible relation. We could get a search tree like the one in Figure 6.1, where the two stars represent relations which determine the same maximal partial clone. These should be detected, and one of them should be thrown away. The most straightforward way would be to generate all relations and then check for duplicates. But this does not scale and occupies too much space and time on the computer even for small k since we effectively have to compare each element of the tree with each other.

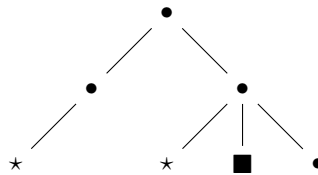


Figure 6.1: Example of a search tree

So we want to be able to detect a duplicate without looking into the list of all coherent relations which were generated before, i.e., a property of the relations to check for. We introduce the notions of a *minimal relation* and of a *quasi-minimal relation* which accomplishes a bit more and also puts isomorphic clones together, i.e., clones which look the same if we permute the elements of the base set E_k . In the example tree this is symbolized by the box, i.e., the clones corresponding to the stars and the box are isomorphic.

This method avoids isomorphism tests and is similar to the methods shown by Read and others in [50, 43, 29, 30].

6.1 Different Relations – Different Clones

Lemma 6.1.1. *Let $\varrho^{(h)}, \chi^{(h)} \in \mathcal{R}_k$ and $\pi \in S_h$ with $\varrho^{[\pi]} = \chi$. Then $\text{pPOL}_k \varrho = \text{pPOL}_k \chi$.*

Proof. Because $\kappa\chi = \kappa(\varrho^{[\pi]}) = (\kappa\varrho)^{[\pi]} \in [\kappa\varrho]_{\mathbb{P}}$ holds by Lemma 4.5.1 (f) we have $\text{pPOL}_k \varrho = \text{pPol}_k(\kappa\varrho) \subseteq \text{pPol}_k(\kappa\chi) = \text{pPOL}_k \chi$ by Theorem 3.8.1 (a).

Since $\varrho = \chi^{[\pi^{-1}]}$ we also obtain $\text{pPOL}_k \chi \subseteq \text{pPOL}_k \varrho$. \square

Thus we call two coherent relations $\varrho^{(h)}$ and $\chi^{(h)}$ *equivalent*, if there is some $\pi \in S_h$ with

$$\varrho^{[\pi]} = \chi.$$

If two clones or the corresponding relations are compared, it suffices to choose one representative of each equivalence class. Let $\varrho^{(h)}$ and $\chi^{(h)}$ be two h -ary relations. If $\varrho \cap \chi$ is considered, then let χ be one of the relations $\chi^{[\pi]}$ with $\pi \in S_h$, such that $|\varrho \cap \chi^{[\pi]}|$ is maximal. Thus χ is chosen such that

$$\chi \in \left\{ \chi' \in S \mid |\varrho \cap \chi'| = \max_{\chi'' \in S} |\varrho \cap \chi''| \right\}$$

holds with

$$S = \left\{ \chi^{[\pi]} \mid \pi \in S_h \right\}.$$

Let

$$\mathcal{W}_k := \left\{ \varrho_1, \varrho_2, \iota_k^h \mid 3 \leq h \leq k \right\} \subseteq \widetilde{\mathcal{R}}_k^{\max}.$$

Lemma 6.1.2. *Let $\varrho^{(h)}, \chi^{(\mu)} \in \widetilde{\mathcal{R}}_k^{\max} \setminus \mathcal{W}_k$ with $\varrho \notin [\chi]_{\zeta, \tau}$.*

Then $\text{pPOL}_k \varrho \neq \text{pPOL}_k \chi$ holds.

Proof. We consider three cases:

- $\mu < h$: Let $s = (s_1, \dots, s_\mu) \in \chi \setminus \iota_k^\mu$ and $v = (v_1, \dots, v_\mu) \in E_k^\mu \setminus \chi$ be arbitrary. Then define a function $f \in \widetilde{P}_k^{(1)}$ by $f(s_i) := v_i$ for all $i \in \{1, \dots, \mu\}$ and $\text{dom}(f) := \{s_1, \dots, s_\mu\}$. Then f is well-defined, because $s_i \neq s_j$ holds for $i \neq j$. Additionally $f \notin \text{pPOL}_k \chi$ follows from $f(s) = v \notin \chi$ and $s \in \chi$.
Assume $f \notin \text{pPOL}_k \varrho$. Then there is some $r \in \varrho$ with $f(r) \in E_k^h \setminus \varrho$. But f is defined only at μ different values, thus r can only have μ different entries. That means, we have $r \in \delta_\varepsilon$ with ε equivalence relation on E_h and $\varepsilon \neq \iota_h^2$. Then follows $\delta_\varepsilon \subset \varrho$ by definition of a coherent relation. Then $r_i = r_j \implies f(r_i) = f(r_j)$ implies $f(r) \in \delta_\varepsilon \subseteq \varrho$, in contradiction to the assumption, i.e., $f \in \text{pPOL}_k \varrho$.
- $h < \mu$. Because the case $h < \mu$ is symmetrical to the case above we have $\text{pPOL}_k \varrho \neq \text{pPOL}_k \chi$ for $\mu \neq h$.
- $\mu = h$. We can assume that the rows of χ are permuted such that $|\varrho \cap \chi|$ is maximal, as motivated by Lemma 6.1.1. Then the following cases need to be checked:

- $\varrho \cap \chi = \emptyset$. Let $s = (s_1, \dots, s_h) \in \chi \setminus \iota_k^h$ and $v = (v_1, \dots, v_h) \in E_k^h \setminus \chi$ be arbitrary. Then define a function $f \in \tilde{P}_k^{(1)}$ by $f(s_i) := v_i$ for all $i \in \{1, \dots, h\}$ and $\text{dom}(f) := \{s_1, \dots, s_h\}$. Then f is well-defined because $s_i \neq s_j$ holds for $i \neq j$. Additionally we have $f \notin \text{pPOL}_k \chi$ because $f(s) = v \notin \chi$ and $s \in \chi$.

Assume $f \notin \text{pPOL}_k \varrho$. Then there is some $r \in \varrho$ with $f(r) \notin \varrho$. The tuple r has less than h different entries. Otherwise r would be a permutation of s and thus there would be a permutation of rows of χ such that $\varrho \cap \chi \neq \emptyset$. But then a contradiction follows as in the case $\mu < h$ above, i.e., $\text{pPOL}_k \varrho \neq \text{pPOL}_k \chi$ holds.

- $\varrho \cap \chi \neq \emptyset, \varrho \not\subseteq \chi$. Let $s = (s_1, \dots, s_h) \in \chi \setminus \iota_k^h$ and $v = (v_1, \dots, v_h) \in \varrho \setminus \chi$ be arbitrary. Define a function $f \in \tilde{P}_k^{(1)}$ by $f(s_i) := v_i$ for all $i \in \{1, \dots, h\}$ and $\text{dom}(f) := \{s_1, \dots, s_h\}$. Then f is well-defined, because $s_i \neq s_j$ holds for $i \neq j$. Additionally we have $f \notin \text{pPOL}_k \chi$ because $f(s) = v \notin \chi$ and $s \in \chi$.

Assume $f \notin \text{pPOL}_k \varrho$. Then there is some $r \in \varrho$ with $f(r) \notin \varrho$. If r has less than h different entries then a contradiction follows as in the case $\mu < h$ above. Thus r has exactly h different entries, i.e., $r = s^{[\alpha]}$ holds for some $\alpha \in \Gamma_{\sigma(\varrho)}$. Because ϱ is coherent we also have

$$f(r) = f\left(s^{[\alpha]}\right) = v^{[\alpha]} \in \varrho.$$

This is a contradiction to $f(r) \notin \varrho$. Thus $f \in \text{pPOL}_k \varrho$ holds and this implies $\text{pPOL}_k \varrho \neq \text{pPOL}_k \chi$.

- The case $\varrho \subset \chi$ is the same as above if the roles of χ and ϱ are exchanged. \square

Lemma 6.1.3. *Let $\varrho^{(h)} \in \mathcal{W}_k$ and $\chi^{(\mu)} \in \tilde{\mathcal{R}}_k^{\max} \setminus \mathcal{W}_k$.*

Then $\text{pPOL}_k \varrho \neq \text{pPOL}_k \chi$ holds.

Proof. Let $s = (s_1, \dots, s_\mu) \in \chi \setminus \iota_k^\mu$ and $v = (v_1, \dots, v_\mu) \in E_k^\mu \setminus \chi$ be arbitrary. Define a function $f \in \tilde{P}_k^{(1)}$ by $f(s_i) := v_i$ for all $i \in \{1, \dots, \mu\}$ and $\text{dom}(f) := \{s_1, \dots, s_\mu\}$. Then f is well-defined because $s_i \neq s_j$ holds for $i \neq j$. Additionally we have $f \notin \text{pPOL}_k \chi$ because $f(s) = v \notin \chi$ and $s \in \chi$.

Assume $f \notin \text{pPOL}_k \varrho$. Then there is some $r = (r_1, \dots, r_h) \in \varrho$ with $f(r) \in E_k^h \setminus \varrho$. From $\varrho = \bigcup_{\varepsilon \in E} \delta_\varepsilon$ where E is a set of equivalence relations on E_h follows $r \in \delta_\varepsilon$ for some $\varepsilon \in E$.

From $r_i = r_j \implies f(r_i) = f(r_j)$ follows $f(r) \in \delta_\varepsilon \subseteq \varrho$, in contradiction to the assumption. Thus $f \in \text{pPOL}_k \varrho$. \square

Lemma 6.1.4. *Let $3 \leq \mu < h \leq k$. Then $\text{pPOL}_k \iota_k^\mu \neq \text{pPOL}_k \iota_k^h$ holds.*

Proof. Define the function $f^{(h)}$ by

x_{*1}	x_{*2}	x_{*3}	\dots	x_{*h}	$f(x)$
0	0	0	\dots	0	0
0	1	1	\dots	1	1
1	1	2	\dots	2	2
2	2	2	\dots	3	3
\vdots	\vdots	\vdots	\ddots	\vdots	\vdots
$h-3$	$h-3$	$h-3$	\dots	$h-2$	$h-2$
$h-2$	$h-2$	$h-2$	\dots	$h-2$	$h-1$
otherwise					∞

We have $f \notin \text{pPOL}_k \iota_k^h$, because the columns x_{*1} to x_{*h} are in ι_k^h , but $f(x)$ is not.

Assume $f \notin \text{pPOL}_k \iota_k^\mu$. Then there are $c_{*1}, \dots, c_{*h} \in \iota_k^\mu$ with $f(c_{*1}, \dots, c_{*h}) =: d \notin \iota_k^\mu$. If the rows $c_{1*}, \dots, c_{\mu*}$ are all pairwise different then there is a column c_{*j} with μ different elements, in contradiction to $c_{*j} \in \iota_k^\mu$. Thus two rows are equal, i.e., two entries of d are equal in contradiction to $d \notin \iota_k^\mu$. Thus $f \in \text{pPOL}_k \iota_k^\mu$. \square

Lemma 6.1.5. *Let $\varrho = \varrho_i$ for some $i \in \{1, 2\}$ and $\chi = \iota_k^\mu$ for some $\mu \in \{3, 4, \dots, k\}$. Then $\text{pPOL}_k \varrho \neq \text{pPOL}_k \chi$.*

Proof. Define the function $f^{(2)}$ by

$$f \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and not defined otherwise. Then $f \notin \text{pPOL}_k \varrho$ because

$$(0, 0, 1, 1), (0, 1, 0, 1) \in \varrho$$

but $(0, 0, 0, 1) \notin \varrho$. Because $|\{0, 1\}| = 2$ we get $f \in \text{pPOL}_k \chi$. \square

Lemma 6.1.6. $\text{pPOL}_k \varrho_1 \neq \text{pPOL}_k \varrho_2$.

Proof. Define the function $f^{(3)}$ by

$$f \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and not defined otherwise. Then $f \notin \text{pPOL}_k \varrho_2$ because

$$(0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0) \in \varrho_2$$

but $(0, 0, 0, 1) \notin \varrho_2$. Because $\{(0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 0)\} \not\subseteq \varrho_1^{[\pi]}$ for all $\pi \in S_4$ we have $f \in \text{pPOL}_k \varrho_1$. \square

Theorem 6.1.7. *The relations in $\widetilde{\mathcal{R}}_k^{\max}$ describe distinct maximal partial clones.*

Proof. Combining the Lemmas 6.1.2, 6.1.3, 6.1.4, 6.1.5 and 6.1.6 implies the theorem. \square

6.2 Shortcuts for Finding Coherent Relations

We want to find all coherent relations $\widetilde{\mathcal{R}}_k^{\max}$, and determine $|p\mathcal{M}_k|$. The trivial approach is to enumerate all subsets of E_k^h for $h \in \{1, \dots, k\}$, check if they are coherent and remove duplicates with respect to equivalence of the relations, see page 46. But this approach needs too much memory and time because all coherent relations have to be in memory at the same time, all have to be checked for coherence, and relations generating the same clone have to be identified. Thus we try to do better.

Definition 6.2.1. Let $v := (v_1, \dots, v_h) \in E_k^h$, $w := (w_1, \dots, w_h) \in E_k^h$. We write $v \prec w$ if

$$\exists i \in \{1, 2, \dots, h\} : (\forall j < i : v_j = w_j) \wedge v_i < w_i.$$

Thus \prec defines a lexicographical order on E_k^h . Let $v \preceq w$ if $v \prec w$ or $v = w$.

Let $V = \{V_1, \dots, V_m\} \subseteq E_k^h$ and $W = \{W_1, \dots, W_n\} \subseteq E_k^h$ with $V_i \prec V_j$ and $W_i \prec W_j$ for all $i < j$. We write $V \prec W$ if

$$\exists i \in \{1, 2, \dots, \min(m, n)\} : (\forall j < i : V_j = W_j) \wedge V_i \prec W_i,$$

or

$$m < n \wedge (\forall i \in \{1, 2, \dots, m\} : V_i = W_i).$$

Thus \prec defines a lexicographical order on the subsets of E_k^h . Let $V \preceq W$ if $V \prec W$ or $V = W$. Then \preceq is a total order.

Let $\min_{\prec} V := x \in V$ with $x \preceq y$ for all $y \in V$.

Definition 6.2.2. Let $\varrho^{(h)} \in \tilde{\mathcal{R}}_k^{\max}$. Let S_k^* be the set of all bijective functions on E_k , i.e.,

$$S_k^* := \{f \in P_k^{(1)} \mid f \text{ is bijective}\}.$$

Define the *quasi-relation-class* $\text{qclass}(\varrho)$ by

$$\text{qclass}(\varrho) := \{\{f(v) \mid v \in \varrho\} \mid f \in S_k^*\}$$

and we call ϱ a *quasi-minimal relation* if $\varrho = \min_{\prec} \text{qclass}(\varrho)$.

Define the *relation-class* $\text{class}(\varrho)$ by

$$\text{class}(\varrho) := \{\{(f(v))^{[\pi]} \mid v \in \varrho\} \mid f \in S_k^*, \pi \in \Sigma(\varrho)\}$$

where

$$\Sigma(\varrho) := \{\pi \in S_h \mid \pi(\varepsilon(\varrho)) = \varepsilon(\varrho)\},$$

i.e., all permutations which leave $\delta(\varrho)$ fixed.

We call ϱ a *minimal relation* if $\varrho = \min_{\prec} \text{class}(\varrho)$.

Every minimal relation is a quasi-minimal relation because $\text{qclass}(\varrho) \subseteq \text{class}(\varrho)$.

The general idea to find all coherent relations $\varrho^{(h)}$ in reasonable time is to

- traverse over all models $M(\varrho)$ where $\varepsilon(\varrho)$ is given in such a way, that all equivalence classes are continuous intervals and sorted in decreasing order of size. The areflexive relations are represented by $\varepsilon(\varrho) = \emptyset$. The relations in \mathcal{S} are treated in a special way and no equivalence relation is given.
- For a given model $M(\varrho)$ check all relations χ with $M(\chi) = M(\varrho)$, i.e., traverse over all $\chi \subseteq E_k^h$ with $M(\chi) = M(\varrho)$. Let

$$A_\varrho := \left\{ \min_{\prec} \left\{ v^{[\pi]} \mid \pi \in \Gamma_{\sigma(\varrho)} \right\} \mid v \in \sigma(E_k^h) \right\},$$

the set of all areflexive h -ary tuples on E_k modulo the group associated with the model $M(\varrho)$. Then any subset of A_ϱ uniquely identifies a relation with the given model $M(\varrho)$. The subsets are determined recursively starting at $A_\varrho = \{\eta_h\}$ (since $\eta_h \in \varrho$ if $\varrho = \min_{\prec} \text{class}(\varrho)$) and adding tuples v_1, v_2, \dots to A_ϱ such that $v_i \prec v_j$ for all $i < j$. Thus every subset of A_ϱ is given exactly once and we can make cut offs with the help of the following lemmas.

- We only print a relation ϱ if it is a minimal relation in the sense given above, and the size is given by

$$|\{\text{pPOL}_k \chi \mid \chi \in \text{class}(\varrho)\}| = |\{\min_{\prec} \text{qclass}(\varrho^{[\pi]}) \mid \pi \in \Sigma(\varrho)\}|.$$

Because there is exactly one minimal relation in each $\text{class}(\varrho)$ we can deduce the number of all coherent relations.

Lemma 6.2.3. *Let $\varrho \in \tilde{\mathcal{R}}_k^{\max}$. Let $\chi \subset \varrho$ with*

- (a) $\emptyset \subset \sigma(\chi) \subset \sigma(\varrho)$,
- (b) $M(\chi) = M(\varrho)$, and
- (c) $\forall \pi \in \Gamma_{\sigma(\chi)} \forall v \in \sigma(\chi) : v^{[\pi]} \in \sigma(\chi)$.

Then $\chi \in \tilde{\mathcal{R}}_k^{\max}$.

Proof. Because of (a) the relation χ is non-trivial. Because of (c) we only have to show, that for every $\emptyset \subset \sigma' \subseteq \sigma(\chi)$ there is a relational homomorphism $\varphi : E_k \rightarrow E_h$ from σ' to $M(\chi)$, such that $\varphi(r) = \eta_h$ for some $r \in \sigma'$, i.e., $(\varphi(r_0), \dots, \varphi(r_{h-1})) = (0, \dots, h-1)$ for some $r := (r_0, \dots, r_{h-1}) \in \sigma'$. But this follows from $\sigma' \subseteq \sigma(\chi) \subset \sigma(\varrho)$, (b) and ϱ coherent. Thus χ is coherent. \square

Lemma 6.2.3 for a fixed model $M(\varrho)$ means that we can cut off the subset tree if we encounter a relation χ which is not coherent, since $\chi \notin \tilde{\mathcal{R}}_k^{\max}$ implies $\chi' \notin \tilde{\mathcal{R}}_k^{\max}$ for all $\chi' \supset \chi$ with $M(\chi') = M(\chi)$.

Lemma 6.2.4. *Let $\varrho, \chi \in \tilde{\mathcal{R}}_k^{\max}$. Let $\chi \subset \varrho$ and $\chi \prec \varrho$ with*

- (a) $\emptyset \subset \sigma(\chi) \subset \sigma(\varrho)$,
- (b) $M(\chi) = M(\varrho)$, and
- (c) ϱ is a quasi-minimal relation.

Then χ is a quasi-minimal relation.

Proof. Let $\varrho =: \{v_1, \dots, v_{|\varrho|}\}$ with $v_i \prec v_j$ for all $i < j$. Then $\chi = \{v_1, \dots, v_{|\chi|}\}$ because $\chi \prec \varrho$ and $\chi \subset \varrho$.

Assume χ is not a quasi-minimal relation. Then there is some $\chi' \in \text{qclass}(\chi)$ with $\chi' \prec \chi$, i.e., there is some $f \in S_k$ with $f(\chi) \prec \chi$. Thus there are $a \in \{1, \dots, |\chi| - 1\}$ and $w \in f(\chi)$ with $v_{a-1} \prec w \prec v_a$ and $\{v_1, \dots, v_{a-1}\} \subseteq f(\chi)$. Then $\{v_1, \dots, v_{a-1}, w\} \subseteq f(\varrho)$ implying $f(\varrho) \prec \varrho$, i.e., ϱ is not a quasi-minimal relation, in contradiction to the assumption. \square

Lemma 6.2.4 for a fixed model $M(\varrho)$ means that we can cut off the subset tree if we encounter a relation χ which is not quasi-minimal, since

- χ not quasi-minimal implies χ' not quasi-minimal for all $\chi' \supset \chi$ with $M(\chi') = M(\chi)$, and
- every minimal relation is also quasi-minimal.

We can not cut off the subtree if we encounter a relation which is not minimal but only quasi-minimal. For example

$$\varrho := \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 2 \end{pmatrix}$$

is quasi-minimal, but not minimal because

$$\begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 2 & 1 \end{pmatrix} = \min_{\prec} \text{class}(\varrho).$$

On the other hand

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 3 \end{pmatrix}$$

is minimal and is found in a subtree starting with ϱ .

6.3 Other Completeness Criteria

The number of coherent relations is quite large and it is computationally difficult to find all of them. The next theorem gives a sufficient condition for completeness.

Theorem 6.3.1. *Let $C := \{f_1^{(h_1)}, \dots, f_l^{(h_l)}\} \subseteq \tilde{P}_k$, $\varphi_1, \dots, \varphi_m \in S_k$, $U_0 := \{\eta_k\}$, and $U_j := \{\varphi(v) \mid v \in U_{j-1}, \varphi \in \langle \varphi_1, \dots, \varphi_j \rangle\}$ for $j \in \{1, \dots, m\}$.*

If

- (a) $\langle \varphi_1, \dots, \varphi_m \rangle = S_k$,
- (b) *there are $i \in \{1, \dots, l\}$ and $v_1, \dots, v_{h_i} \in U_{j-1}$ with $f_i(v_1, \dots, v_{h_i}) = \varphi_j(\eta_k)$ for all $j \in \{1, \dots, m\}$,*
- (c) *there are $i \in \{1, \dots, l\}$ and $v_1, \dots, v_{h_i} \in U_m$ with $f_i(v_1, \dots, v_{h_i}) \in \delta_\varepsilon^{(k)}$ for some non-trivial equivalence relation ε such that there is some $x \in E_k$ with $(x, y) \notin \varepsilon$ for all $y \in E_k \setminus \{x\}$, i.e., ε has a singular equivalence class,*
- (d) *for all $\chi \in \mathcal{W}_k$ there is some $f \in C$ with $f \notin \text{pPOL}_k \chi$, and*
- (e) *there is some $f \in C$ with $f \notin P_k \cup C_\infty$,*

then $[C]_P = \tilde{P}_k$.

Proof. Assume $[C]_P \neq \tilde{P}_k$. Then there is some $M \in p\mathcal{M}_k$ with $C \subseteq M$. Because of (e) there is some coherent relation $\varrho^{(h)} \in \tilde{\mathcal{R}}_k^{\max}$ with $M = \text{pPOL}_k \varrho$.

- We first consider $\sigma(\varrho) = \emptyset$. Then $\varrho \in \mathcal{W}_k$ and thus there is some $f \in C$ with $f \notin \text{pPOL}_k \varrho$, i.e. $C \not\subseteq M$.
- Now we see that there is some $v \in \sigma(\varrho)$ and we can assume $v = \eta_h$. That means $\{v\} = \text{pr}_h U_0 := \text{pr}_{0,1,\dots,h-1} U_0$ and by (2) there is some function $f \in C$ such that $\varphi_1(v) = f(v, \dots, v) \in \varrho$ because $f \in \text{pPOL}_k \varrho$. Doing this repeatedly we get $\text{pr}_h U_1 = \{\varphi(v) \mid \varphi \in \langle \varphi_1 \rangle\} \subseteq \varrho$.

By iteration we get $\text{pr}_h U_2, \text{pr}_h U_3, \dots, \text{pr}_h U_m \subseteq \varrho$. Thus $\sigma(E_k^h) \subseteq \sigma(\varrho)$ because $\langle \varphi_1, \dots, \varphi_m \rangle = S_k$ and therefore $U_m = \sigma(E_k^h)$. This implies that $G_{\sigma(\varrho)} = S_h$, i.e. ϱ is totally-symmetric.

If $h = 1$ then $\sigma(E_k^h) = E_k$ and thus $\varrho = E_k$, i.e. $\varrho \notin \tilde{\mathcal{R}}_k^{\max}$, in contradiction to the assumption.

For $h \geq 2$ we know by (3) that $\delta(\varrho) \neq \emptyset$. This means that $\delta(\varrho) = \iota_k^2$ because ϱ is coherent. But then $\varrho = E_k^2$ in contradiction to the assumption that ϱ is coherent.

For $h \geq 3$ we even know that $\delta(\varrho) \neq \delta_\varepsilon$ for all ε without singleton class. Assume $\delta(\varrho) = \delta_\varepsilon$ and ε has at least one singleton class. But then there is some $\pi \in S_h$ with $\delta(\varrho)^{[\pi]} \neq \delta(\varrho)$ in contradiction to ϱ coherent.

Thus $\delta(\varrho) \in \{\iota_k^h, \varrho_1, \varrho_2\}$. If $h = 4$ then $\delta(\varrho) \neq \varrho_i$ for $i \in \{1, 2\}$ because of (3), i.e., $f_i(v_1, \dots, v_{h_i}) = (x, x, y, z) \in \delta(E_k^4) \setminus \varrho_i$. Thus we see that $\delta(\varrho) = \iota_k^h = \delta(E_k^h)$. But then $\varrho = \sigma(\varrho) \cup \delta(\varrho) = E_k^h$ contradicting ϱ coherent.

Thus we can conclude that $C \not\subseteq M$ for all $M \in p\mathcal{M}_k$ and by Theorem 5.3.2 we get $[C]_P = \tilde{P}_k$. \square

6.4 Conclusion

The lists of all coherent relations for $k \in \{2, 3, 4, 5, 6\}$ can be found at

<http://www.math.uni-rostock.de/~schoelzel/papers/numbers/>

together with the source code of the program used to generate them. The source code is also printed in Appendix A.1. The program is written in Haskell, is single-threaded and needs about 52 hours on a SunFire V490 to compute all coherent relations for $k = 6$. In Table 6.1 we show the number of maximal partial clones $|p\mathcal{M}_k|$ compared to the number of maximal total clones $|\mathcal{M}_k|$ as given in [40]. The number of coherent relations is given in the following tables. A recursive formula for the number of maximal total clones has been given by Rosenberg [57]. Further investigations in this area might look into deriving a similar formula for the number of maximal partial clones.

k	$ \mathcal{M}_k $	$ p\mathcal{M}_k $
2	5	8
3	18	58
4	82	1 102
5	643	325 722
6	15 182	5 242 621 816
7	7 848 984	?
8	549 761 933 169	?

Table 6.1: Number of maximal (partial) clones

Chapter 7

Uniqueness of Minimal Covering of Maximal Partial Clones

In the last chapter we have shown that the number of maximal partial clones increases very fast if k increases. If we want to check if a set A of partial functions is complete we have to check all maximal partial clones if A is not a subset of any them. One special situation is the case that A consists of only one function.

A function $f \in \tilde{P}_k$ is called a *Sheffer partial function* (or *partial Sheffer function*) if it generates the partial clone \tilde{P}_k of all partial functions, i.e.,

$$f \text{ is a Sheffer partial function} \iff [f]_{\mathcal{P}} = \tilde{P}_k.$$

The name ‘‘Sheffer function’’ is named after H.M. Sheffer who proved in 1913 that the Boolean function NAND (not and) and its dual NOR (exclusive or) suffice to generate all Boolean functions; see [68]. They are defined by the following table.

x	y	NAND(x, y)	NOR(x, y)
0	0	1	0
0	1	0	1
1	0	0	1
1	1	1	0

The first known work to deal with such functions dates back to 1880 and is due to C.S. Peirce but only published in 1933 [46]. In 1935 D.L. Webb showed that P_k can be generated by one binary function for every $k \geq 2$ generalizing the result of Sheffer; see [72]. Then in 1967 and 1969 a general and minimal characterization of all Sheffer functions was given by Rousseau and Schofield [62, 65] just a few years after Rosenberg had characterized all the maximal total clones with his completeness theorem [55, 56]. Most Sheffer functions can be redefined on some places of their domain in an arbitrary way. This was considered by Rosenberg [58] and Romov [53]. Both of them use the notion of a partial Sheffer function but different from our one. So we can not use their results.

The notion of a *minimal covering* of the maximal clones of P_k has been introduced. This is a subset of the set of all maximal clones, such that for a single function f it suffices to check if f is in any member of the minimal covering to know if f is a Sheffer function. It is minimal in the sense that no clone can be removed from the minimal covering without destroying the previous property. For the maximal total clones it was shown that there is a unique minimal covering for every $k \geq 2$ and all members were identified.

Considering this, it is interesting to investigate the same route for Sheffer partial functions, i.e., try to show that the minimal covering of the maximal partial clones for some k is unique and determine all its members. The characterization of the maximal partial clones (see Theorem 5.3.2) due to Haddad and Rosenberg [14, 22, 24] has been done about twenty years ago and the question about minimal coverings is still not solved in the partial case. So it seems to be really more difficult than the total case.

For small k the problem is solved; for $k = 2$ in 1991 by Haddad and Rosenberg [23], for $k = 3$ in 2007 by Haddad and Lau [17], and for $k = 4$ in 2008 by the author in his diploma thesis [66] (published in [67]) based on the concrete list of the maximal partial clones given in [26]. That means there is a unique minimal covering $p\mathcal{C}_k$ of $p\mathcal{M}_k$ for $k \in \{2, 3, 4\}$ and the members are determined. The following table lists their sizes in comparison to the number of all maximal partial clones $|p\mathcal{M}_k|$.

k	$ p\mathcal{M}_k $	$ p\mathcal{C}_k $
2	8	5
3	58	26
4	1102	449

Furthermore some results are included in these papers that give members and non-members of minimal coverings for all $k \geq 2$. For example

- every maximal partial clone $p\text{POL}_k \varrho$ with $\emptyset \subset \varrho \subset E_k$ belongs to every minimal covering [17];
- every maximal partial clone $p\text{POL}_k \leq$ is in no minimal covering if $\leq \subset \tilde{E}_k^2$ is a reflexive and anti-symmetric relation such that there is some element $c \in E_k$ comparable to every other element, i.e., $c \leq a$ or $a \leq c$ for all $a \in E_k$; [66, 67];
- if the maximal partial clone C can be generated by a single function f , i.e., $C = [f]_{\text{P}}$, then C belongs to every minimal covering [17].

In this chapter we show that there is a unique minimal covering of the maximal partial clones. The ultimate goal to determine all the members of the minimal covering for every $k \geq 2$ seems still to be far away since the characterization of the maximal partial clones is considerably more difficult than the one for the maximal total clones.

7.1 Minimal Covering

Definition 7.1.1. A set $\mathcal{X} \subseteq p\mathcal{M}_k$ is a *minimal covering* of $p\mathcal{M}_k$, iff

- for every $f \in \tilde{P}_k$

$$[f]_{\text{P}} = \tilde{P}_k \iff \forall B \in \mathcal{X} : f \notin B$$

holds, and

- for each $C \in \mathcal{X}$ there is some $f_C \in \tilde{P}_k$ with

$$[f_C]_{\text{P}} \neq \tilde{P}_k \text{ and } (\forall B \in \mathcal{X} \setminus \{A\} : f_C \notin B).$$

Lemma 7.1.2. Let C be a maximal partial clone and there is some $f \in C$ with $f \notin B$ for all $B \in p\mathcal{M}_k \setminus \{C\}$. Then C is in every minimal covering of $p\mathcal{M}_k$.

Proof. Let $f \in C \in p\mathcal{M}_k$ with $f \notin B$ for all $B \in p\mathcal{M}_k \setminus \{C\}$. Assume there is a minimal covering \mathcal{X} of $p\mathcal{M}_k$ with $C \notin \mathcal{X}$. Then $[f]_P \subseteq C \subset \tilde{P}_k$ and $f \notin A$ for each $A \in \mathcal{X} \subseteq p\mathcal{M}_k \setminus \{C\}$, in contradiction to the first condition of a minimal covering. \square

Lemma 7.1.3. *Let $C \in p\mathcal{M}_k$ and $\mathcal{C} \subseteq p\mathcal{M}_k \setminus \{C\}$ such that every $C' \in \mathcal{C}$ is in every minimal covering of $p\mathcal{M}_k$ and for all $f \in C$ there is some $C_f \in \mathcal{C}$ with $f \in C_f$. Then C is in no minimal covering of $p\mathcal{M}_k$.*

Proof. Assume C is in some minimal covering \mathcal{X} of $p\mathcal{M}_k$. Then there is some $f_C \in \tilde{P}_k$ with $[f_C]_P \neq \tilde{P}_k$ and $f_C \notin B$ for all $B \in \mathcal{X} \setminus \{C\}$. From $\mathcal{C} \subseteq \mathcal{X}$ follows $f_C \notin C$. Thus $f \notin B$ for all $B \in \mathcal{X}$, and $[f]_P \neq \tilde{P}_k$ contradicting the assumption that \mathcal{X} is a minimal covering of $p\mathcal{M}_k$. Thus C is in no minimal covering. \square

7.2 Some Definitions

Definition 7.2.1. Let $\omega(v)$ be the set of entries of $v = (v_1, \dots, v_h) \in E_k^h$. That means we let

$$\omega(v) = \omega((v_1, \dots, v_h)) := \{v_1, \dots, v_h\}.$$

Additionally let

$$\omega(\varrho) = \bigcup_{v \in \varrho} \omega(v).$$

Definition 7.2.2. The set of coherent relations $\tilde{\mathcal{R}}_k^{\max}$ can be divided into the following sets:

$$\begin{aligned} \mathcal{U} &:= \{\chi^{(\mu)} \in \tilde{\mathcal{R}}_k^{\max} \mid \mu = 1\}, \\ &\quad \text{(non-trivial unary relations)} \\ \mathcal{A} &:= \{\chi^{(\mu)} \in \tilde{\mathcal{R}}_k^{\max} \mid \mu \geq 2 \text{ and } \chi \text{ is areflexive}\}, \\ &\quad \text{(areflexive coherent relations)} \\ \mathcal{Q} &:= \{\chi^{(\mu)} \in \tilde{\mathcal{R}}_k^{\max} \mid \mu \geq 2 \text{ and } \chi \text{ is quasi-diagonal}\}, \\ &\quad \text{(quasi-diagonal coherent relations)} \\ \mathcal{S} &:= \{\chi^{(\mu)} \in \tilde{\mathcal{R}}_k^{\max} \mid \mu \geq 3 \text{ and } \delta(\chi) = \iota_k^\mu\}, \\ &\quad \text{(totally symmetric, totally reflexive coherent relations)} \\ \mathcal{L} &:= \{\chi^{(\mu)} \in \tilde{\mathcal{R}}_k^{\max} \mid \mu = 4 \text{ and } \delta(\chi) \in \{\varrho_1, \varrho_2\}\}. \\ &\quad \text{(quasi-linear coherent relations)} \end{aligned}$$

Definition 7.2.3. Let $\varrho^{(h)} \in \mathcal{Q}$ be arbitrary and

$$\varrho^* := \text{pr}_0(\varepsilon(\varrho) \setminus \iota_h^2),$$

i.e., the elements of E_h which are in no singleton class of $\varepsilon(\varrho)$. We define

$$\begin{aligned} \text{pp } \varrho &:= \text{pr}_{\varrho^*} \varrho, \\ \|\varrho\| &:= |\varrho^*|, \\ \mathcal{Q}_0 &:= \{\chi^{(\mu)} \in \mathcal{Q} \mid \varepsilon(\chi) \text{ has no singular equivalence class}\} \\ &= \{\chi^{(\mu)} \in \mathcal{Q} \mid \text{pp } \chi = \chi\} = \{\chi^{(\mu)} \in \mathcal{Q} \mid \|\chi\| = \mu\}, \\ \mathcal{Q}_1 &:= \mathcal{Q} \setminus \mathcal{Q}_0. \end{aligned}$$

If $\varrho \in \mathcal{Q}_1$ then define

$$\mathcal{Q}_\varrho := \left\{ \chi \in \mathcal{Q}_1 \mid \begin{array}{l} (\|\chi\| < \|\varrho\|) \text{ or} \\ (\|\chi\| = \|\varrho\| \text{ and } \text{pp } \chi \not\subseteq \text{pp } \varrho) \end{array} \right\}.$$

Because $\text{pPOL}_k \varrho = \text{pPOL}_k \varrho^{[\pi]}$ for all $\pi \in S_h$ we use the convention that $\text{pp } \varrho = \text{pr}_{E_{\|\varrho\|}} \varrho$ for all $\varrho \in \mathcal{Q}$.

The relations in \mathcal{Q}_1 are exactly the coherent quasi-diagonal relations ϱ where $\varepsilon(\varrho)$ has at least one singular class.

Example 7.2.4. Let $k = 10$ and

$$\varrho^{(5)} := \begin{pmatrix} 0 & 5 \\ 1 & 6 \\ 2 & 7 \\ 3 & 8 \\ 4 & 9 \end{pmatrix} \cup \delta_{\{0,1\},\{2,3\}}^{(5)} \in \mathcal{Q}.$$

Then

$$\begin{aligned} \varepsilon(\varrho) &= \begin{pmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 3 & 2 & 0 & 1 & 2 & 3 & 4 \end{pmatrix}, \\ \varrho^* &= \{0, 1, 2, 3\}, \\ \text{pp } \varrho &= \begin{pmatrix} 0 & 5 \\ 1 & 6 \\ 2 & 7 \\ 3 & 8 \end{pmatrix} \cup \delta_{\{0,1\},\{2,3\}}^{(4)}, \\ \|\varrho\| &= 4. \end{aligned}$$

Then $\varrho \in \mathcal{Q}_1$ and $\text{pp } \varrho \in \mathcal{Q}_0$.

7.3 A Product of Functions

Definition 7.3.1. Let $D' \in E_k^{a \times b}$ be an (a, b) -matrix on E_k , i.e.,

$$D' = \begin{pmatrix} d_{11} & \dots & d_{1b} \\ \vdots & \ddots & \vdots \\ d_{a1} & \dots & d_{ab} \end{pmatrix}$$

with $d_{ij} \in E_k$ for all i, j .

If a function $f^{(n)} \in \tilde{P}_k$ is defined by

$$f(D') := v = (v_1, \dots, v_a)^T$$

then

$$\begin{aligned} n &:= b, \\ \text{dom } f &:= D := \{(d_{i1}, \dots, d_{ib}) \mid i \in \{1, \dots, a\}\}, \\ f(d_{i1}, \dots, d_{ib}) &:= v_i \end{aligned}$$

for all $i \in \{1, \dots, a\}$. If the domain $D := \text{dom } f$ is given then D' is a matrix whose rows are the entries of D in lexicographical order.

Let $\chi^{(h)} \in \mathcal{R}_k$ and $f^{(n)} \in \tilde{P}_k$ be defined by $f(\chi) = v$ then assume χ be given as a matrix as explained before, i.e., $n = |\chi|$ and $v \in E_k^h$.

Definition 7.3.2. Let $f^{(n)} \in \tilde{P}_k$ with $D = \text{dom } f$ and $g^{(m)} \in \tilde{P}_k$ with $E = \text{dom } g$. Then $D' \in E_k^{|D| \times n}$ and $E' \in E_k^{|E| \times m}$.

Let $N := n \cdot m$. Define $D' \otimes E' \in E_k^{(|D|+|E|) \times N}$ by

$$D' \otimes E' = \begin{pmatrix} D'_{*1} & \cdots & D'_{*1} & D'_{*2} & \cdots & D'_{*2} & \cdots & D'_{*n} \cdots & D'_{*n} \\ E'_{*1} & \cdots & E'_{*m} & E'_{*1} & \cdots & E'_{*m} & \cdots & E'_{*1} \cdots & E'_{*m} \end{pmatrix}.$$

Define $F^{(N)} := (f \otimes g) \in \tilde{P}_k^{(N)}$ by

$$F(D' \otimes E') := F \left(\frac{D'_{*1} \mid \cdots \mid D'_{*n}}{E' \mid \cdots \mid E'} \right) := \left(\frac{f(D')}{g(E')} \right).$$

We assume E' has no constant rows so F is well-defined. Then

$$\begin{aligned} \text{dom } F &= \{ \underbrace{(a_1, \dots, a_1)}_{m \text{ times}}, \dots, \underbrace{(a_i, \dots, a_i)}_{m \text{ times}}, \dots, \underbrace{(a_n, \dots, a_n)}_{m \text{ times}} \mid (a_1, \dots, a_n) \in D \} \\ &\cup \{ (b_1, \dots, b_m, b_1, \dots, b_m, \dots, b_1, \dots, b_m) \mid (b_1, \dots, b_m) \in E \}. \end{aligned}$$

Let $c = (c_1, \dots, c_N) \in \text{dom } F$. Then we say it is from the E -part or g -part of F if $c = \text{pr}_i(E', E', \dots, E')$ for some i . Otherwise we say it is from the D -part or f -part of F .

Likewise we inductively set $f \otimes g_1 \otimes \cdots \otimes g_{l-1} \otimes g_l := (f \otimes g_1 \otimes \cdots \otimes g_{l-1}) \otimes g_l$ with $g_i \in \tilde{P}_k$ for all $i \in \{1, \dots, l\}$.

Example 7.3.3. Let $\chi^{(2)}$ be the relation given by

$$\chi = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}.$$

If we define f by

$$f(\chi) := \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

then we mean

$$\begin{aligned} f \left(\begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \right) &:= \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \\ D = \text{dom } f &:= \{(0, 0), (2, 1)\}, \\ D' &= \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}. \end{aligned}$$

Let $g \in \tilde{P}_k$ be given by

$$g \left(\begin{pmatrix} 0 & 2 & 3 \\ 2 & 4 & 5 \end{pmatrix} \right) := \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

with $E = \text{dom } g = \{(0, 2, 3), (2, 4, 5)\}$. Then

$$(f \otimes g) \left(\begin{pmatrix} 0 & 0 & 0 & \mid & 0 & 0 & 0 \\ 2 & 2 & 2 & \mid & 1 & 1 & 1 \\ 0 & 2 & 3 & \mid & 0 & 2 & 3 \\ 2 & 4 & 5 & \mid & 2 & 4 & 5 \end{pmatrix} \right) = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 0 \end{pmatrix}$$

and

$$\text{dom}(f \otimes g) = \{(0, 0, 0, 0, 0, 0), (2, 2, 2, 1, 1, 1), \\ (0, 2, 3, 0, 2, 3), (2, 4, 5, 2, 4, 5)\}.$$

7.4 Some Simple Cases

For the rest of this chapter we assume $k \geq 3$ as the case $k = 2$ would complicate things a bit and is already solved:

Lemma 7.4.1 ([23]). *The set*

$$p\mathcal{C}_2 := \left\{ \text{pPOL}_2\{0\}, \text{pPOL}_2\{1\}, \text{pPOL}_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, P_2 \cup C_\infty \right\}$$

is the unique minimal covering of $p\mathcal{M}_2$.

Lemma 7.4.2 (Lemma 4 [17]). *The maximal partial clone $P_k \cup C_\infty$ belongs to every minimal covering of $p\mathcal{M}_k$.*

Lemma 7.4.3 (Lemmas 5, 7 [17]). *Let $\varrho \in \mathcal{U}$, i.e., $\emptyset \subset \varrho \subset E_k$. Then $\text{pPOL}_k \varrho$ belongs to every minimal covering of $p\mathcal{M}_k$.*

Lemma 7.4.4. *Let $\varrho^{(h)} \in \tilde{\mathcal{R}}_k^{\max}$ with $h \geq 2$ and $f^{(n)} \in \tilde{P}_k$. Let $c_{*1}, c_{*2}, \dots, c_{*n} \in \varrho$ with $c_{1*}, \dots, c_{n*} \in \text{dom}(f)$ and $c_{i'*} = c_{i''*}$ for some $1 \leq i' < i'' \leq h$. Then $d := (d_1, \dots, d_h) := f(c_{*1}, c_{*2}, \dots, c_{*n}) \in \varrho$.*

Proof. Because two rows are equal we have $c_{*i} \in \delta(\varrho) \subseteq \iota_k^h$ for all $i \in \{1, 2, \dots, n\}$.

Because ϱ is coherent there are the following cases for $\delta := \delta(\varrho)$:

- $\delta = \emptyset$, i.e., ϱ is areflexive. Then $c_{*1} \notin \varrho$ contradicting the assumption.
- $\delta = \delta_\varepsilon$ for some equivalence relation $\varepsilon \neq \iota_h^2$. Then $c_{*1}, c_{*2}, \dots, c_{*n} \in \delta_\varepsilon$ and thus $d \in \delta_\varepsilon \subseteq \varrho$.
- $\delta = \iota_k^h$. Then $d_{i'} = f(c_{i'*}) = f(c_{i''*}) = d_{i''}$, i.e., $d \in \iota_k^h \subseteq \varrho$.
- $\delta = \varrho_1$, i.e., $\delta = \{(a, a, a, a) \mid a \in E_k\} \cup \{(a, a, b, b) \mid a, b \in E_k, a \neq b\} \cup \{(a, b, a, b) \mid a, b \in E_k, a \neq b\}$.

– If $i' = 1$ and $i'' = 2$ then

$$c_{*j} \in \delta \setminus \{(a, b, a, b) \mid a, b \in E_k, a \neq b\} = \delta_{\{0,1\}, \{2,3\}}$$

for all $j \in \{1, 2, \dots, n\}$ and thus $d \in \delta_{\{0,1\}, \{2,3\}} \subset \varrho_1 \subseteq \varrho$. The case $i' = 3$ and $i'' = 4$ is analogous.

– If $i' = 1$ and $i'' = 3$ then

$$c_{*j} \in \delta \setminus \{(a, a, b, b) \mid a, b \in E_k, a \neq b\} = \delta_{\{0,2\}, \{1,3\}}$$

for all $j \in \{1, 2, \dots, n\}$ and thus $d \in \delta_{\{0,2\}, \{1,3\}} \subset \varrho_1 \subseteq \varrho$. The case $i' = 2$ and $i'' = 4$ is analogous.

– If $i' = 1$ and $i'' = 4$ then $c_{*j} \in \{(a, a, a, a) \mid a \in E_k\} = \delta_{\{0,1,2,3\}}$ for all $j \in \{1, 2, \dots, n\}$ and thus $d \in \delta_{\{0,1,2,3\}} \subset \varrho_1 \subseteq \varrho$. The case $i' = 2$ and $i'' = 3$ is analogous.

- $\delta = \varrho_2$ is done analogously. □

Lemma 7.4.5. Let $\varrho^{(h)} \in \mathcal{S}$ with either

$$h \geq 4, \text{ or} \quad (7.1)$$

$$h = 3 \text{ and } \exists x \in \sigma(E_k^2) \forall a \in E_k \setminus \omega(x) \exists y \in \sigma(\varrho) : \omega(x) \cup \{a\} = \omega(y). \quad (7.2)$$

Then

$$\forall f \in \text{pPOL}_k \varrho \exists \gamma \in \mathcal{U} \cup \{\chi\} : f \in \text{pPOL}_k \gamma \quad (7.3)$$

with

$$\chi := \{x \in E_k^{h-1} \mid \{x\} \times E_k \subseteq \varrho\}.$$

Furthermore $\text{pPOL}_k \chi \in p\mathcal{M}_k$ holds.

Proof. The definition of χ implies that χ is totally symmetric and totally reflexive. We have to show that χ is non-diagonal. $\chi \neq \iota_k^2$ for $h = 3$ because of (7.2). Assume $\chi = E_k^{h-1}$. Since $\varrho \neq E_k^h$ there is an $x := (x_1, x_2, \dots, x_h) \in E_k^h \setminus \varrho$. This contradicts the construction of χ , because $(x_1, x_2, \dots, x_{h-1}) \in \chi$ and $x \notin \varrho$ hold. Thus χ is a non-diagonal totally symmetric totally reflexive relation and thus $\text{pPOL}_k \chi \in p\mathcal{M}_k$.

Let $f^{(n)} \in \text{pPOL}_k \varrho$ be arbitrary. Assume, $f \notin \text{pPOL}_k \gamma$ for all $\gamma \in \mathcal{U} \cup \{\chi\}$. Then there are $c_{*1}, \dots, c_{*n} \in \chi$ with $c := f(c_{*1}, \dots, c_{*n}) \in E_k^{h-1} \setminus \chi$. This means,

$$\exists q \in E_k \setminus \omega(c) \forall y \in \sigma(\varrho) : \omega(c) \cup \{q\} \neq \omega(y).$$

Because of $f \notin \text{pPOL}_k(E_k \setminus \{q\})$ there are some $q_1, \dots, q_n \in E_k \setminus \{q\}$ with $f(q_1, \dots, q_n) = q$. Thus follows

$$f \begin{pmatrix} c_{*1} & \dots & c_{*n} \\ q_1 & \dots & q_n \end{pmatrix} = \begin{pmatrix} c \\ q \end{pmatrix} =: t$$

with $|\omega(t)| = h$, and therefore $t \notin \iota_k^h$. Because of $\omega(t) \neq \omega(y)$ for every $y \in \sigma(\varrho)$ by construction, $t \notin \varrho$ holds. But c_{*1}, \dots, c_{*n} are chosen with $\begin{pmatrix} c_{*i} \\ q_i \end{pmatrix} \in \varrho$ for all $i \in \{1, \dots, n\}$ contradicting $f \in \text{pPOL}_k \varrho$. Thus (7.3) holds. \square

The set \mathcal{S}' consists of all relations in \mathcal{S} which do not fulfill the conditions of Lemma 7.4.5, i.e., $\mathcal{S}' := \{\chi^{(\mu)} \mid \chi \in \mathcal{S}, \mu = 3 \text{ and } (7.2) \text{ is not fulfilled by } \chi\}$.

Lemma 7.4.6. Let $\varrho^{(h)} \in \mathcal{Q}_0$ with $h = 2$ and

$$\exists x \in E_k \exists \pi \in S_2 : \{x\} \times E_k \subseteq \varrho^{[\pi]}.$$

Then $\text{pPOL}_k \varrho$ is in no minimal covering of $p\mathcal{M}_k$.

Proof. Let $f^{(n)} \in \text{pPOL}_k \varrho$ be arbitrary. Assume $f \notin \text{pPOL}_k \theta$ for all $\theta \in \mathcal{U}$.

Let $A \subset E_k$ be a maximal set with $A \times E_k \subseteq \varrho^{[\pi]}$. Let $y \in E_k$ be arbitrary. Because $f \notin \text{pPOL}_k A$ and $f \notin \text{pPOL}_k(E_k \setminus \{y\})$ there are rows $c_A \in E_k^n$ and $c_y \in E_k^n \setminus \{y\}$ with $f(c_A) =: a \in E_k \setminus A$ and $f(c_y) = y$. Thus $(a, y) \in \varrho^{[\pi]}$ and because y is arbitrary we get $(A \cup \{a\}) \times E_k \subseteq \varrho^{[\pi]}$ contradicting A maximal.

Thus the assumption is wrong and $f \in \text{pPOL}_k \theta$ for some $\theta \in \mathcal{U}$. This implies $\text{pPOL}_k \varrho$ is in no minimal covering because $\text{pPOL}_k \theta$ is in every minimal covering of $p\mathcal{M}_k$ by Lemma 7.4.3. \square

The set \mathcal{Q}'_0 consists of all relations in \mathcal{Q}_0 which do not fulfill the conditions of Lemma 7.4.6.

7.5 Sorting the Minimal Coverings

Definition 7.5.1. Let $\varrho, \chi \in \widetilde{\mathcal{R}}_k^{\max}$ with $\varrho \neq \chi$, i.e., $\text{pPOL}_k \varrho \neq \text{pPOL}_k \chi$ by definition of $\widetilde{\mathcal{R}}_k^{\max}$. We write $\varrho \ll \chi$ iff

$$\forall f \in \text{pPOL}_k \varrho \exists g \in \text{pPOL}_k \chi \left((g \notin \text{pPOL}_k \chi) \text{ and } (\forall \psi \in \widetilde{\mathcal{R}}_k^{\max} (f \notin \text{pPOL}_k \psi \implies g \notin \text{pPOL}_k \psi)) \right)$$

Lemma 7.5.2. Let $X = \text{pPOL}_k \varrho \in p\mathcal{M}_k$, $f \in X$, and $\mathcal{Y} \subseteq p\mathcal{M}_k$ with $f \notin Y$ for all $Y \in \mathcal{Y}$ and

$$\mathcal{Z} := \{\text{pPOL}_k \psi \mid \psi \in \widetilde{\mathcal{R}}_k^{\max} \text{ and } \varrho \ll \psi\} \subseteq p\mathcal{M}_k.$$

Then there is some $F \in X$ with $F \notin Y$ for all $Y \in \mathcal{Y} \cup \mathcal{Z}$.

Proof. Let $\{\text{pPOL}_k \psi_1, \dots, \text{pPOL}_k \psi_l\} := \mathcal{Z}$ and $f_0 := f$.

Let $f_i \in X$ with $f_i \notin Y$ for all $Y \in \mathcal{Y}$ and $f_i \notin \text{pPOL}_k \psi_j$ for all $j \leq i$. Because $\varrho \ll \psi_{i+1}$ there is some $f_{i+1} \in X$ with $f_{i+1} \notin Y$ for all $Y \in \mathcal{Y}$ and $f_{i+1} \notin \text{pPOL}_k \psi_j$ for all $j \leq i+1$ by definition of \ll .

Thus by induction there is some $F := f_l \in X$ with $F \notin Y$ for all $Y \in \mathcal{Y}$ and $F \notin Z$ for all $Z \in \mathcal{Z}$, i.e., $F \notin Y$ for all $Y \in \mathcal{Y} \cup \mathcal{Z}$. \square

Remark 7.5.3. With the help of \ll we can define a directed graph $\mathcal{G} = (p\mathcal{M}_k, E)$ such that $(X, Y) \notin E$ for all $X, Y \in p\mathcal{M}_k$ with $X = \text{pPOL}_k \varrho$, $Y = \text{pPOL}_k \psi$ and $\varrho \ll \psi$. We show that \mathcal{G} is acyclic and conclude that there is a unique minimal covering of $p\mathcal{M}_k$.

If $X \in p\mathcal{M}_k$ is a sink in \mathcal{G} then X is in every minimal covering of $p\mathcal{M}_k$. Assume this is false. Then there is a minimal covering \mathcal{Y} of $p\mathcal{M}_k$ with $X \notin \mathcal{Y}$ and

$$\forall f \in X \exists Y \in \mathcal{Y} : f \in Y.$$

Since X is a sink, i.e., $(X, Y) \notin E$, we have $\varrho \ll \psi$ for all $Y = \text{pPOL}_k \psi \in \mathcal{Y}$ and thus by Lemma 7.5.2 there is some $F \in X$ with $F \notin Y$ for all $Y \in \mathcal{Y}$ contradicting \mathcal{Y} is a covering of $p\mathcal{M}_k$. Thus X is in every minimal covering of $p\mathcal{M}_k$.

If $X \in p\mathcal{M}_k$ is not a sink in \mathcal{G} then X is covered by its successors $U(X) := \{Y \in p\mathcal{M}_k \mid (X, Y) \in E\}$, i.e.,

$$X \subseteq \bigcup_{Y \in U(X)} Y.$$

Assume this is false. Then there is some $f \in X$ with $f \notin X'$ for all $X' \in U(X)$. By Lemma 7.5.2 there is some $F \in X$ with $F \notin X'$ for all $X' \in U(X)$ and $F \notin Z$ for all $Z \in p\mathcal{M}_k$ with $(X, Z) \in E$. Thus $F \notin Y$ for all $Y \in \mathcal{Y} \setminus \{X\}$. But then $U(X) = \emptyset$ because of the existence of F , i.e., X is a sink. Thus X is covered by $U(X)$.

Then we show in following sections that \mathcal{G} is acyclic. This implies if X is not a sink then X is covered by sinks since \mathcal{G} is transitive and finite, i.e., X is in no minimal covering. Thus there is only one minimal covering.

Remark 7.5.4. Sometimes we write $\chi \subset \varrho$ to mean $\chi \subset \varrho^{[\pi]}$ for some $\pi \in S_h$. Because $\text{pPOL}_k \varrho = \text{pPOL}_k \varrho^{[\pi]}$ we can assume $\pi = \text{id}$ in most cases.

Similarly if we write $\chi \not\subseteq \varrho$ then $\chi \not\subseteq \varrho^{[\pi]}$ for all $\pi \in S_h$.

Lemma 7.5.5. *Let $\varrho^{(h)} \in \mathcal{A} \cup \mathcal{Q}'_0 \cup \mathcal{L}$ and $\chi^{(\mu)} \in (\mathcal{Q} \cup \mathcal{L} \cup \mathcal{S}) \setminus \{\varrho\}$. Then $\varrho \ll \chi$.*

Proof. Let $\sigma := \sigma(\varrho)$ and $\delta := \delta(\varrho)$. Let $f \in \text{pPOL}_k \varrho$ be arbitrary. If $f \notin \text{pPOL}_k \chi$ then $g := f$ fulfills the conditions of $\varrho \ll \chi$. Thus assume $f \in \text{pPOL}_k \chi$.

First case: $\mu \leq h$ or $\chi \in \mathcal{S}$.

Let $g_0(\chi) := v$ (see Definition 7.3.1) for some $v \in \varrho^{[\pi]} \setminus \chi$ if $\chi \subset \varrho^{[\pi]}$ for some $\pi \in S_h$ (w.l.o.g. $\pi = \text{id}$) and $v \in E_k^\mu \setminus \chi$ otherwise. Then $g_0 \notin \text{pPOL}_k \chi$.

We have to show $g_0 \in \text{pPOL}_k \varrho$. Assume $g_0^{(n)} \notin \text{pPOL}_k \varrho$. Then there are some rows c_{1*}, \dots, c_{h*} with $c_{*1}, \dots, c_{*n} \in \varrho$ and $g_0(c_{*1}, \dots, c_{*n}) =: d \notin \varrho$. Because of Lemma 7.4.4 all rows have to be different. Thus if $\mu = h$ then $\{c_{*1}, \dots, c_{*n}\} \subseteq \chi^{[\pi']}$ for some $\pi' \in S_h$.

- Let $\mu < h$. Because g_0 is only defined on μ different rows Lemma 7.4.4 applies and we have $d \in \varrho$.
- Let $\mu = h$ and $\chi \subset \varrho$. Then $\pi' \in \Gamma_{\sigma(\varrho)}$ because $\chi^{[\pi']} \subset \varrho$ and $\chi \subset \varrho$. Thus $d = v^{[\pi']} \in \varrho$ because $v \in \varrho$.
- Let $\mu = h$ and $\chi \not\subseteq \varrho^{[\pi]}$ for all $\pi \in S_h$. Thus there is some $j \in \{1, 2, \dots, n\}$ with $c_{*j} \notin \varrho$ contradicting the assumption.
- Let $\chi \in \mathcal{S}$ and $\mu > h$. Then $E_k^h = \text{pr}_A t_k^\mu = \{c_{*1}, \dots, c_{*n}\} \subseteq \varrho$ contradicting ϱ coherent.

Thus $g_0 \in \text{pPOL}_k \varrho$.

Let $G_0 := f \otimes g_0$ and $L = 0$. By construction $G_0 \notin \text{pPOL}_k \chi$.

Second case: $\mu > h$ and $\chi \notin \mathcal{S}$.

Let $\sigma_0 := \sigma$ and define $\sigma_1, \sigma_2, \dots, \sigma_l, \sigma_{l+1}$ recursively until $\sigma_{l+1} = \emptyset$ holds with $\emptyset \notin \{\sigma_0, \sigma_1, \dots, \sigma_l\}$.

Let $\emptyset \subset \sigma_i \subseteq \sigma$ be given. Because ϱ is coherent, there is a relational homomorphism $\varphi_i : E_k \rightarrow E_h$ from σ_i to $M(\varrho)$ and an $s_i \in \sigma_i$ with $\varphi_i(s_i) = \eta_h$. Let $\sigma_{i+1} := \{s \in \sigma_i \mid \varphi_i(s) \in \delta \cap E_h^h\}$. From $\varphi_i(s_i) = \eta_h \notin \delta \cap E_h^h$ follows $|\sigma_{i+1}| < |\sigma_i|$. Because $|\sigma|$ is finite there is an $l \in \mathbb{N}$ with $\sigma_{l+1} = \emptyset$ and $\emptyset \notin \{\sigma_0, \sigma_1, \dots, \sigma_l\}$.

Define $\varphi_* : E_h \rightarrow E_k$ by $\varphi_*(\eta_h) := s_0$. Then define for $i \in \{0, 1, \dots, l\}$ the function $q_i : E_k \rightarrow E_k$ with $q_i(x) := \varphi_*(\varphi_i(x))$. Then q_i is a relational homomorphism from σ_i to ϱ . For $v := (v_0, v_1, \dots, v_{m-1}) \in E_k^m$ and $m \in \{1, 2, \dots, k\}$ let $Q_{-1}(v) := \delta_{E_m}^m$,

$$Q_i(v) := Q_{i-1}(v) \cup \{(x_{v_0}, x_{v_1}, \dots, x_{v_{m-1}}) \in E_k^m \mid \varphi_i(a) = \varphi_i(b) \implies x_a = x_b\}$$

and $Q_i(v) =: Q_{l+j}(v)$ for all $j \geq 1$. Because of $|\varphi_i(E_k)| = h$ we have $|\omega(x)| \leq h$ for all $x \in Q_i(\eta_k)$.

Let

$$\begin{aligned} \forall i \in \{0, 1, 2, \dots, l\} : g_i(\{\eta_k\} \cup Q_{i-1}(\eta_k)) &:= q_i(\eta_k), \\ \forall j \in \{1, 2, \dots, |E_k^k|\} : g_{l+j}(\{\eta_k\} \cup Q_l(\eta_k)) &:= w_j, \end{aligned}$$

where $\{w_1, w_2, \dots, w_{|E_k^k|}\} = E_k^k$. Let $L := l + |E_k^k|$.

We now show $g_i \in \text{pPOL}_k \varrho$ for $i \in \{0, 1, \dots, L\}$. Assume $g_i^{(n)} \notin \text{pPOL}_k \varrho$. Then there are rows c_{1*}, \dots, c_{h*} with $c_{*1}, \dots, c_{*n} \in \varrho$ and $g_i(c_{*1}, \dots, c_{*n}) =: d \in E_k^h \setminus \varrho$. By construction of g_i we can w.l.o.g. assume that $c' := c_{*1} = \text{pr}_{p_1, \dots, p_h} \eta_k$ with pairwise different coordinates p_1, \dots, p_h . Thus $c' \in \sigma(\varrho)$.

- If $c' \in \sigma_i$ (i.e., $i \leq l$ because $\sigma_{l+1} = \emptyset$) then $d = g_i(c', c_{*2}, \dots, c_{*n}) = q_i(c') \in \varrho$ because q_i is a relational homomorphism from σ_i to ϱ .
- Now let $c' \in \varrho \setminus \sigma_i$. Then there is some $j < i$ with $c' \in \sigma_j$ and $c' \notin \sigma_{j+1}$. Then $\varphi_j(c') \in \sigma(E_k^h)$. Thus

$$E_k^h = Q_j(c') \subseteq Q_{i-1}(c') = \text{pr}_{p_1, \dots, p_h} Q_{i-1}(\eta_k) = \{c_{*2}, \dots, c_{*n}\} \subseteq \varrho,$$

i.e., $\varrho = E_k^h$ in contradiction to ϱ coherent.

Thus $g_i \in \text{pPOL}_k \varrho$ for all $i \in \{0, 1, \dots, L\}$.

Let $G_i := f \otimes g_0 \otimes g_1 \otimes \dots \otimes g_i$ for $i \in \{0, 1, \dots, L\}$.

We show $G_L \notin \text{pPOL}_k \chi$.

- If $g_i \notin \text{pPOL}_k \chi$ for some $i \in \{0, 1, \dots, L\}$, then $G_L \notin \text{pPOL}_k \chi$.
- Otherwise $Q_l(v) \subseteq \chi$ for some $v \in \chi$ by construction of $Q_l(v)$ and $g_i \in \text{pPOL}_k \chi$ for $i \in \{0, 1, \dots, L\}$. Then $g_{l+j}(\{v\} \cup \delta_{E_\mu}^\mu \cup Q_l(v)) \in \chi$ for all $j \in \{1, 2, \dots, |E_k^k|\}$ and thus $E_k^\mu \subseteq \chi$ in contradiction to χ coherent. Thus $G_L \notin \text{pPOL}_k \chi$.

Let $G_{-1} = f$ and $g = G_L$. Now we show that $g \in \text{pPOL}_k \varrho$ by induction over $i \in \{-1, 0, 1, \dots, L\}$.

The basis $G_{-1} = f \in \text{pPOL}_k \varrho$ is given by choice of f .

The induction goes from $i - 1$ to i for $i \in \{0, 1, \dots, L\}$. Let $G_{i-1} \in \text{pPOL}_k \varrho$. We want to show $G_i = G_{i-1} \otimes g_i \in \text{pPOL}_k \varrho$. Let $D := \text{dom } G_{i-1}$, $E := \text{dom } g_i$ and D', E' the associated matrices (see Definition 7.3.1).

Assume $G_i \notin \text{pPOL}_k \varrho$. Then there are some rows c_{1*}, \dots, c_{h*} from $D' \otimes E'$ with

$$F \begin{pmatrix} c_{1*} \\ \vdots \\ c_{h*} \end{pmatrix} =: \begin{pmatrix} s_1 \\ \vdots \\ s_h \end{pmatrix} =: s \notin \varrho \quad (7.4)$$

and

$$\forall j \in \{1, 2, \dots, N\} : c_{*j} \in \varrho. \quad (7.5)$$

By Lemma 7.4.4 all the rows c_{i*} are pairwise different.

Because $G_{i-1} \in \text{pPOL}_k \varrho$ and $g_i \in \text{pPOL}_k \varrho$ some rows have to be from the D -part and some from the E -part of $D' \otimes E'$.

Assume w.l.o.g. c_{1*} is from the D -part and c_{h*} is from the E -part.

From $\delta(\chi) \neq \emptyset$ and χ coherent follows

$$\delta_{E_\mu}^{(\mu)} = \{(x_1, \dots, x_\mu) \in E_k^\mu \mid x_1 = x_2 = \dots = x_\mu\} \subseteq \delta(\chi) \subseteq \chi.$$

- $\varrho \in \mathcal{A}$. Because $E_k \subseteq Q_i(x) \subseteq c_{h*}$ for some $x \in E_k$ there is some column j with $c_{1j} = c_{hj}$ in contradiction to ϱ areflexive.
- $\varrho \in \mathcal{Q}'_0$. If $h = 2$ then $\{x\} \times E_k \subseteq \varrho$ in contradiction to $\varrho \in \mathcal{Q}'_0$, i.e., ϱ does not fulfill the conditions of Lemma 7.4.6.

Let $h \geq 3$ and $i \in \{2, \dots, h-1\}$ be arbitrary. Because $c_{1*} \neq c_{i*}$ there is some column j with $(c_{1j}, c_{ij}, c_{hj})^T = (x, y, y)^T$ and $x \neq y$. Because i is arbitrary and $\delta(\varrho) = \delta_\varepsilon$ for some equivalence relation ε we get that $(0, a) \notin \varepsilon = \varepsilon(\varrho)$ for all $a \neq 0$, i.e., $\varepsilon(\varrho)$ has a singular equivalence class contradicting $\varrho \in \mathcal{Q}_0$.

- $\varrho \in \mathcal{L}$. If c_{1*}, c_{2*}, c_{3*} (which are pairwise different) from D and c_{4*} from E , then there is a column $c_{*j} = (x, y, z, w)^T \notin \varrho$ with $|\{x, y, z\}| \geq 2$ and $|\{x, y, z, w\}| = 3$ in contradiction to (7.5). Otherwise there is $c_{*j} = (x, y, y, y)^T \notin \varrho$ with $x \neq y$ contradicting (7.5).

Thus $G_i \in \text{pPOL}_k \varrho$ and by induction $g = G_L \in \text{pPOL}_k \varrho$. Because $g \notin \text{pPOL}_k \chi$ we get $\varrho \ll \chi$. \square

Lemma 7.5.6. *Let $\varrho^{(h)} \in \mathcal{Q}_1 \cup \mathcal{S}'$ and $\chi^{(\mu)} \in \mathcal{S} \setminus \{\varrho\}$. Then $\varrho \ll \chi$.*

Proof. Let $f^{(n)} \in \text{pPOL}_k \varrho$ be arbitrary. If $f \notin \text{pPOL}_k \chi$ then $g := f$ fulfills the conditions of \ll . Thus assume $f \in \text{pPOL}_k \chi$.

Let $g_\chi^{(m)}(\chi) := v$ for some $v \in \varrho \setminus \chi$ if $\chi \subseteq \varrho$ and $v \in E_k^\mu \setminus \chi$ otherwise. Then $g_\chi \in \text{pPOL}_k \varrho$ and let $F^{(N)} := f \otimes g_\chi$ with $N = n \cdot m$. We get $g \notin \text{pPOL}_k \chi$ because $g_\chi \notin \text{pPOL}_k \chi$. Let $D := \text{dom } f$ and $E := \text{dom } g_\chi$ as in Definition 7.3.2, $n = |D|$, $m = |E|$ and $N = |D| \cdot |E|$.

It suffices to show $F \in \text{pPOL}_k \varrho$.

Assume this is false. Then there are some rows c_{1*}, \dots, c_{h*} from $D \otimes E$ with $s_i := F(c_{i*})$, $s := (s_1, \dots, s_h)^T \notin \varrho$ and $c_{*j} \in \varrho$ for all $j \in \{1, 2, \dots, N\}$. By Lemma 7.4.4 all the rows c_{i*} are pairwise different.

Because $f \in \text{pPOL}_k \varrho$ and $g_\chi \in \text{pPOL}_k \varrho$ some rows have to be from the D -part and some from the E -part of $D' \otimes E'$. Then w.l.o.g. ϱ is given such that c_{1*} is from the D -part and c_{h*} is from the E -part. Let $c_{i_1*}, \dots, c_{i_q*}$ with $q < h$ be the rows of the E -part. Then $\delta_{E_q}^{(q)} \subseteq (c_{ij})_{i=i_1, \dots, i_q, j=1, \dots, N}$.

- $\varrho \in \mathcal{Q}_1$. Let $c_{i_1*}, c_{i_2*}, c_{h*}$ be three different rows. There are columns j_1 and j_2 with $(c_{i_1 j_1}, c_{i_2 j_1}, c_{h j_1})^T = (x, y, x)^T$, $(c_{i_1 j_2}, c_{i_2 j_2}, c_{h j_2})^T = (x, y, y)^T$ and $x \neq y$. Thus $(i_1 - 1, i_2 - 1), (i_1 - 1, h - 1), (i_2 - 1, h - 1) \notin \varepsilon(\varrho)$. Because $i_1 \neq i_2 \neq h$ are arbitrary it follows that $\delta(\varrho) = \delta_{i_h}^{(h)} = E_k^h$ in contradiction to ϱ coherent.
- $\varrho \in \mathcal{S}'$. If only c_{1*} is from D , then $\{x\} \times E_k^2 \subseteq \varrho$ for some $x \in c_{1*}$, and thus $(x, y)^T \times E_k \subseteq \varrho$ with $x \neq y$. If only c_{h*} is from E then $(x, y)^T \times E_k \subseteq \varrho$ with $x \neq y$, $x \in c_{1*}$ and $y \in c_{2*}$. Thus Lemma 7.4.5 applies in both cases contradicting $\varrho \in \mathcal{S}'$. \square

Lemma 7.5.7. *Let $\varrho^{(h)} \in \mathcal{Q}_1$ and $\chi^{(\mu)} \in \mathcal{Q}_\varrho$. Then $\varrho \ll \chi$.*

Proof. Let $f^{(n)} \in \text{pPOL}_k \varrho$ be arbitrary. If $f \notin \text{pPOL}_k \chi$ then $g := f$ fulfills the conditions of \ll . Thus assume $f \in \text{pPOL}_k \chi$.

Let $g_\chi^{(m)}(\chi) := v$ for some $v \in E_k^\mu \setminus \chi$ and let $F^{(N)} := f \otimes g_\chi$ with $N = n \cdot m$. We get $F \notin \text{pPOL}_k \chi$ because $g_\chi \notin \text{pPOL}_k \chi$.

It suffices to show

$$F^{(N)} := (f \otimes g_\chi) \in \text{pPOL}_k \varrho. \quad (7.6)$$

Let $D := \text{dom } f$ and $E := \text{dom } g$ as in Definition 7.3.2.

Assume (7.6) is false. Then there are c_{1*}, \dots, c_{h*} from $D \otimes E$ with $s_i := F(c_{i*})$, $s := (s_1, \dots, s_h)^T \notin \varrho$ and $c_{*j} \in \varrho$ for all $j \in \{1, 2, \dots, N\}$. By Lemma 7.4.4 all the rows c_{i*} are pairwise different.

If all rows c_{i*} are from the D -part of $D \otimes E$, then $f(c_{*1m}, c_{*2m}, c_{*nm}) = s \notin \varrho$ in contradiction to $f \in \text{pPOL}_k \varrho$. Thus at least one row is from the E -part.

Let $l := \|\varrho\|$. Let w.l.o.g. $\text{pp } \varrho$ be the first l rows of ϱ . Assume there is some row c_{i_1*} with $1 \leq i_1 \leq l$ such that c_{i_1*} is not from the part of E representing $\text{pp } \chi$. Let $i_2 \neq i_1$ with $1 \leq$

$i_2 \leq l$ be arbitrary. Then there are columns j_1 and j_2 with $(c_{i_1 j_1}, c_{i_2 j_1}, c_{h j_1})^T = (x, y, x)^T$, $(c_{i_1 j_2}, c_{i_2 j_2}, c_{h j_2})^T = (x, y, y)^T$ and $x \neq y$. Thus $(i_2 - 1, i_1 - 1) \notin \varepsilon(\varrho)$, i.e., there is a singleton class in $\varepsilon(\text{pp } \varrho)$ in contradiction to $\text{pp } \varrho \in \mathcal{Q}_\varrho$.

So we need $\text{pp } \chi \subseteq \text{pp } \varrho$ in contradiction to the definition of \mathcal{Q}_ϱ . \square

For the proof of Theorem 7.5.13 some lemmas are needed using the following condition on $\varrho^{(h)} \in \mathcal{A}$.

$$\begin{aligned} \exists \varphi \in \text{Pol}_k^{(1)} \varrho \forall l \in \{1, 2, \dots, h-1\} \forall D \subseteq \sigma(E_k^l) \forall v \in \sigma(E_k^{h-l}) \\ \forall \pi \in S_h \exists m \geq 0 : D \times \{\varphi^m(v)\} \not\subseteq \varrho^{[\pi]}. \end{aligned} \quad (7.7)$$

Proposition 7.5.8. *Let $\varrho^{(h)} \in \mathcal{A}$ and ϱ fulfills (7.7). Then there is some $\varphi' \in \text{Pol}_k^{(1)} \varrho$ which suffices the conditions in (7.7) and $\varphi' \notin \text{pPOL}_k \{x\}$ for all $x \in E_k$.*

Proof. There is some $\varphi \in \text{Pol}_k^{(1)} \varrho$ which fulfills (7.7). Let

$$\varphi'(x) = \begin{cases} y & \text{for some } y \in E_k \setminus \{x\}, \text{ if } x \in E_k \setminus \omega(\varrho), \\ \varphi(x) & \text{otherwise.} \end{cases}$$

Let $x \in E_k \setminus \omega(\varrho)$. If $x \in \omega(D) \cup \omega(v)$ then $D \times \{(\varphi')^0(v)\} = D \times \{v\} \not\subseteq \varrho^{[\pi]}$ for all π . Thus φ' fulfills the conditions of (7.7) because it coincides with φ on $\omega(\varrho)$.

Let $x \in \omega(\varrho)$. Then there is some $D \subseteq \sigma(E_k^{h-1})$ with $D \times \{x\} \subseteq \varrho^{[\pi]}$ for some $\pi \in S_h$. But there is some $m \geq 0$ with $D \times \{(\varphi')^m(x)\} \not\subseteq \varrho^{[\pi]}$ because of (7.7). Thus $\varphi'(x) \neq x$. For $x \notin \omega(\varrho)$ follows $\varphi'(x) \neq x$ by definition of φ' . \square

Lemma 7.5.9. *Let $\varrho^{(h)} \in \mathcal{A}$ and ϱ fulfills (7.7). Then $\text{pPOL}_k \varrho$ is in every minimal covering of $p\mathcal{M}_k$.*

Proof. Because of Lemma 7.5.5 we just have to find a function $f \in \text{pPOL}_k \varrho$ with

$$\forall \chi \in (\mathcal{U} \cup \mathcal{A}) \setminus \{\varrho\} : f \notin \text{pPOL}_k \chi.$$

Then there is some function $g \in \text{pPOL}_k \varrho$ with

$$\forall \chi \in \tilde{\mathcal{R}}_k^{\max} \setminus \{\varrho\} : g \notin \text{pPOL}_k \chi$$

and by construction also $g \notin P_k \cup C_\infty$. Thus $\text{pPOL}_k \varrho$ is in every minimal covering of $p\mathcal{M}_k$ by Lemma 7.1.2.

We now construct the function f mentioned above.

We can assume $\pi = \text{id} \in S_h$ in (7.7) because $\text{pPOL}_k \varrho = \text{pPOL}_k \varrho^{[\pi]}$. Because of Proposition 7.5.8 we can assume $\varphi \notin \text{pPOL}_k \{x\}$ for all $x \in E_k$.

Let $f_0 := \varphi$ and define $f_j := f_{j-1} \otimes f_{\chi_j}$ recursively with

$$X := \{\chi_1, \dots, \chi_N\} := \{\chi \in \mathcal{U} \cup \mathcal{A} \mid \varphi \in \text{pPOL}_k \chi\}.$$

Let $\chi^{(\mu)} := \chi_j \in X$. There are two cases:

$\mu \leq h$: Let $f_\chi(\chi) := z$ with $z \in \varrho \setminus \chi$ if $\chi \subset \varrho$ and $z \in E_k^\mu \setminus \chi$ otherwise. Then $f_\chi \notin \text{pPOL}_k \chi$ and by construction $f_\chi \in \text{pPOL}_k \varrho$.

$\mu > h$: Because ϱ is coherent there is a relational homomorphism $\theta_0 : E_k \rightarrow E_k$ from ϱ to $M(\varrho)$. Let $\theta_1 : E_h \rightarrow E_k$ with $\theta_1(\eta_h) = v_1$ for some $v_1 \in \varrho$. Then $\theta \in \text{pPOL}_k \varrho$ for $\theta : E_k \rightarrow E_k$ with $\theta(x) = \theta_1(\theta_0(x))$.

Let $v \in \chi$ be arbitrary and let $f_\chi(x) = \theta(v)$. Because $|\omega(\theta(v))| \leq h < \mu$ we have $\theta \notin \text{pPOL}_k \chi$, and thus $f_\chi \notin \text{pPOL}_k \chi$.

By construction θ is a relational homomorphism from ϱ to ϱ . Thus $f_\chi \in \text{pPOL}_k \varrho$.

Because $f_j = f_{j-1} \otimes f_{\chi_j}$ and $f_{\chi_j} \notin \text{pPOL}_k \chi_j$ we get $f_j \notin \text{pPOL}_k \chi_j$.

Assume $f_j^{(n)} \notin \text{pPOL}_k \varrho$. Then there are rows c_{1*}, \dots, c_{h*} with $c_{*1}, \dots, c_{*n} \in \varrho$ and $f(c_{*1}, \dots, c_{*n}) = d \in E_k^h \setminus \varrho$. Then the rows c_{i*} are pairwise different and some rows belong to the f_{j-1} part of f_j and some to the f_χ part. The rows can w.l.o.g. be sorted in a way such that the first l rows for some $l \in \{1, 2, \dots, h-1\}$ are from the f_{j-1} part of $f_j = f_{j-1} \otimes f_{\chi_j}$.

Let $D := \text{pr}_{0, \dots, l-1} \{c_{*1}, \dots, c_{*n}\}$ and $W := \text{pr}_{l, \dots, h-1} \{c_{*1}, \dots, c_{*n}\}$. Because the rows $c_{l*}, c_{l+1*}, \dots, c_{h-1*}$ are from the f_χ part and f_χ is only defined on χ we obtain $W = \text{pr}_{p_l, p_{l+1}, \dots, p_{h-1}} \chi$ for pairwise different p_i . Let $v \in W$ be arbitrary. Then there is some $v' \in \chi$ with $v = \text{pr}_{p_l, p_{l+1}, \dots, p_{h-1}} v'$. Thus $\{\varphi^m(v) \mid m \geq 0\} \subseteq W$ because $\varphi \in \text{pPOL}_k \chi$, i.e., $\{\varphi^m(v') \mid m \geq 0\} \subseteq \chi$.

But then $D \times \{\varphi^m(v) \mid m \geq 0\} \subseteq \varrho$ in contradiction to (7.7). \square

Lemma 7.5.10. Let $\varrho^{(h)} \in \mathcal{A}$, $\bar{\chi}^{(l)} \subseteq E_k^l$, $\bar{\chi} \neq \emptyset$, $V \subseteq E_k^{h-l}$, $V \neq \emptyset$, $l \in \{1, \dots, h-1\}$ and $\bar{\chi} \times V \subseteq \varrho$. Let $\Gamma' := \{\pi \in \Gamma_{\sigma(\varrho)} \mid \forall x \in E_h \setminus E_l : \pi(x) = x\}$ and $\chi' := \{c^{[\pi]} \mid c \in \bar{\chi}, \pi \in \Gamma'_{|E_l}\}$.

Then χ' is coherent and $\chi' \times V \subseteq \varrho$.

Proof. From the definition of Γ' and ϱ coherent follows $\chi' \times V \subseteq \varrho$.

We now show χ' is coherent.

- If $\chi' = E_k^l$ then there is some $(x_1, x_2, \dots, x_h) \in \chi' \times V \subseteq \varrho$ with $x_1 = x_j$ for some $j \geq 2$ contradicting $\varrho \in \mathcal{A}$. Thus $\chi' \neq E_k^l$.
- $\chi' \subseteq \sigma(E_k^l)$, i.e., χ' is areflexive and $1 \leq l < k$,
- $r^{[\pi]} \in \chi'$ for all $r \in \chi'$ and $\pi \in \Gamma_{\chi'}$ because $\pi \in \Gamma'$ for any $\pi \in \Gamma_{\chi'}$, and because ϱ is coherent.
- $M(\chi') = \{\eta_l^{[\pi]} \mid \pi \in \Gamma_{\chi'} = \Gamma'_{|E_l}\}$. Let $\emptyset \subset \psi \subseteq \chi'$ and $w \in V$ be arbitrary. Because ϱ is coherent there exists a relational homomorphism $\lambda : E_k \rightarrow E_h$ from $\psi \times \{w\}$ to $M(\varrho)$ with

$$\lambda \begin{pmatrix} c \\ w \end{pmatrix} = \eta_h,$$

i.e., $\lambda(c) = \eta_l$, for some $c \in \psi$. For any $c' \in \psi$ we have

$$\lambda \begin{pmatrix} c' \\ w \end{pmatrix} \in M(\varrho) = \{\eta_h^{[\pi]} \mid \pi \in \Gamma_{\sigma(\varrho)}\}$$

and because $\lambda(w) = (l, \dots, h-1)^T$ we get

$$\lambda \begin{pmatrix} c' \\ w \end{pmatrix} \in \{\eta_h^{[\pi]} \mid \pi \in \Gamma'\}$$

and thus $\lambda(c') \in M(\chi')$.

Let $\lambda' : E_k \rightarrow E_l$ be defined by

$$\lambda'(x) := \begin{cases} \lambda(x) & \text{if } x \in \omega(\chi'), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\lambda' : E_k \rightarrow E_l$ is a relational homomorphism from χ' to $M(\chi')$ with $\lambda'(c) = \eta_l$ for some $c \in \chi'$.

Thus χ' is coherent, i.e., $\chi' \in \mathcal{U} \cup \mathcal{A}$. □

Lemma 7.5.11. *Let $\varrho^{(h)} \in \mathcal{A}$ and ϱ does not fulfill (7.7).*

Then for all $f \in \text{pPOL}_k \varrho$ there is some $\chi^{(\mu)} \in \mathcal{U} \cup \mathcal{A}$ with $\mu < h$ and $f \in \text{pPOL}_k \chi$.

Proof. We want to show by an indirect proof that

$$\forall f \in \text{pPOL}_k \varrho \exists \chi^{(\mu)} \in \mathcal{U} \cup \mathcal{A} : (\mu < h \text{ and } f \in \text{pPOL}_k \chi).$$

Let $f^{(n)} \in \text{pPOL}_k \varrho$ be arbitrary with

$$\forall \chi^{(\mu)} \in \mathcal{U} \cup \mathcal{A} : (\mu < h \implies f \notin \text{pPOL}_k \chi). \quad (7.8)$$

Let $\varphi(x) := f(x, \dots, x)$. Then $\varphi \in \text{pPOL}_k^{(1)} \varrho$ and by (7.8) we have $\varphi(x) \in E_k \setminus \{x\}$ for all $x \in E_k$, implying $\varphi \in \text{Pol}_k^{(1)} \varrho$. Because (7.7) is false, there are $l, D = \{D_{*1}, \dots, D_{*|D|}\} \subseteq \sigma(E_k^l)$, $v \in \sigma(E_k^{h-l})$ and $\pi \in S_h$ with $0 < l < h$ and

$$\forall m \geq 0 : D \times \{\varphi^m(v)\} \subseteq \varrho^{[\pi]}.$$

Because $\text{pPOL}_k \varrho = \text{pPOL}_k \varrho^{[\pi]}$ we assume w.l.o.g. $\pi = \text{id}$, i.e.,

$$\forall m \geq 0 : D \times \{\varphi^m(v)\} \subseteq \varrho. \quad (7.9)$$

We show $\varphi^m(v) \in \sigma(E_k^{h-l})$ for all $m \geq 0$. Assume otherwise. Then

$$\begin{pmatrix} D_{*1} \\ \varphi^m(v) \end{pmatrix} \in l_k^h$$

for some m , but this contradicts (7.9) because $\varrho \subseteq \sigma(E_k^h)$.

Because $\sigma(E_k^{h-l})$ is finite, there are $0 \leq m_1 < m_2$ such that $\varphi^{m_1}(v) = \varphi^{m_2}(v)$. Let $V := \{\varphi^{m_1+m}(v) \mid m \geq 0\}$. Then for any $w \in V$ there is some $w' \in V$ with $\varphi(w') = w$.

Let $\chi \in \{\psi^{(\mu)} \in \mathcal{U} \cup \mathcal{A} \mid \mu = l\}$ with $\chi \times V \subseteq \varrho$. Then there are rows c_{*1}, \dots, c_{l*} with $c_{*1}, \dots, c_{*n} \in \chi$ and $f(c_{*1}, \dots, c_{*n}) =: d \in E_k^l \setminus \chi$.

Let $w' \in V$ arbitrary and $w = \varphi(w') \in V$. Then $\chi \times \{w'\} \subseteq \varrho$ and

$$f \begin{pmatrix} c_{*1} & \dots & c_{*n} \\ w' & \dots & w' \end{pmatrix} = \begin{pmatrix} d \\ w \end{pmatrix} \in E_k^h,$$

i.e.,

$$\begin{pmatrix} d \\ w \end{pmatrix} \in \varrho$$

because $f \in \text{pPOL}_k \varrho$. Thus $(\chi \cup \{d\}) \times V \subseteq \varrho$. This also implies $\chi \cup \{d\} \subseteq \sigma(E_k^l)$ as shown before.

Let $\Gamma' := \{\pi \in \Gamma_{\sigma(\varrho)} \mid \forall x \in E_h \setminus E_l : \pi(x) = x\}$ and $\chi' := \{c^{[\pi]} \mid c \in \chi \cup \{d\}, \pi \in \Gamma'_{|E_l}\}$.

By Lemma 7.5.10 with $\bar{\chi} = \chi \cup \{d\}$ we get χ' coherent, i.e., $\chi' \in \mathcal{U} \cup \mathcal{A}$, and $\chi \subset \chi'$ with $\chi' \times V \subseteq \varrho$.

Now let $\chi_0 := \{D_{*1}\}$ then χ_0 is coherent and $\chi_0 \times V \subseteq \varrho$. By the argument above there is an infinite chain $\chi_0 \subset \chi_1 \subset \chi_2 \subset \dots$ with $\chi_i \in \mathcal{U} \cup \mathcal{A}$ and $\chi_i \times V \subseteq \varrho$ for all $i \in \mathbb{N}$. But this contradicts $|\mathcal{U} \cup \mathcal{A}| < \infty$ and thus the assumption (7.8) is wrong.

Thus for any $f \in \text{pPOL}_k \varrho$ there is some $\chi^{(\mu)} \in \mathcal{U} \cup \mathcal{A}$ with $\mu < h$ and $f \in \text{pPOL}_k \chi$. \square

Lemma 7.5.12. *Let $\varrho^{(h)} \in \mathcal{A}$ and ϱ does not fulfill (7.7).*

Then $\text{pPOL}_k \varrho$ is in no minimal covering of $p\mathcal{M}_k$.

Proof. Let $f \in \text{pPOL}_k \varrho$ be arbitrary. Let $\chi_0^{(\mu_0)} := \varrho$ and $\mu_0 := h$.

Let $l \geq 0$.

- If $\chi_l \in \mathcal{U}$ then $\text{pPOL}_k \chi_l$ belongs to every minimal covering by Lemma 7.4.3.
- If $\chi_l \in \mathcal{A}$ and χ_l fulfills the conditions in (7.7) then $\text{pPOL}_k \chi_l$ belongs to every minimal covering by Lemma 7.5.9.
- If $\chi_l \in \mathcal{A}$ and χ_l does not fulfill the conditions in (7.7) then by Lemma 7.5.11 there is some $\chi_{l+1}^{(\mu_{l+1})} \in \mathcal{U} \cup \mathcal{A}$ with $\mu_{l+1} < \mu_l$ and $f \in \text{pPOL}_k \chi_{l+1}$.

Since $\mu_0 = h$ is finite and $\mu_0 > \mu_1 > \dots > \mu_l > 0$ the last case can only a finite number of times. Thus there is some $l \geq 0$ such that $\chi_l^{(\mu_l)} \in \mathcal{U} \cup \mathcal{A}$, $f \in \text{pPOL}_k \chi_l$ and $\text{pPOL}_k \chi$ belongs to every minimal covering of $p\mathcal{M}_k$.

Thus $\text{pPOL}_k \varrho$ is in no minimal covering of $p\mathcal{M}_k$ by Lemma 7.1.3. \square

Theorem 7.5.13. *Let $\varrho^{(h)} \in \mathcal{A}$. Then $\text{pPOL}_k \varrho$ is in any minimal covering of $p\mathcal{M}_k$ if and only if ϱ fulfills (7.7).*

Proof. If ϱ fulfills (7.7) then $\text{pPOL}_k \varrho$ is in every minimal covering by Lemma 7.5.9. If ϱ does not fulfill (7.7) then $\text{pPOL}_k \varrho$ is no minimal covering by Lemma 7.5.12. \square

7.6 Uniqueness of Minimal Coverings

Lemma 7.6.1. *Let \mathcal{X}, \mathcal{Y} be different minimal coverings of $p\mathcal{M}_k$. Then $\text{pPOL}_k \varrho \in \mathcal{X}$ if and only if $\text{pPOL}_k \varrho \in \mathcal{Y}$ for all $\varrho \in \mathcal{U} \cup \mathcal{A}$.*

Proof. By Lemma 7.4.3 we have $\text{pPOL}_k \varrho \in \mathcal{X}$ and $\text{pPOL}_k \varrho \in \mathcal{Y}$ for all $\varrho \in \mathcal{U}$. By Theorem 7.5.13

$$\forall \varrho \in \mathcal{A} (\text{pPOL}_k \varrho \in \mathcal{X} \iff \text{pPOL}_k \varrho \in \mathcal{Y}). \quad \square$$

Lemma 7.6.2. *Let \mathcal{X}, \mathcal{Y} be different minimal coverings of $p\mathcal{M}_k$. Then $\text{pPOL}_k \varrho \in \mathcal{X}$ if and only if $\text{pPOL}_k \varrho \in \mathcal{Y}$ for all $\varrho \in \mathcal{Q}_0 \cup \mathcal{L}$.*

Proof. Assume this is false. Then there is some $\varrho \in \mathcal{Q}_0 \cup \mathcal{L}$ with $X := \text{pPOL}_k \varrho \in \mathcal{X} \setminus \mathcal{Y}$. Because X is in some minimal covering of $p\mathcal{M}_k$ we get $\varrho \in \mathcal{Q}'_0 \cup \mathcal{L}$. By Lemma 7.5.5 we have

$$\mathcal{X} := \{\text{pPOL}_k \psi \mid \psi \in \tilde{\mathcal{R}}_k^{\max} \text{ and } \varrho \ll \psi\} \supseteq \{\text{pPOL}_k \psi \mid \psi \in (\mathcal{Q} \cup \mathcal{L} \cup \mathcal{S}) \setminus \{\varrho\}\}.$$

Since \mathcal{X} is a minimal covering there is some $f \in X$ with $f \notin X'$ for all $X' \in \mathcal{X} \setminus \{X\}$. By Lemma 7.5.2 there is some $F \in X$ with $F \notin X'$ for all $X' \in \mathcal{X} \cup \mathcal{Z}$ and $X' \neq X$. Since \mathcal{Y} is a covering there is some $Y \in \mathcal{Y}$ with $F \in Y$. But then $Y \in \mathcal{Y} \setminus (\mathcal{X} \cup \mathcal{Z}) = (\mathcal{Y} \setminus \mathcal{Z}) \setminus \mathcal{X} = (\mathcal{Y} \cap \{\text{pPOL}_k \chi \mid \chi \in \mathcal{U} \cup \mathcal{A}\}) \setminus \mathcal{X} = \emptyset$ by Lemma 7.6.1. This is a contradiction. \square

Lemma 7.6.3. *Let $\mathcal{X} \subseteq p\mathcal{M}_k$ be a minimal covering of $p\mathcal{M}_k$. Then $\text{pPOL}_k \varrho \notin \mathcal{X}$ for all $\varrho \in \mathcal{S} \setminus \mathcal{S}'$.*

Proof. Assume $X := \text{pPOL}_k \varrho \in \mathcal{X}$ for some $\varrho \in \mathcal{S} \setminus \mathcal{S}'$. Then there is some $f \in X$ with $f \notin Y$ for all $Y \in \mathcal{X} \setminus \{X\}$. Applying Lemma 7.4.5 recursively on X implies $f \in \text{pPOL}_k \chi$ for some $\chi \in \mathcal{U} \cup \mathcal{Q} \cup \mathcal{S}'$. By Lemmas 7.4.3, 7.5.5 and 7.5.6 there is some $g \in \text{pPOL}_k \chi$ with $g \notin Y$ for all $Y \in \mathcal{X}$ in contradiction to \mathcal{X} minimal covering. \square

Lemma 7.6.4. *Let \mathcal{X}, \mathcal{Y} be different minimal coverings of $p\mathcal{M}_k$. Then $\text{pPOL}_k \varrho \in \mathcal{X}$ if and only if $\text{pPOL}_k \varrho \in \mathcal{Y}$ for all $\varrho \in \mathcal{S}$.*

Proof. Assume this is false. Then there is some $\varrho \in \mathcal{S}$ with $X := \text{pPOL}_k \varrho \in \mathcal{X} \setminus \mathcal{Y}$. In particular $\varrho \in \mathcal{S}'$ by Lemma 7.6.3. Then there is some $f \in X$ with $f \notin X'$ for all $X' \in \mathcal{X} \setminus \{X\}$.

Then $f \notin Y$ for all $Y \in \mathcal{Y}$ with $Y = \text{pPOL}_k \chi$ and $\chi \in \mathcal{U} \cup \mathcal{A} \cup \mathcal{Q}_0 \cup \mathcal{L}$ by Lemmas 7.6.1 and 7.6.2. Thus there is some $Z \in \mathcal{Y}$ with $Z = \text{pPOL}_k \psi$ and $\psi \in \mathcal{Q}_1 \cup \mathcal{S}'$ and $f \in Z$. By Lemma 7.5.6 there is some $g \in Z$ with $g \notin X$ and $g \notin X'$ for all $X' \in \mathcal{X} \setminus \{X\}$, i.e., $g \notin X'$ for all $X' \in \mathcal{X}$. This contradicts \mathcal{X} minimal covering because $g \in Z \in \mathcal{Y}$. \square

Theorem 7.6.5. *Let \mathcal{X}, \mathcal{Y} be different minimal coverings of $p\mathcal{M}_k$. Then $\mathcal{X} \setminus \mathcal{Y} \subseteq \{\text{pPOL}_k \psi \mid \psi \in \mathcal{Q}_1\}$.*

Proof. The theorem follows from Lemmas 7.6.1, 7.6.2 and 7.6.4, and Lemma 7.4.2 for the partial clone $P_k \cup C_\infty$. \square

Lemma 7.6.6. *Let \mathcal{X}, \mathcal{Y} be different minimal coverings of $p\mathcal{M}_k$. Let $X := \text{pPOL}_k \varrho \in \mathcal{X} \setminus \mathcal{Y}$ for some $\varrho \in \mathcal{Q}_1$. Then there is some $\chi \in \mathcal{Q}_1$ with $Y := \text{pPOL}_k \chi \in \mathcal{Y} \setminus \mathcal{X}$ and $\text{pp}\chi = \text{pp}\varrho$.*

Proof. By $\mathcal{X} \neq \mathcal{Y}$ and Theorem 7.6.5 we have $\emptyset \subset \mathcal{X} \setminus \mathcal{Y} \subseteq \{\text{pPOL}_k \psi \mid \psi \in \mathcal{Q}_1\}$. Let $X := \text{pPOL}_k \varrho \in \mathcal{X} \setminus \mathcal{Y}$ be arbitrary with $\varrho \in \mathcal{Q}_1$. Then there is some $f \in X$ with $f \notin X'$ for all $X' \in \mathcal{X} \setminus \{X\}$. Then $f \in Y$ with $Y := \text{pPOL}_k \chi \in \mathcal{Y} \setminus \mathcal{X}$ for some $\chi \in \mathcal{Q}_1$.

Assume $\text{pp}\chi \neq \text{pp}\varrho$. Then $\chi \in \mathcal{Q}_\varrho$ or $\varrho \in \mathcal{Q}_\chi$. If $\chi \in \mathcal{Q}_\varrho$ then there is some $g \in X$ with $g \notin X'$ for all $X' \in \mathcal{X} \setminus \{X\}$ and $g \notin \text{pPOL}_k \chi$ by Lemma 7.5.7. Thus $\varrho \in \mathcal{Q}_\chi$ has to be true. But then there is some $G \in Y = \text{pPOL}_k \chi$ with $G \notin X'$ for all $X' \in \mathcal{X}$ by Lemma 7.5.7 again contradicting \mathcal{X} minimal covering. \square

Definition 7.6.7. Let $\varrho^{(h)} \in \mathcal{Q}_1$. We call ϱ *irreducible* iff

$$\forall \emptyset \subset A \subset E_h \forall v \in \sigma(E_k^{h-|A|}) \forall \pi \in S_h : (\text{pr}_A \varrho) \times \{v\} \not\subseteq \varrho^{[\pi]}.$$

Otherwise we call it *reducible*.

Example 7.6.8. Let $k = 4$ and $h = 3$. Let

$$\varrho = \begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 2 & 3 \end{pmatrix} \cup \delta_{\{0,1\}}^{(3)}.$$

We show that ϱ is irreducible.

- Let $|A| = 1$. Then $\text{pr}_A \varrho = E_4$ and $v = (v_1, v_2) \in \sigma(E_4^2)$. Assume $(\text{pr}_A \varrho) \times \{v\} \subseteq \varrho^{[\pi]}$ for some $\pi \in S_3$. Then $(v_1, v_1, v_2), (v_2, v_1, v_2) \in \varrho^{[\pi]}$. Thus $\delta_{\{0,1\}}^{(3)} \cup \delta_{\{0,2\}}^{(3)} \subseteq \varrho^{[\pi]}$ because ϱ is coherent. But this contradicts $\varrho \in \mathcal{Q}_1$.
- Let $A = \{0, 1\}$. If $\pi \neq \text{id}$ then $\delta_X^{(3)} \subseteq \varrho$ with $X \subset E_3$, $|X| = 2$ and $X \neq \{0, 1\}$ in contradiction to $\varrho \in \mathcal{Q}_1$. Thus $\pi = \text{id}$.

Because for all $v \in E_k$

$$\begin{pmatrix} 0 & 0 \\ 1 & 2 \\ v & v \end{pmatrix} \not\subseteq \varrho \quad \text{and} \quad \text{pr}_A \varrho = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} \cup \delta_{\{0,1\}}^{(2)}$$

we get $(\text{pr}_A \varrho) \times \{v\} \not\subseteq \varrho$.

- Let $|A| = 2$ and $A \neq \{0, 1\}$. Then $\text{pr}_A \varrho = E_4^2$. Assume $(\text{pr}_A \varrho) \times \{x\} \subseteq \varrho^{[\pi]}$ for some $x \in E_k$. Then $(x, y, x), (y, x, x), (y, y, x) \in \varrho^{[\pi]}$ and thus $\iota_4^3 \subseteq \varrho^{[\pi]}$ because ϱ is coherent. But this contradicts $\varrho \in \mathcal{Q}_1$.

Thus ϱ is irreducible.

Now let

$$\varrho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 2 \end{pmatrix} \cup \delta_{\{0,1\}}^{(3)}.$$

Then ϱ is reducible because

$$(\text{pr}_A \varrho) \times \{v\} = \begin{pmatrix} 0 & 1 & 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 1 & 2 & 3 \\ 2 & 2 & 2 & 2 & 2 & 2 \end{pmatrix} \subseteq \varrho = \varrho^{[\pi]}$$

holds with $A = \{0, 1\}$, $v = 2$ and $\pi = \text{id}$.

Lemma 7.6.9. Let $\varrho^{(h)} \in \mathcal{Q}_1$ be reducible. Then for every $f \in \text{pPOL}_k \varrho$ there is some

$$\chi \in \mathcal{X}_\varrho := \{\{a\} \mid a \in E_k\} \cup \{\psi^{(\mu)} \in \mathcal{Q} \mid \text{pp } \psi = \text{pp } \varrho \text{ and } \mu < h\}$$

with $f \in \text{pPOL}_k \chi$.

Proof. Let $\sigma := \sigma(\varrho)$. Assume there is some $f^{(n)} \in \text{pPOL}_k \varrho$ such that $f \notin \text{pPOL}_k \chi$ for all $\chi \in \mathcal{X}_\varrho$. Then $f(x, \dots, x) \in E_k \setminus \{x\}$ for each $x \in E_k$.

Because ϱ is reducible there are some A with $\emptyset \subset A \subset E_h$, and $\pi \in S_h$ and $v \in \sigma(E_k^{h-|A|})$ such that

$$(\text{pr}_A \varrho) \times \{v\} \subseteq \varrho^{[\pi]}.$$

Because $\text{pPOL}_k \varrho = \text{pPOL}_k \varrho^{[\pi]}$ we assume w.l.o.g. $\pi = \text{id}$.

We show that $\text{pp pr}_A \varrho = \text{pp } \varrho$. If $|A| = 1$ then $\text{pr}_A \varrho = E_k$ and thus $E_k \times \{v\} \subseteq \varrho$. This implies $\delta_{\{0,i\}}^{(h)} \subseteq \varrho$ for all $i \in E_h \setminus \{0\}$ contradicting $\varrho \in \mathcal{Q}$. Let $|A| \geq 2$. We know

$s_0 = (0, \dots, 0), s_1 = (1, \dots, 1) \in \delta(\text{pr}_A \varrho)$. Then $\{s_0, s_1\} \times \{v\} \subseteq \delta(\varrho)$, i.e., for all $i \in E_h$ and $j \in E_h \setminus (A \cup \{i\})$ we get $(i, j) \notin \varepsilon(\varrho)$. Thus all non-singular classes of $\varepsilon(\varrho)$ are covered by A , i.e., the projection pr_A preserves them, and this implies $\text{pp pr}_A \varrho = \text{pp } \varrho$.

We show that $(\text{pr}_A \sigma) \times \{v\} \subseteq \sigma$. Assume otherwise, then there is some $s \in \text{pr}_A \sigma$ with $\{s\} \times \{v\} \subseteq \delta(\varrho)$. But this contradicts $\text{pr}_A \delta(\varrho) \cap \text{pr}_A \sigma = \emptyset$ because $\text{pp pr}_A \varrho = \text{pp } \varrho$. So $s \notin \text{pr}_A \delta(\varrho)$ in contradiction to the assumption. We proved $(\text{pr}_A \sigma) \times \{v\} \subseteq \sigma$, and thus $\omega(\text{pr}_A \sigma) \cap \omega(v) = \emptyset$.

Now we show that $\gamma := \text{pr}_A \varrho \in \mathcal{Q}$, i.e., that it is coherent. Let $\theta \in \Gamma_{\sigma(\gamma)}$ and $w \in \gamma$ arbitrarily. There is some $\hat{w} \in \gamma$ with $\hat{w}^{[\theta]} \in \gamma$. Then $\{\hat{w}, \hat{w}^{[\theta]}\} \times \{v\} \subseteq \varrho$, i.e., $\theta \in \Gamma_\sigma$ and thus $\{w, w^{[\theta]}\} \times \{v\} \subseteq \varrho$. This implies $w^{[\theta]} \in \gamma$.

$M(\gamma) = \text{pr}_A M(\varrho)$ because $\text{pp pr}_A \varrho = \text{pp } \varrho$.

Let $\gamma' \subseteq \sigma(\gamma)$. Then $\gamma' \times \{v\} \subseteq \sigma$. Thus there is a relational homomorphism $\varphi : E_k \rightarrow E_h$ from $\gamma' \times \{v\}$ to $M(\varrho)$ and some $w \in \gamma'$ with $\varphi \begin{pmatrix} w \\ v \end{pmatrix} = \eta_h$. Let $\hat{\varphi} : E_k \rightarrow E_{|A|}$ be given by

$$\hat{\varphi}(x) = \begin{cases} \varphi(x) & \text{if } \varphi(x) \in E_{|A|}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\hat{\varphi}$ is a relational homomorphism from γ' to $M(\gamma)$ with $\hat{\varphi}(w) = \eta_{|A|}$. Thus γ is a coherent relation and $\gamma \in \mathcal{X}_\varrho$ because $\text{pp } \gamma = \text{pp } \varrho$ and $|A| < h$. Since $f \notin \text{pPOL}_k \chi$ for all $\chi \in \mathcal{X}_\varrho$ there are rows $c_{1*}, \dots, c_{|A|*}$ with $c_{*1}, \dots, c_{*n} \in \gamma$ and $f(c_{*1} \dots c_{*n}) = d \in E_k^{|A|} \setminus \gamma$.

Then

$$f \begin{pmatrix} c_{*1} & \dots & c_{*n} \\ v & \dots & v \end{pmatrix} \in E_k^h \setminus \varrho,$$

i.e., $f \notin \text{pPOL}_k \varrho$ contradicting the assumption. \square

Proposition 7.6.10. *Let $\varrho^{(h)}, \chi^{(\mu)} \in \mathcal{Q}$ with $\mu \geq 3$, $f, g \in \text{pPOL}_k \chi$ with $g(\varrho) \in E_k^h$ and g is not defined anywhere else, and $F^{(n)} := f \otimes g \notin \text{pPOL}_k \chi$.*

Then there are rows $c_{1}, \dots, c_{\mu*}$*

- (a) *with $c_{*1}, \dots, c_{*n} \in \chi$ and $F(c_{*1} \dots c_{*n}) = d \in E_k^\mu \setminus \chi$, and*
- (b) *there is some j with $c_{*j} \in \sigma(E_k^\mu)$, and*
- (c) *the rows $c_{1*}, \dots, c_{\|\chi\|*}$ belong to the g -part of F , and*
- (d) *if $\text{pp } \varrho = \text{pp } \chi$ the rows $c_{1*}, \dots, c_{\|\chi\|*}$ belong to the first $\|\chi\|$ rows of the g -part of F .*

Proof. Statement (a) follows directly from $F \notin \text{pPOL}_k \chi$. Choose some rows $c_{1*}, \dots, c_{\mu*}$ such that (a) holds.

(b) Assume (b) is false. Then $\{c_{*1}, \dots, c_{*n}\} \subseteq \delta(\chi) = \delta_{\varepsilon(\chi)}$ contradicting all rows c_{i*} are pairwise different by Lemma 7.4.4. Thus for any two rows there is a column in which they differ, i.e. there is some column $c_{*j} \in \chi \setminus \delta(\chi)$.

(c) Because $\varrho \in \mathcal{Q}$ we have $\delta_{E_h}^{(h)} \subseteq \varrho$. Because $g \in \text{pPOL}_k \chi$ there is at least one row from the f -part of F and because $f \in \text{pPOL}_k \chi$ there is at least one row from the g -part of F . Let c_{i_f*} be an arbitrary row from the f -part and c_{i_g*} be an arbitrary row from the g -part. Because $\mu \geq 3$ there is a third row $c_{i'*}$ different from c_{i_f*} and c_{i_g*} . Let $c_{i'*}$ be arbitrary with this condition.

There are two cases to consider:

- The row $c_{i' *}$ is from the f -part. Then there is some column j in which the rows $c_{i_f *}$ and $c_{i' *}$ differ, i.e., $c_{i_f j} = x$, $c_{i' j} = y$ and $x \neq y$. By construction and $\delta_{E_h}^{(h)} \subset \varrho$, we can choose j more specifically such that

$$\begin{pmatrix} c_{i_f j} \\ c_{i' j} \\ c_{i_g j} \end{pmatrix} = \begin{pmatrix} x \\ y \\ y \end{pmatrix}.$$

- The row $c_{i' *}$ is from the g -part. Then there is some j with

$$\begin{pmatrix} c_{i_f j} \\ c_{i' j} \\ c_{i_g j} \end{pmatrix} = \begin{pmatrix} x \\ y \\ y \end{pmatrix}$$

and $x \neq y$ by construction and $\delta_{E_h}^{(h)} \subset \varrho$.

Thus $(i_f, i_g), (i_f, i') \notin \varepsilon(\chi)$. Because i_f, i_g and i' are chosen arbitrarily any row $c_{i_f *}$ from the f -part belongs to a singular class of $\varepsilon(\chi)$. Because the first $\|\chi\|$ rows of χ belong to non-singular classes of $\varepsilon(\chi)$ the first $\|\chi\|$ rows $c_{1*}, \dots, c_{\|\chi\|*}$ belong to the g -part of F . Thus (c) is true.

- (d) Let $\text{pp } \varrho = \text{pp } \chi$. Assume one of the rows $c_{1*}, \dots, c_{\|\chi\|*}$ does not belong to the first $\|\chi\|$ rows of the g -part of F , w.l.o.g. let this be the row c_{1*} . As shown before c_{1*} belongs to the g -part of F . Because $\text{pp } \varrho = \text{pp } \chi$ the row c_{1*} belongs to a singular class of $\varepsilon(\varrho)$. Now let $c_{i_1 *}, c_{i_2 *}$ be two arbitrarily chosen different rows.

- If $c_{i_1 *}$ and $c_{i_2 *}$ are both from the f -part then they differ at some point and by construction we get columns $c_{*j}, c_{*j'}$ with

$$\begin{pmatrix} c_{1j} & c_{1j'} \\ c_{i_1 j} & c_{i_1 j'} \\ c_{i_2 j} & c_{i_2 j'} \end{pmatrix} = \begin{pmatrix} x & y \\ x & x \\ y & y \end{pmatrix}$$

and $x \neq y$.

- If $c_{i_1 *}$ is from the f -part and $c_{i_2 *}$ from the g -part then by construction and because c_{1*} belongs to a singular class of $\varepsilon(\varrho)$ there is some column c_{*j} with

$$\begin{pmatrix} c_{1j} \\ c_{i_1 j} \\ c_{i_2 j} \end{pmatrix} = \begin{pmatrix} x \\ y \\ y \end{pmatrix}$$

with $x \neq y$.

- If $c_{i_1 *}$ and $c_{i_2 *}$ are both from the g -part then because c_{1*} belongs to a singular class of $\varepsilon(\varrho)$ there is some column c_{*j} with

$$\begin{pmatrix} c_{1j} \\ c_{i_1 j} \\ c_{i_2 j} \end{pmatrix} = \begin{pmatrix} x \\ y \\ y \end{pmatrix}$$

with $x \neq y$.

Thus for all cases $(1, i_1), (1, i_2) \notin \varepsilon(\chi)$. Because i_1 and i_2 are chosen arbitrarily the row c_{1*} belongs to a singular class of $\varepsilon(\chi)$ in contradiction to the convention that the first $\|\chi\|$ rows of χ belong to the non-singular classes of $\varepsilon(\chi)$, see Definition 7.2.3. Thus (d) is true. \square

Definition 7.6.11. Let $\varrho \in \mathcal{Q}_1$. Define $\mathcal{T}_\varrho \subseteq \tilde{\mathcal{R}}_k^{\max}$ by

$$\mathcal{T}_\varrho := \{\psi \in \mathcal{Q}_1 \mid \text{pp } \psi = \text{pp } \varrho\}.$$

Lemma 7.6.12. Let $\varrho \in \mathcal{Q}_1$, $\mathcal{T} \subseteq \mathcal{T}_\varrho$, $|\mathcal{T}| \geq 2$ and $f \in \tilde{P}_k$ with

$$\left(\forall \psi \in \tilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T} : f \notin \text{pPOL}_k \psi \right) \text{ and } (\exists \chi \in \mathcal{T} : f \in \text{pPOL}_k \chi).$$

Then

(a) there are $\chi_0 \in \mathcal{T}$ and $F \in \text{pPOL}_k \chi_0$ with

$$\forall \psi \in \tilde{\mathcal{R}}_k^{\max} \setminus \{\chi_0\} : F \notin \text{pPOL}_k \psi,$$

or

(b) there are $F \in \tilde{P}_k$ and $\mathcal{T}' \subset \mathcal{T}$, $\mathcal{T}' \neq \emptyset$ with

$$\left(\forall \psi \in \tilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}' : F \notin \text{pPOL}_k \psi \right) \text{ and } (\exists \chi \in \mathcal{T}' : F \in \text{pPOL}_k \chi).$$

Proof. Assume (a) is false. By assumption there is some $\psi_1 \in \mathcal{T}$ with $f \in \text{pPOL}_k \psi_1$. Then there is some $\psi_2 \in \tilde{\mathcal{R}}_k^{\max} \setminus \{\psi_1\}$ with $f \in \text{pPOL}_k \psi_2$ because (a) is false. Because $f \notin \text{pPOL}_k \psi$ for all $\psi \in \tilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}$ we get $\psi_2 \in \mathcal{T}$. Thus there are $\psi_1^{(\mu_1)}, \psi_2^{(\mu_2)} \in \mathcal{T}$ with $\psi_1 \neq \psi_2$ and $f \in (\text{pPOL}_k \psi_1) \cap (\text{pPOL}_k \psi_2)$. We can choose ψ_1 such that μ_1 is minimal. This implies

$$\forall \chi \in \{\{a\} \mid a \in E_k\} \cup \{\psi^{(\mu)} \in \mathcal{Q} \mid \text{pp } \psi = \text{pp } \varrho \text{ and } \mu < \mu_1\} : f \notin \text{pPOL}_k \chi.$$

Thus ψ_1 is irreducible because $f \in \text{pPOL}_k \psi_1$ and Lemma 7.6.9.

Furthermore $\mu_1 \leq \mu_2$. If $\mu_2 = \mu_1$ then ψ_2 is also irreducible by the same argument.

We construct a function $F^{(n)} := f \otimes g$ (where g is given for each case separately) such that $F \in \text{pPOL}_k \psi_1$ and $F \notin \text{pPOL}_k \psi_2$ holds (or the other way round). We consider some cases (a)–(f):

(a) $\exists \emptyset \subset A \subset E_{\mu_1} \exists v \in \sigma(E_k^{\mu_2 - |A|}) \exists \pi \in S_{\mu_2} : (\text{pr}_A \psi_1) \times \{v\} \subseteq \psi_2^{[\pi]}$ (w.l.o.g. $\pi = \text{id}$).

Assume $\text{pp}(\text{pr}_A \psi_1) \neq \text{pp } \psi_1$. Then $\|(\text{pr}_A \psi_1) \times \{v\}\| = \|\text{pr}_A \psi_1\| < \|\psi_1\| = \|\psi_2\|$ in contradiction to $\delta(\text{pr}_A \psi_1) \times \{v\} \subseteq \delta((\text{pr}_A \varrho) \times \{v\}) \subseteq \delta(\psi_2)$. Thus $\text{pp}(\text{pr}_A \psi_1) = \text{pp } \psi_1 = \text{pp } \psi_2$.

Let $g(\text{pr}_A \psi_1) := d \in E_k^{|A|}$ with $g(\text{pp}(\text{pr}_A \psi_1)) \in E_k^{\|\psi_2\|} \setminus \text{pp } \psi_2$. Then $F \notin \text{pPOL}_k \psi_2$ because

$$F \left(\underbrace{\begin{pmatrix} \text{pr}_A \psi_1 \\ v \end{pmatrix}}_{\subseteq \psi_2} \right) \in (E_k^{\|\psi_2\|} \setminus \text{pp } \psi_2) \times E_k^{\mu_2 - \|\psi_2\|} \subseteq E_k^{\mu_2} \setminus \psi_2.$$

We have $g \in \text{pPOL}_k \psi_1$ because g is defined on less than μ_1 rows. Assume $F \notin \text{pPOL}_k \psi_1$. Then there are rows $c_{1*}, \dots, c_{\mu_1*}$ with $c_{*1}, \dots, c_{*n} \in \psi_1$ and the first $\|\psi_1\|$ rows belong to the g -part of F , and a column $c_{*j} \in \sigma(E_k^{\mu_1})$ by Proposition 7.6.10. Let w.l.o.g. the rows c_{1*}, \dots, c_{l*} belong to the g -part of F and $c_{l+1*}, \dots, c_{\mu_1*}$ to the f -part of F with $\|\chi\| \leq l < \mu_1$. Then let

$$v := \begin{pmatrix} c_{l+1j} \\ \dots \\ c_{\mu_1j} \end{pmatrix}$$

and

$$c' := \text{pr}_{1,2,\dots,l}\{c_{*1}, \dots, c_{*n}\}.$$

Then $c' = \text{pr}_{A'} \psi_1$ for some $\emptyset \subset A' \subseteq A \subset E_{\mu_1}$ by construction of g . By construction of F we get

$$(\text{pr}_{A'} \psi_1 \times \{v\}) = c' \times \{v\} \subseteq \{c_{*1}, \dots, c_{*n}\} \subseteq \psi_1$$

contradicting ψ_1 irreducible.

Thus there is some \mathcal{T}' with $\{\psi_1\} \subseteq \mathcal{T}' \subseteq \mathcal{T} \setminus \{\psi_2\}$ and

$$\left(\forall \psi \in \tilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}' : F \notin \text{pPOL}_k \psi \right) \text{ and } (\exists \chi \in \mathcal{T}' : F \in \text{pPOL}_k \chi).$$

- (b) $\mu_1 = \mu_2$ and $\exists \emptyset \subset A \subset E_{\mu_2} \exists v \in \sigma(E_k^{\mu_1 - |A|}) \exists \pi \in S_{\mu_1} : (\text{pr}_A \psi_2) \times \{v\} \subseteq \psi_1^{[\pi]}$.

This is a restriction of the previous case with the roles of ψ_1 and ψ_2 switched. Thus there is some \mathcal{T}' with $\{\psi_2\} \subseteq \mathcal{T}' \subseteq \mathcal{T} \setminus \{\psi_1\}$ and

$$\left(\forall \psi \in \tilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}' : F \notin \text{pPOL}_k \psi \right) \text{ and } (\exists \chi \in \mathcal{T}' : F \in \text{pPOL}_k \chi).$$

- (c) $\mu_1 < \mu_2$ and $\exists v \in \sigma(E_k^{\mu_2 - \mu_1}) \exists \pi \in S_{\mu_2} : \psi_1 \times \{v\} \subseteq \psi_2^{[\pi]}$ (w.l.o.g. $\pi = \text{id}$).

Because ψ_1 is coherent there is some relational homomorphism $\varphi : E_k \rightarrow E_{\mu_1}$ from $\sigma(\psi_1)$ to $M(\psi_1)$ and some $s \in \sigma(\psi_1)$ with $\varphi(s) = \eta_{\mu_1}$. Define $\varphi^* : E_{\mu_1} \rightarrow E_k$ by $\varphi^*(\eta_{\mu_1}) = s$.

Let

$$g((\sigma(\psi_1) \times \{v\}) \cup \delta(\psi_2)) := d := \varphi^* \left(\varphi \left(\begin{array}{c} s \\ v \end{array} \right) \right).$$

Then $g \in \text{pPOL}_k \psi_1$ by construction.

Assume $g \in \text{pPOL}_k \psi_2$. Then $d \in \delta(\psi_2)$ because $|\omega(d)| = |\omega(s)| = \mu_1 < \mu_2$. But $|\omega(\text{pr}_{E_{\|\psi_2\|}} d)| = \|\psi_2\|$ in contradiction to the assumption that the first $\|\psi_2\|$ rows belong to the non-singular classes of $\varepsilon(\psi_2)$. Thus $g \notin \text{pPOL}_k \psi_2$ and this implies $F \notin \text{pPOL}_k \psi_2$.

Because $\sigma(\psi_1) \times \{v\} \subseteq \psi_2$ and the first $\|\psi_1\|$ rows belong to the non-singular classes of $\varepsilon(\psi_2)$ we get $\sigma(\psi_1) \times \{v\} \subseteq \sigma(\psi_2)$ and thus $\omega(v) \cap \omega(\sigma(\psi_1)) = \emptyset$. Assume $F \notin \text{pPOL}_k \psi_1$. Then there are $c_{1*}, \dots, c_{\mu_1*}$ with $c_{*1}, \dots, c_{*n} \in \psi_1$ and $F(c_{*1}, \dots, c_{*n}) \in E_k^{\mu_1} \setminus \psi_1$. By Proposition 7.6.10 the rows $c_{1*}, \dots, c_{\|\psi_1\|*}$ are the first rows in the definition of g . Thus the other rows can not belong to the last $(\mu_2 - \mu_1)$ rows in the definition of g because $\omega(v) \cap \omega(\sigma(\psi_1)) = \emptyset$. Thus this part of the definition of g can be ignored here, and thus $F \in \text{pPOL}_k \psi_1$ because ψ_1 is irreducible.

Thus there is some \mathcal{T}' with $\{\psi_1\} \subseteq \mathcal{T}' \subseteq \mathcal{T} \setminus \{\psi_2\}$ and

$$\left(\forall \psi \in \tilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}' : F \notin \text{pPOL}_k \psi \right) \text{ and } (\exists \chi \in \mathcal{T}' : F \in \text{pPOL}_k \chi).$$

- (d) $\mu_1 = \mu_2$ and $\exists \pi \in S_{\mu_2} : \psi_1 \subset \psi_2^{[\pi]}$ (w.l.o.g. $\pi = \text{id}$).

Let $g(\psi_2) = d \in E_k^{\mu_2} \setminus \psi_2$. Because $\text{pr}_A \psi_1 \subseteq \text{pr}_A \psi_2$ for all $\emptyset \subset A \subset E_{\mu_1}$, ψ_1 irreducible and $g \in \text{pPOL}_k \psi_1$ we get $F \in \text{pPOL}_k \psi_1$. Furthermore $g \notin \text{pPOL}_k \psi_2$ implies $F \notin \text{pPOL}_k \psi_2$.

Thus there is some \mathcal{T}' with $\{\psi_1\} \subseteq \mathcal{T}' \subseteq \mathcal{T} \setminus \{\psi_2\}$ and

$$\left(\forall \psi \in \tilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}' : F \notin \text{pPOL}_k \psi \right) \text{ and } (\exists \chi \in \mathcal{T}' : F \in \text{pPOL}_k \chi).$$

(e) $\mu_1 = \mu_2$ and $\exists \pi \in S_{\mu_1} : \psi_2 \subset \psi_1^{[\pi]}$ (w.l.o.g. $\pi = \text{id}$).

Analogous to the previous case because ψ_2 is irreducible in this case. Thus there is some \mathcal{T}' with $\{\psi_2\} \subseteq \mathcal{T}' \subseteq \mathcal{T} \setminus \{\psi_1\}$ and

$$\left(\forall \psi \in \tilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}' : F \notin \text{pPOL}_k \psi \right) \text{ and } (\exists \chi \in \mathcal{T}' : F \in \text{pPOL}_k \chi).$$

(f) Otherwise we have

$$\begin{aligned} \forall \emptyset \subset A \subset E_{\mu_1} \forall v \in \sigma(E_k^{\mu_2 - |A|}) \forall \pi \in S_{\mu_2} : (\text{pr}_A \psi_1) \times \{v\} \not\subseteq \psi_2^{[\pi]}, \\ \mu_1 < \mu_2 \text{ or } \forall \emptyset \subset A \subset E_{\mu_2} \forall v \in \sigma(E_k^{\mu_1 - |A|}) \forall \pi \in S_{\mu_1} : (\text{pr}_A \psi_2) \times \{v\} \not\subseteq \psi_1^{[\pi]}, \\ \mu_1 = \mu_2 \text{ or } \forall v \in \sigma(E_k^{\mu_2 - \mu_1}) \forall \pi \in S_{\mu_2} : \psi_1 \times \{v\} \not\subseteq \psi_2^{[\pi]}, \\ \mu_1 < \mu_2 \text{ or } \forall \pi \in S_{\mu_2} : \psi_1 \not\subseteq \psi_2^{[\pi]}, \\ \mu_1 < \mu_2 \text{ or } \forall \pi \in S_{\mu_1} : \psi_2 \not\subseteq \psi_1^{[\pi]}. \end{aligned}$$

Let $g(\psi_1) = d \in E_k^{\mu_1} \setminus \psi_1$. Then $g \in \text{pPOL}_k \psi_2$ because $\mu_1 \leq \mu_2$ and if $\mu_1 = \mu_2$ then $\psi_2 \not\subseteq \psi_1^{[\pi]}$ for all $\pi \in S_{\mu_1}$. Assume $F^{(n)} = f \otimes g \notin \text{pPOL}_k \psi_2$. Then there are $c_{1*}, \dots, c_{\mu_2*}$ with $c_{*1}, \dots, c_{*n} \in \psi_2$ and $F(c_{*1}, \dots, c_{*n}) \notin \psi_2$ and the rows $c_{1*}, \dots, c_{\|\psi_2\|*}$ belong to the g -part of F by Proposition 7.6.10, i.e., one of the following cases apply

- there is some $A \subset E_{\mu_1}$ and $v \in \sigma(E_k^{\mu_2 - |A|})$ with $(\text{pr}_A \psi_1) \times \{v\} \subseteq \psi_2$ contradicting the first assumption, or
- $\mu_1 < \mu_2$ and there is some $v \in \sigma(E_k^{\mu_2 - |A|})$ with $\psi_1 \times \{v\} \subseteq \psi_2$ contradicting the third assumption.

Thus $F \in \text{pPOL}_k \psi_2$. Furthermore $F \notin \text{pPOL}_k \psi_1$ because $g \notin \text{pPOL}_k \psi_1$.

Thus there is some \mathcal{T}' with $\{\psi_2\} \subseteq \mathcal{T}' \subseteq \mathcal{T} \setminus \{\psi_1\}$ and

$$\left(\forall \psi \in \tilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}' : F \notin \text{pPOL}_k \psi \right) \text{ and } (\exists \chi \in \mathcal{T}' : F \in \text{pPOL}_k \chi).$$

Thus in every case there is some $\mathcal{T}' \subset \mathcal{T}$ with

$$\left(\forall \psi \in \tilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}' : F \notin \text{pPOL}_k \psi \right) \text{ and } (\exists \chi \in \mathcal{T}' : F \in \text{pPOL}_k \chi),$$

i.e., (b) is true. □

Theorem 7.6.13. *For every $k \geq 2$ there is exactly one minimal covering of $\text{p}\mathcal{M}_k$.*

Proof. For $k = 2$ this statement is given by Lemma 7.4.1. Thus we can assume $k \geq 3$. Assume the statement is false. Then there are pairwise different minimal coverings $\mathcal{X}_1, \dots, \mathcal{X}_l$ with $l \geq 2$. Choose $\varrho \in \tilde{\mathcal{R}}_k^{\max}$ with $\text{pPOL}_k \varrho \in \mathcal{X}_1 \setminus \mathcal{X}_2$ arbitrarily. Then $\varrho \in \mathcal{Q}_1$ because of Theorem 7.6.5. Let

$$\mathcal{T} := \{\psi \in \mathcal{Q}_1 \mid \text{pp } \psi = \text{pp } \varrho \text{ and } (\exists a, b \in \{1, \dots, l\} : \text{pPOL}_k \psi \in \mathcal{X}_a \setminus \mathcal{X}_b)\} \subseteq \mathcal{T}_\varrho.$$

Then $\varrho \in \mathcal{T}$ and $|\mathcal{T}| \geq 2$ by Lemma 7.6.6. Additionally there is some $f \in \tilde{P}_k \setminus (P_k \cup C_\infty)$ with

$$(\forall \psi \in \tilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T} : f \notin \text{pPOL}_k \psi) \text{ and } (\exists \chi \in \mathcal{T} : f \in \text{pPOL}_k \chi). \quad (7.10)$$

Otherwise $\text{pPOL}_k \varrho$ would be in no minimal covering contradicting the assumption.

Now we can assume that $\widehat{\mathcal{T}} \subseteq \mathcal{T}$ has minimal size $|\widehat{\mathcal{T}}| \geq 2$ and fulfills (7.10) (with $\widehat{\mathcal{T}}$ instead of \mathcal{T}).

By Lemma 7.6.12 there are two cases:

- (a) there are $\chi_0 \in \widehat{\mathcal{T}}$ and $F \in \text{pPOL}_k \chi_0$ with

$$\forall \psi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \{\chi_0\} : F \notin \text{pPOL}_k \psi.$$

Then $\text{pPOL}_k \chi_0$ is in every minimal covering of $p\mathcal{M}_k$ by Lemma 7.1.2 in contradiction to the definition of \mathcal{T} and the assumption.

- (b) there are $F \in \widetilde{P}_k$, \mathcal{T}' with $\emptyset \subset \mathcal{T}' \subset \widehat{\mathcal{T}}$ with

$$(\forall \psi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \mathcal{T}' : F \notin \text{pPOL}_k \psi) \text{ and } (\exists \chi \in \mathcal{T}' : F \in \text{pPOL}_k \chi).$$

Because $\widehat{\mathcal{T}}$ is minimal under the condition $|\widehat{\mathcal{T}}| \geq 2$ we conclude $|\mathcal{T}'| = 1$. Then $\mathcal{T}' = \{\chi_0\}$, $F \in \text{pPOL}_k \chi_0$ and

$$\forall \psi \in \widetilde{\mathcal{R}}_k^{\max} \setminus \{\chi_0\} : F \notin \text{pPOL}_k \psi.$$

Thus $\text{pPOL}_k \chi_0$ is in every minimal covering of $p\mathcal{M}_k$ by Lemma 7.1.2, in contradiction to the definition of \mathcal{T} and the assumption.

Thus there are no two different minimal coverings of $p\mathcal{M}_k$. □

Let $p\mathcal{C}_k$ be the unique minimal covering of $p\mathcal{M}_k$. Using the uniqueness of minimal coverings we can improve the statements of Lemmas 7.1.3 and 7.1.2.

Lemma 7.6.14. *Let $C \in p\mathcal{M}_k$ and $\mathcal{C} \subseteq p\mathcal{M}_k \setminus \{C\}$ such that for all $f \in C$ there is some $C' \in \mathcal{C}$ with $f \in C'$. Then $C \notin p\mathcal{C}_k$.*

Proof. Assume C is in the minimal covering $p\mathcal{C}_k$ of $p\mathcal{M}_k$. Let $\mathcal{Y} := (p\mathcal{C}_k \setminus \{C\}) \cup \mathcal{C}$. Then \mathcal{Y} is a covering of $p\mathcal{M}_k$ because for all $f \in X \in p\mathcal{M}_k$ there is

- some $Y \in p\mathcal{C}_k \setminus \{C\}$ with $f \in Y$, or
- $f \in C$ and then there is some $Y \in \mathcal{C}$ with $f \in Y$.

Then there is some minimal covering $\widetilde{\mathcal{Y}} \subseteq \mathcal{Y}$ of $p\mathcal{M}_k$. But $\widetilde{\mathcal{Y}} \cap p\mathcal{C}_k \subset p\mathcal{C}_k$ and thus $\widetilde{\mathcal{Y}} \neq p\mathcal{C}_k$ contradicting Theorem 7.6.13. □

Lemma 7.6.15. *Let $C \in p\mathcal{M}_k$. Then*

$$C \in p\mathcal{C}_k \iff (\exists f \in C \forall B \in p\mathcal{M}_k \setminus \{C\} : f \notin B).$$

Proof. We split the proof into two directions:

\Leftarrow Follows from Lemma 7.1.2 and Theorem 7.6.13.

\Rightarrow Let $C \in p\mathcal{C}_k$. Assume,

$$\forall f \in C \exists B \in p\mathcal{M}_k \setminus \{C\} : f \in B.$$

By Lemma 7.6.14 with $\mathcal{C} = p\mathcal{M}_k \setminus \{C\}$ follows $C \notin p\mathcal{C}_k$ in contradiction to the assumption. □

Lemma 7.6.16. *Let $\varrho^{(h)} \in \mathcal{Q}_1$ be reducible. Then $\text{pPOL}_k \varrho$ is not in the minimal covering $p\mathcal{C}_k$ of $p\mathcal{M}_k$.*

Proof. This follows directly from Lemma 7.6.9 with the help of Lemma 7.6.14. □

7.7 Conclusion

Now we know that there is exactly one minimal covering $p\mathcal{C}_k$ of $p\mathcal{M}_k$ and the last three lemmas show that this can help in determining some members and non-members of $p\mathcal{C}_k$. Additionally the technique of the product of functions \otimes proved to be a very good tool in the solution of this problem and might be used for solving other questions regarding partial clones, not necessarily the maximal ones. It is a operation not possible in the realm of total clones since the size of the domain of $f \otimes g$ is the sum of sizes of the domains of f and g , i.e., the domain of $f \otimes g$ is not the full domain in nearly every case.

Chapter 8

Intervals of Partial Clones

In the previous two chapters we were concerned about the maximal partial clones on a k -element set with $k \geq 2$. Now we go down in the lattice of partial clones $\mathbb{L}(\tilde{P}_k)$ below the maximal partial clones. Then we can use the structure of the lattice of all total clones $\mathbb{L}(P_k)$ (see Figure 8.1 for the case $k = 2$) to determine parts of the lattice $\mathbb{L}(\tilde{P}_k)$. Since P_k is a subset of \tilde{P}_k we can look at $C \cap P_k$ for a given partial clone $C \in \mathbb{L}(\tilde{P}_k)$ and try to get some information about the partial clones from the knowledge about the total clones.

First note that $C \cap P_k$ is a total clone, i.e., we can find it in the lattice $\mathbb{L}(P_k)$. One possible way to capture this is by the notion of an interval.

Let $A, C \in \mathbb{L}(\tilde{P}_k)$. The *interval* between A and C is the set of all partial clones which contain A and are contained in C , i.e., the set

$$[A, C] := \{B \in \mathbb{L}(\tilde{P}_k) \mid A \subseteq B \subseteq C\}.$$

Specifically we take $A \in \mathbb{L}(P_k)$ and $C \notin \mathbb{L}(P_k)$ with $A \subset C$. Then the interval is not empty and we can try to determine the size and the structure of sublattice $[A, C] \subseteq \mathbb{L}(\tilde{P}_k)$.

Since A and C might be taken arbitrarily, studies in this direction concentrate on different questions with respect to C :

- Let $C = \tilde{P}_k$, i.e., one looks at the interval $[A, \tilde{P}_k]$ for some total clone A ; see e.g. [27, 28, 15].
- Let A be a total clone which is finitely generated with *order* n (see e.g. [40] Chapter 11), i.e., if $A = [A^{(n)}]$ holds for some $n \in \mathbb{N}$. Then $A = \text{Pol}_k G_n(A)$.
Let $C := \text{pPOL}_k G_n(A)$ and consider the interval $[C, \text{pPOL}_k G_n(C)]$; see e.g. [19].

The second interval type naturally occurs in the study of partial clones where $A \in \mathbb{L}(P_k)$ is a fixed *total component*, i.e., if we study the set

$$\mathcal{I}_k(A) := \left\{ B \in \mathbb{L}(\tilde{P}_k) \mid B \cap P_k = A \right\}.$$

These sets were first studied by Alekseev and Voronenko in [1], by Strauch in [69, 70, 71], and others in [6, 16, 18]. Some results already follow from the determination of the submaximal partial clones [38].

Theorem 4 in [19] (see also Theorem 2 in [71]) states that

$$\mathcal{I}_k(A) = [A, \text{pPOL}_k G_n(A)]$$

holds for every finitely generated total clone $A \in \mathbb{L}(P_k)$ with order n .

Therefore the sets $\mathcal{I}_k(A)$ are also called intervals and we only consider some of this type of interval in this chapter. Additionally we stick to the Boolean case, i.e., we take $k = 2$. Then all $A \in \mathbb{L}(P_2)$ are finitely generated as shown by Post in [48]. Thus there is always a maximal element in $\mathcal{I}_2(A)$.

In this chapter we show that the interval $\mathcal{I}_2(S \cap T)$ is finite and then we identify all partial clones in that interval. The clone $S \cap T$ is the clone of all selfdual and idempotent functions. This concludes the investigation of all the finite intervals on the 2-element set.

8.1 Definitions

We need to define some clones of P_2 now.

$$\begin{aligned}
 M &:= \text{Pol}_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \\
 S &:= \text{Pol}_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
 L &:= \text{Pol}_2 \varrho_2 \\
 T_{a,i} &:= \text{Pol}_2(E_2^i \setminus (E_2 \setminus \{a\})^i) \text{ for } a \in E_2, i \geq 1 \\
 T_a &:= T_{a,1} = \text{Pol}_2\{a\} \text{ for } a \in E_2, \\
 T &:= T_0 \cap T_1, \\
 T_{a,\infty} &:= \bigcap_{i \geq 1} T_{a,i} \text{ for } a \in E_2, \\
 V &:= [\vee, c_0, c_1], \\
 \Lambda &:= [\wedge, c_0, c_1], \\
 &[P_2^{(1)}].
 \end{aligned}$$

Example 8.1.1.

$$T_{0,2} = \text{Pol}_2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Theorem 8.1.2 (Post's Theorem; [47, 48]). *The set of clones in P_2 is countably infinite. The clones of P_2 are*

- $P_2, T_a, T,$
- $M, M \cap T_a, M \cap T,$
- $T_{a,\mu}, T_{a,\mu} \cap T, T_{a,\mu} \cap M, T_{a,\mu} \cap M \cap T,$
- $S, S \cap T, S \cap M,$
- $L, L \cap S, L \cap T_a, L \cap T,$
- $[P_2^{(1)}], [P_2^{(1)}] \cap S,$
- $\Lambda, \Lambda \cap T_a, \Lambda \cap T,$
- $V, V \cap T_a, V \cap T,$
- $\Lambda \cap V, \Lambda \cap V \cap T_a, \Lambda \cap V \cap T$

where $a \in \{0, 1\}$ and $\mu \in \{2, 3, \dots, \infty\}$.

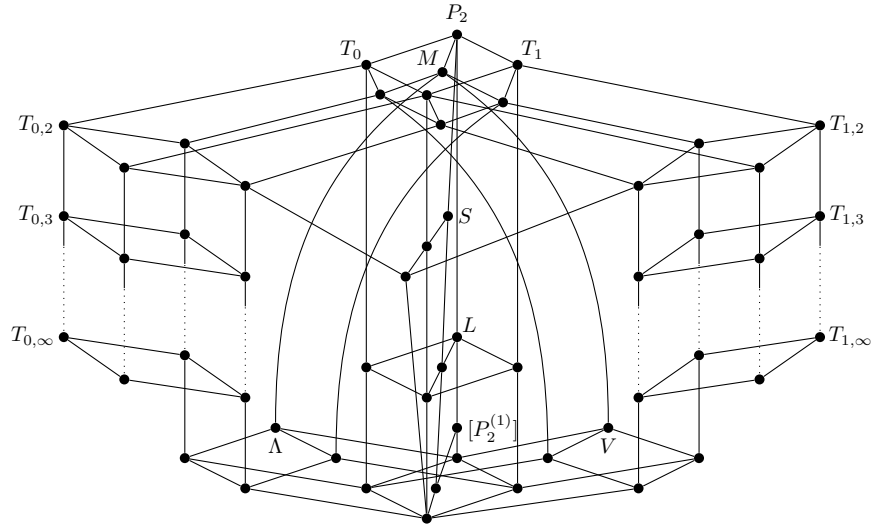


Figure 8.1: Post's lattice

There are numerous new proofs of Theorem 8.1.2; see for example [40, 12, 74].

Theorem 8.1.3. *Let $A \subseteq P_2$ be a total clone. Then $\mathcal{I}_2(A)$ is finite if $S \cap T_0 \cap T_1 \subseteq A$ or $M \cap T_0 \cap T_1 \subseteq A$.*

Furthermore it holds:

A	$ \mathcal{I}_2(A) $
P_2	3
T_a ($a \in \{0, 1\}$)	6
M	6
S	6
T	30
$M \cap T_a$ ($a \in \{0, 1\}$)	15
$M \cap T$	101
$S \cap T$	380

Proof. The finiteness of the set $\mathcal{I}_2(S \cap T)$ is shown in Theorem 8.2.3 in the following section. The statement $|\mathcal{I}_2(S \cap T)| = 380$ is the result of Section 8.3 and a computer program.

The proofs of the remaining statements can be found in [13, 1, 71, 69, 19, 28, 15]. □

Theorem 8.1.4. *Let $A \subseteq P_2$ be a total clone with*

$$A \subseteq B \in \{L, V, \Lambda, T_{0,\infty}, T_{1,\infty}\}.$$

Then the set $\mathcal{I}_2(A)$ has the cardinality of the continuum.

Proof. The theorem has been proven in [70] for $B \in \{V, \Lambda, T_{0,\infty}, T_{1,\infty}\}$, in [1] for $A = L$, and is a conclusion of [1] for $A \in \{L \cap T_0, L \cap T_1, L \cap S, L \cap T_0 \cap S\}$. □

Theorem 8.1.5. *Let $A \subseteq P_2$ be a total clone with $S \cap T_0 \cap T_1 \not\subseteq A$ and $M \cap T_0 \cap T_1 \not\subseteq A$. Then the set $\mathcal{I}_2(A)$ is infinite.*

Proof. The theorem has been proven in [41]. □

8.2 The Interval $\mathcal{I}_2(S \cap T)$

Let $s : E_2 \rightarrow E_2$ with $s(0) = 1$ and $s(1) = 0$,

$$\begin{aligned} s^\circ &:= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \varrho_s &:= \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ S_{max} &:= \text{pPOL}_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ T_{max} &:= \text{pPOL}_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ and} \\ (ST)_{max} &:= \text{pPOL}_2 \varrho_s. \end{aligned}$$

Furthermore, for $\{a_0, a_1\} \subseteq \tilde{E}_2$ define

$$U(a_0, a_1) := \bigcup_{n \geq 1} \left\{ f^{(n)} \in \tilde{P}_2 \mid \forall x \in \{0, 1\} : f(x, x, \dots, x) = a_x \right\}.$$

Obviously, for all

$$(a_0, a_1) \in \{(\alpha, \infty), (\infty, \alpha), (\infty, \infty) \mid \alpha \in E_2\}$$

we have

$$\begin{aligned} U(a_0, a_1) &\subseteq (ST)_{max}, \\ U(0, 1) &\not\subseteq (ST)_{max}, \\ S_{max} \cap T_{max} &\subset (ST)_{max}, \\ (ST)_{max} \cap P_2 &= S \cap T, \text{ and} \\ (ST)_{max} &= (S_{max} \cap T_{max}) \cup \bigcup_{(a_0, a_1) \in \tilde{E}_2^2 \setminus E_2^2} U(a_0, a_1). \end{aligned} \quad (8.1)$$

Lemma 8.2.1. *The partial clone $(ST)_{max}$ is the maximal element of the set $\mathcal{I}_2(S \cap T)$.*

Proof. Assume $(ST)_{max}$ is not the maximal element of the set $\mathcal{I}_2(S \cap T)$. Then there is some function $f^{(n)} \in \tilde{P}_2 \setminus (ST)_{max}$ with $[\{f\} \cup (S \cap T)]_P \cap P_2 = S \cap T$.

Then there are $a_1, \dots, a_n, b_0, \dots, b_3 \in E_2$ with

$$f \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ s(a_1) & s(a_2) & \dots & s(a_n) \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} \in E_2^4 \setminus \varrho_2.$$

There are binary functions $g_1, \dots, g_n \in S \cap T$ with $g_i(0, 0) = 0$ and $g_i(0, 1) = a_i$ for all $i \in \{1, \dots, n\}$. We define the binary function g by $g := f(g_1, g_2, \dots, g_n)$. Then

$$g \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = f \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ s(a_1) & s(a_2) & \dots & s(a_n) \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} \in E_2^4 \setminus \varrho_s.$$

Thus $g \in P_2$, but $g \notin \text{Pol}_2 \varrho_s = S \cap T$ in contradiction to $[\{f\} \cup (S \cap T)]_{\mathbb{P}} \cap P_2 = S \cap T$. \square

For every $f^{(n)} \in S_{max} \cap T_{max}$ set

$$\chi(f) := \{(f(\mathbf{a}), f(s(\mathbf{a}))) \mid \mathbf{a} \in E_2^n\}$$

where $s(\mathbf{a}) = (s(a_1), \dots, s(a_n))$ for $\mathbf{a} = (a_1, \dots, a_n) \in E_2^n$. For $(a_0, a_1) \in ({}^0_1 \infty \infty \infty \infty)$ and some $R \subseteq \tilde{E}_2^2$ with $(a_0, a_1) \in R$, let

$$U_R(a_0, a_1) := \{g \in S_{max} \cap T_{max} \mid \chi(g) \subseteq R, g(\mathbf{0}) = a_0, g(\mathbf{1}) = a_1\}.$$

Since $\chi(f)$ is symmetric by definition, we can assume that R is symmetric. Furthermore we note that $U(a_0, a_1) = U_{\tilde{E}_2^2}(a_0, a_1)$.

Obviously,

$$S_{max} \cap T_{max} = \bigcup_{(a_0, a_1) \in \{(0,1)\} \cup (\tilde{E}_2^2 \setminus E_2^2)} \bigcup_{\substack{R \\ (a_0, a_1) \in R \subseteq \tilde{E}_2^2}} U_R(a_0, a_1) \quad (8.2)$$

holds.

Let $V_n := \{(0, a_2, \dots, a_n) \in E_k^n\}$. Because $f(s(\mathbf{a})) = s(f(\mathbf{a}))$ for all $f^{(n)} \in S$ the functions in $S \cap T$ are fully determined by the values on the tuples in V_n .

Lemma 8.2.2. *Let $f \in \tilde{P}_2$ with $(f(\mathbf{0}), f(\mathbf{1})) = (\alpha, \beta)$.*

(a) *If $f \in (ST)_{max} \setminus (S_{max} \cap T_{max})$ then $U(\alpha, \beta) \subseteq [\{f\} \cup (S \cap T)]_{\mathbb{P}}$.*

(b) *If $f \in S_{max} \cap T_{max}$ then $U_{\chi(f)}(\alpha, \beta) \subseteq [\{f\} \cup (S \cap T)]_{\mathbb{P}}$.*

Proof. (a): Let $f^{(n)} \in (ST)_{max} \setminus (S_{max} \cap T_{max})$. Then $(\alpha, \beta) \in \tilde{E}_2^2 \setminus E_2^2$. We can assume $\alpha = \infty$. Furthermore there is some tuple $(a_1, \dots, a_n) \in V_n \setminus \{\mathbf{0}\}$ with

$$(a, b) := (f(a_1, \dots, a_n), f(s(a_1), \dots, s(a_n))) \in E_2^2 \setminus s^\circ.$$

Let $G^{(m)} \in U(\alpha, \beta)$ be arbitrary. We want to show that $G \in [\{f\} \cup (S \cap T)]_{\mathbb{P}}$.

There are functions

$$f_1^{(m)}, \dots, f_n^{(m)}, r^{(m)}, t^{(2)} \in S \cap T$$

with the properties

- $f_i(v) = a_i$ for all $i \in \{1, \dots, n\}$ and all $v \in V_m \setminus \{\mathbf{0}\}$,
- $r(v) = 0$ for all $v \in V_m \setminus \{\mathbf{0}\}$, and
- for some $c \in E_2$ with $(c, \beta) \notin s^\circ$

$$t \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} = \begin{pmatrix} c \\ c \end{pmatrix}.$$

Thus

$$q^{(m)} := t(r, f(f_1, \dots, f_n)) \in [\{f\} \cup (S \cap T)]_{\mathbb{P}},$$

where

$$q(\mathbf{x}) = \begin{cases} \alpha & \text{for } \mathbf{x} = \mathbf{0}, \\ \beta & \text{for } \mathbf{x} = \mathbf{1}, \\ c & \text{otherwise.} \end{cases}$$

Let

$$U^*(\alpha, \beta) := \bigcup_{m \geq 1} \{g^{(m)} \in U(\alpha, \beta) \mid \forall \mathbf{x} \in E_2^m \setminus \{\mathbf{0}, \mathbf{1}\} : g(\mathbf{x}) \in E_2\}.$$

Let $g^{(m)} \in U^*(\alpha, \beta)$ be arbitrary. There is some function $h^{(m+1)} \in S \cap T$ with

$$h(\beta, 1, 1, \dots, 1) = \beta \text{ (for } \beta = \infty \text{ this is true for any partial function)}$$

and

$$\forall (x_1, \dots, x_m) \in E_2^m \setminus \{\mathbf{0}, \mathbf{1}\} : h(c, x_1, \dots, x_m) = g(x_1, \dots, x_m).$$

Since $h, q \in [\{f\} \cup (S \cap T)]_{\mathbb{P}}$ and

$$\forall \mathbf{x} \in E_2^m : g(\mathbf{x}) = h(q(\mathbf{x}), \mathbf{x})$$

we have $g \in [\{f\} \cup (S \cap T)]_{\mathbb{P}}$, i.e., $U^*(\alpha, \beta) \subseteq [\{f\} \cup (S \cap T)]_{\mathbb{P}}$.

Let $\mathbf{d} \in E_2^m \setminus \{\mathbf{0}, \mathbf{1}\}$. There are functions $p_1^{(m)}, \dots, p_n^{(m)} \in U^*(\alpha, \beta)$ with

$$\begin{aligned} p_i(\mathbf{d}) &= 0, \\ p_i(\mathbf{x}) &= a_i, \text{ for all } \mathbf{x} \in E_2^m \setminus \{\mathbf{0}, \mathbf{1}, \mathbf{d}\}. \end{aligned}$$

Then we can set $u_{\mathbf{d}} := f(p_1, \dots, p_n) \in [\{f\} \cup (S \cap T)]_{\mathbb{P}}$ and we have

$$\begin{aligned} u_{\mathbf{d}}(\mathbf{d}) &= f(0, \dots, 0) = \alpha = \infty, \\ u_{\mathbf{d}}(\mathbf{0}) &= \alpha, \\ u_{\mathbf{d}}(\mathbf{1}) &= \beta, \\ u_{\mathbf{d}}(\mathbf{x}) &= f(a_1, \dots, a_n) = a \in E_2 \text{ for all } \mathbf{x} \in E_2^m \setminus \{\mathbf{0}, \mathbf{1}, \mathbf{d}\}. \end{aligned}$$

Now we take the arbitrary function $G^{(m)} \in U(\alpha, \beta)$. Then there is some function $g^{(m)} \in U^*(\alpha, \beta)$ with $G(\mathbf{x}) = g(\mathbf{x})$ for all $\mathbf{x} \in E_2^m$ with $G(\mathbf{x}) \in E_2$. Let

$$\{\mathbf{d}_1, \dots, \mathbf{d}_l\} := \{\mathbf{x} \in E_2^m \setminus \{\mathbf{0}, \mathbf{1}\} \mid G(\mathbf{x}) = \infty\}.$$

Then

$$G = e_1^{(1+l)}(g, u_{\mathbf{d}_1}, \dots, u_{\mathbf{d}_l}),$$

i.e., $G \in [\{f\} \cup (S \cap T)]_{\mathbb{P}}$ and thus $U(\alpha, \beta) \subseteq [\{f\} \cup (S \cap T)]_{\mathbb{P}}$.

(b): Let $G^{(m)} \in U_{\chi(f)}(\alpha, \beta)$ be arbitrary. Then for every $\mathbf{v} \in V_m$ there is some $\mathbf{b}_{\mathbf{v}} := (b_{\mathbf{v},1}, \dots, b_{\mathbf{v},n}) \in V_n$ such that $(G(\mathbf{v}), G(s(\mathbf{v}))) = (f(\mathbf{b}_{\mathbf{v}}), f(s(\mathbf{b}_{\mathbf{v}})))$ holds. It is easy to check that there are functions $g_1^{(m)}, \dots, g_n^{(m)} \in S \cap T$ with $g_i(\mathbf{v}) = b_{\mathbf{v},i}$ for all $\mathbf{v} \in V_m$ and all $i \in \{1, \dots, n\}$. Then we have $G = f(g_1, \dots, g_n)$, i.e. (b) holds. \square

Theorem 8.2.3. *The set $\mathcal{I}_2(S \cap T)$ is finite.*

Proof. This follows from (8.1), (8.2), Lemma 8.2.1 and Lemma 8.2.2. \square

8.3 The Partial Clones of $\mathcal{I}_2(S \cap T)$

Now we determine the elements of $\mathcal{I}_2(S \cap T)$.

Since $\chi(f)$ is symmetric by definition we only need consider symmetric R in the sets $U_R(a_0, a_1)$. Because we want to save space we omit the symmetric equivalents of pairs; e.g., we write $(0, \infty)$ and omit $(\infty, 0)$. That means we only need to consider subsets of $\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & \infty \\ 1 & 0 & 1 & \infty & \infty & \infty \end{pmatrix}$. As exception we write \tilde{E}_2^2 where we do not omit the symmetry.

Lemma 8.3.1. *Let $C \in \mathcal{I}_2(S \cap T)$ and $U_R(a_0, a_1) \subseteq C$ for some $R \subseteq \begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}$, a_0, a_1 and $(0, \infty) \in R$.*

Then $U_{R'}(a_0, a_1) \subseteq C$ with $R' := R \cup \begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}$.

Proof. If $(0, 1) \in R$ then let $(c_0, c_1) := (0, 1)$. Otherwise let $(c_0, c_1) := (0, \infty)$. Let $f^{(5)} \in S \cap T$ and $g_1^{(4)}, g_2^{(4)} \in U_R(a_0, a_1)$ as defined below belong to C and since it is a clone also the function $h^{(4)} := f(e_1^{(4)}, e_2^{(4)}, e_3^{(4)}, e_4^{(4)}, g_1^{(4)}, g_2^{(4)})$ belongs to C .

x_1	x_2	x_3	x_4	x_5	x_6	$f(x_1, \dots, x_6)$
0	0	0	0	0	0	0
0	0	0	0	1	1	1
0	0	0	1			0
0	0	1	0			1
0	0	1	1			1
0	1	0	0			0
1	0	1	1			1
1	1	0	0			0
1	1	0	1			0
1	1	1	0			1
1	1	1	1	0	0	0
1	1	1	1	1	1	1

x_1	x_2	x_3	x_4	$g_1(x_1, \dots, x_4)$	$g_2(x_1, \dots, x_4)$	$h(x_1, \dots, x_4)$
0	0	0	0	a_0	a_0	a_0
0	0	0	1	0	0	0
0	0	1	0	0	0	1
0	0	1	1	0	∞	∞
0	1	0	0	c_0	c_0	c_0
1	0	1	1	c_1	c_1	c_1
1	1	0	0	∞	0	∞
1	1	0	1	∞	∞	∞
1	1	1	0	∞	∞	∞
1	1	1	1	a_1	a_1	a_1

Since $\chi(h) = R \cup \begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix} = R'$ we have $U_{R'}(a_0, a_1) \subseteq C$ by Lemma 8.2.2. \square

By Lemma 8.3.1 and Lemma 8.2.2 we know that for $U_R(a_0, a_1)$ the set R can only be $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} \infty \\ \infty \end{pmatrix}$, $\begin{pmatrix} 0 & \infty \\ 1 & \infty \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}$ or \tilde{E}_2^2 . Furthermore $(a_0, a_1) \in R$ implies that the only possible combinations for $U_R(a_0, a_1)$ are

$$\begin{aligned}
U_R(a_0, a_1) \in & \{U_T(0, 1) \mid T \in \{\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & \infty \\ 1 & \infty \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}\} \cup \\
& \left\{U_T(a_0, a_1) \mid (a_0, a_1) \in \begin{pmatrix} 0 & 1 & \infty & \infty \\ \infty & \infty & 0 & 1 \end{pmatrix}, T \in \left\{\begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}, \tilde{E}_2^2\right\}\right\} \cup \\
& \left\{U_T(\infty, \infty) \mid T \in \left\{\begin{pmatrix} \infty \\ \infty \end{pmatrix}, \begin{pmatrix} 0 & \infty \\ 1 & \infty \end{pmatrix}, \begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}, \tilde{E}_2^2\right\}\right\}.
\end{aligned}$$

Lemma 8.3.2. Denote by $f_{a_0, a_1, R} \in U_R(a_0, a_1)$ and $f_{b_0, b_1, S} \in U_S(b_0, b_1)$ arbitrary functions.

Then

$$\begin{aligned} f_{a_0, a_1, R} \star f_{0, 1, S} &\in U_{R \star S}(a_0, a_1) \\ f_{a_0, a_1, R} \star f_{0, \infty, S} &\in U_{R \star S}(a_0, \infty) \\ f_{a_0, a_1, R} \star f_{1, \infty, S} &\in \bigcup_{c \in \tilde{E}_2} U_{R \star S}(c, \infty) \\ f_{a_0, a_1, R} \star f_{\infty, 1, S} &\in U_{R \star S}(\infty, a_1) \\ f_{a_0, a_1, R} \star f_{\infty, 0, S} &\in \bigcup_{c \in \tilde{E}_2} U_{R \star S}(\infty, c) \\ f_{a_0, a_1, R} \star f_{\infty, \infty, S} &\in U_{R \star S}(\infty, \infty) \end{aligned}$$

where $R \star S$ is given by

$R \star S$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 & \infty \\ 1 & \infty \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}$	\tilde{E}_2^2	$\begin{pmatrix} \infty \\ \infty \end{pmatrix}$
$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 & \infty \\ 1 & \infty \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}$	\tilde{E}_2^2	$\begin{pmatrix} \infty \\ \infty \end{pmatrix}$
$\begin{pmatrix} 0 & \infty \\ 1 & \infty \end{pmatrix}$	$\begin{pmatrix} 0 & \infty \\ 1 & \infty \end{pmatrix}$	$\begin{pmatrix} 0 & \infty \\ 1 & \infty \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}$	\tilde{E}_2^2	$\begin{pmatrix} \infty \\ \infty \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}$	\tilde{E}_2^2	$\begin{pmatrix} \infty \\ \infty \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}$	\tilde{E}_2^2	$\begin{pmatrix} \infty \\ \infty \end{pmatrix}$
E_2^2	E_2^2	E_2^2	E_2^2	$\begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}$	\tilde{E}_2^2	$\begin{pmatrix} \infty \\ \infty \end{pmatrix}$
$\begin{pmatrix} \infty \\ \infty \end{pmatrix}$	$\begin{pmatrix} \infty \\ \infty \end{pmatrix}$	$\begin{pmatrix} \infty \\ \infty \end{pmatrix}$	$\begin{pmatrix} \infty \\ \infty \end{pmatrix}$	$\begin{pmatrix} \infty \\ \infty \end{pmatrix}$	$\begin{pmatrix} \infty \\ \infty \end{pmatrix}$	$\begin{pmatrix} \infty \\ \infty \end{pmatrix}$

Proof. Let $f := f_{a_0, a_1, R}$ and $g := f_{b_0, b_1, S}$.

We first look at the values $c_0 := (f \star g)(\mathbf{0})$ and $c_1 := (f \star g)(\mathbf{1})$.

- $(b_0, b_1) = (0, 1)$. Then

$$\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = (f \star g) \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix} = f \begin{pmatrix} g(\mathbf{0}) & \mathbf{0} \\ g(\mathbf{1}) & \mathbf{1} \end{pmatrix} = f \begin{pmatrix} 0 & \mathbf{0} \\ 1 & \mathbf{1} \end{pmatrix} = f \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

- $(b_0, b_1) = (0, \infty)$. Then

$$\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = (f \star g) \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix} = f \begin{pmatrix} g(\mathbf{0}) & \mathbf{0} \\ g(\mathbf{1}) & \mathbf{1} \end{pmatrix} = f \begin{pmatrix} 0 & \mathbf{0} \\ \infty & \mathbf{1} \end{pmatrix} = \begin{pmatrix} a_0 \\ \infty \end{pmatrix}$$

- $(b_0, b_1) = (1, \infty)$. Then

$$\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = (f \star g) \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix} = f \begin{pmatrix} g(\mathbf{0}) & \mathbf{0} \\ g(\mathbf{1}) & \mathbf{1} \end{pmatrix} = f \begin{pmatrix} 1 & \mathbf{0} \\ \infty & \mathbf{1} \end{pmatrix} = \begin{pmatrix} c \\ \infty \end{pmatrix}$$

where $c \in \tilde{E}_2$.

- $(b_0, b_1) = (\infty, 1)$. Then

$$\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = (f \star g) \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix} = f \begin{pmatrix} g(\mathbf{0}) & \mathbf{0} \\ g(\mathbf{1}) & \mathbf{1} \end{pmatrix} = f \begin{pmatrix} \infty & \mathbf{0} \\ 1 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \infty \\ a_1 \end{pmatrix}$$

- $(b_0, b_1) = (\infty, 0)$. Then

$$\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = (f \star g) \begin{pmatrix} \mathbf{0} \\ \mathbf{1} \end{pmatrix} = f \begin{pmatrix} g(\mathbf{0}) & \mathbf{0} \\ g(\mathbf{1}) & \mathbf{1} \end{pmatrix} = f \begin{pmatrix} \infty & \mathbf{0} \\ 0 & \mathbf{1} \end{pmatrix} = \begin{pmatrix} \infty \\ c \end{pmatrix}$$

where $c \in \tilde{E}_2$.

Let $f^{(n+1)}, g^{(m)} \in \tilde{P}_2$ and $\mathbf{x} \in E_2^m, \mathbf{y} \in E_2^n$ be arbitrary. We now look at the possibilities for $R \star S$, i.e. $\chi(f \star g)$ depending on $\chi(f)$ and $\chi(g)$.

Then let

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := (f \star g) \begin{pmatrix} \mathbf{x} & \mathbf{y} \\ s(\mathbf{x}) & s(\mathbf{y}) \end{pmatrix} = f \begin{pmatrix} g(\mathbf{x}) & \mathbf{y} \\ g(s(\mathbf{x})) & s(\mathbf{y}) \end{pmatrix} = f \begin{pmatrix} w_1 & \mathbf{y} \\ w_2 & s(\mathbf{y}) \end{pmatrix}$$

where $(w_1, w_2) \in \chi(g)$.

- If $(w_1, w_2) = (\infty, \infty)$ then $(u_1, u_2) = (f(\infty, \mathbf{y}), f(\infty, s(\mathbf{y}))) = (\infty, \infty)$.
- If $(w_1, w_2) = (a, \infty)$ for some $a \in E_2$ then

$$(u_1, u_2) = (f(a, \mathbf{y}), f(\infty, s(\mathbf{y}))) = (c, \infty)$$

for some $c \in \tilde{E}_2$.

- If $(w_1, w_2) = (0, 1)$ then

$$(u_1, u_2) = (f(0, \mathbf{y}), f(1, s(\mathbf{y}))) = (f((0, \mathbf{y})), f(s(0, \mathbf{y}))) \in \chi(f).$$

- If $(w_1, w_2) = (0, 0)$ then $(u_1, u_2) = (f(0, \mathbf{y}), f(0, s(\mathbf{y})))$. Since $f(0, \mathbf{y})$ and $f(0, s(\mathbf{y}))$ can be chosen independently the values (u_1, u_2) can be anything from \tilde{E}_2^2 unless $\chi(f) = \{(\infty, \infty)\}$.

The table for $R \star S$ can now be deduced from this. □

The implications in Lemma 8.3.2 are strict, which the following Lemma shows.

Lemma 8.3.3. *Let $C \in \mathcal{I}_2(S \cap T)$ with $U_R(a_0, a_1) \cup U_S(b_0, b_1) \subseteq C$.*

Then

- $U_{R \star S}(a_0, a_1) \subseteq C$ if $(b_0, b_1) = (0, 1)$;
- $U_{R \star S}(a_0, \infty) \subseteq C$ if $(b_0, b_1) = (0, \infty)$;
- $\bigcup_{c \in \tilde{E}_2} U_{R \star S}(c, \infty) \subseteq C$ if $(b_0, b_1) = (1, \infty)$;
- $U_{R \star S}(\infty, a_1) \subseteq C$ if $(b_0, b_1) = (\infty, 1)$;
- $\bigcup_{c \in \tilde{E}_2} U_{R \star S}(\infty, c) \subseteq C$ if $(b_0, b_1) = (\infty, 0)$;
- $U_{R \star S}(\infty, \infty) \subseteq C$ if $(b_0, b_1) = (\infty, \infty)$.

Proof. Note that $U_T(c_0, c_1) = \emptyset$ if $(c_0, c_1) \notin T$. This applies in the case that $R = (\infty)$ or $S = (\infty)$. Then $R \star S = (\infty)$ and therefore $\bigcup_{c \in \tilde{E}_2} U_{R \star S}(c, \infty) = U_{(\infty)}(\infty, \infty)$.

Furthermore if $R = (\infty)$ (i.e., we know $(a_0, a_1) = (\infty, \infty)$) or $S = (\infty)$ (i.e., we know $(b_0, b_1) = (\infty, \infty)$) then $R \star S = (\infty)$ and nothing is left to show since $U_{R \star S}(\infty, \infty) = U_R(\infty, \infty) \subseteq C$ or $U_{R \star S}(\infty, \infty) = U_S(\infty, \infty) \subseteq C$. Thus we can assume that $R \neq (\infty)$ and $S \neq (\infty)$ holds in the rest of the proof.

Let $f^{(3)} \in U_R(a_0, a_1)$, $g^{(3)} \in U_S(b_0, b_1)$ and $h_1^{(3)}, h_2^{(3)} \in S \cap T$ be defined by the following table. We can define $G^{(3)} := f(h_1, h_2, g)$. Let $G_{xyz} := G(x, y, z)$ for any $x, y, z \in E_2$.

x_1	x_2	x_3	$f(\mathbf{x})$	$h_1(\mathbf{x})$	$h_2(\mathbf{x})$	$g(\mathbf{x})$
0	0	0	a_0	0	0	b_0
1	1	1	a_1	1	1	b_1
0	0	1	c_0			
1	1	0	c_1			
0	1	0	α_0	0	1	v_0
1	0	1	α_1	1	0	v_1
1	0	0	β_0	1	0	w_0
0	1	1	β_1	0	1	w_1

The values for $(\alpha_0, \alpha_1), (\beta_0, \beta_1) \in R$ and $(v_0, v_1), (w_0, w_1) \in S$ are chosen according to the tables below.

R	S	α_0	α_1	β_0	β_1	S	v_0	v_1	w_0	w_1
$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	= $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	0	1	0	1	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	0	1	0	1
$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \infty$		0	1	∞	∞	$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \infty$	0	1	∞	∞
$\begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}$	$\neq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	0	1	0	1	$\begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}$	0	1	0	∞
$\begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}$		0	∞	0	∞	$\begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}$	0	∞	0	∞
\tilde{E}_2^2		0	0	0	0	\tilde{E}_2^2	0	0	0	0

The values of (c_0, c_1) depend on (b_0, b_1) .

- $(b_0, b_1) = (1, \infty)$ let $c \in \tilde{E}_2$ be arbitrary. Then we can set $c_0 := c$ since $R \neq (\infty)$. Thus $(G_{000}, G_{111}) = (c, \infty)$.
- $(b_0, b_1) = (\infty, 0)$ let $c \in \tilde{E}_2$ be arbitrary. Then we can set $c_1 := c$ since $R \neq (\infty)$. Thus $(G_{000}, G_{111}) = (\infty, c)$.
- Otherwise, the values (c_0, c_1) are irrelevant.

Then (G_{000}, G_{111}) coincides with the possibilities in the statement of this lemma.

Then the following table shows the values of the tuple $(G_{010}, G_{101}, G_{100}, G_{011})$ depending on R and S .

R	S	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \infty$	$\begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}$	\tilde{E}_2^2
$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	(0, 1, 0, 1)	(0, 1, ∞ , ∞)	(0, 1, 0, ∞)	(0, ∞ , 0, ∞)	(0, 0, 0, 0)
$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \infty$		(0, 1, ∞ , ∞)	(0, 1, ∞ , ∞)	(0, 1, 0, ∞)	(0, ∞ , 0, ∞)	(0, 0, 0, 0)
$\begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}$	$\neq \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	(0, 1, 0, ∞)	(0, 1, ∞ , ∞)	(0, 1, 0, ∞)	(0, ∞ , 0, ∞)	(0, 0, 0, 0)
$\begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}$		(0, ∞ , 0, ∞)	(0, ∞ , ∞ , ∞)	(0, ∞ , 0, ∞)	(0, ∞ , 0, ∞)	(0, 0, 0, 0)
\tilde{E}_2^2		(0, 0, 0, 0)	(0, 0, ∞ , ∞)	(0, 0, 0, ∞)	(0, ∞ , 0, ∞)	(0, 0, 0, 0)

Then we know that $\{(G_{010}, G_{101}), (G_{100}, G_{011})\} \subseteq \chi(G)$. By Lemmas 8.2.2 and 8.3.1 we can conclude that $U_T(G_{000}, G_{111}) \subseteq C$ holds with T determined by the following table.

(G_{010}, G_{101})	(G_{100}, G_{011})	T
(0, 1)	(0, 1)	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
(0, 1)	(∞ , ∞)	$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \infty$
(0, 1)	(0, ∞)	$\begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}$
(0, ∞)		$\begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}$
(0, 0)		\tilde{E}_2^2

This proves the statement. □

The computer program in Section B.1 in the Appendix implements the implications and generated the list in Section B.2 in the Appendix. This yields the following theorem.

Theorem 8.3.4. *There are 380 different partial clones in the interval $\mathcal{I}_2(S \cap T)$.*

Part of the interval $\mathcal{I}_2(S \cap T)$ is shown in Figure 8.2 where each node represents the union of all the nodes below it and the node name at that label. For example the node $U_{\left(\begin{smallmatrix} 0 & 1 \\ \infty & \infty \end{smallmatrix}\right)}(1, \infty)$ represents the partial clone

$$(S \cap T) \cup U_{\left(\begin{smallmatrix} 0 & 1 \\ \infty & \infty \end{smallmatrix}\right)}(1, \infty) \cup U_{\left(\begin{smallmatrix} 0 & 1 \\ \infty & \infty \end{smallmatrix}\right)}(0, \infty) \cup U_{\left(\begin{smallmatrix} 0 & 1 \\ \infty & \infty \end{smallmatrix}\right)}(\infty, \infty) \cup U_{\left(\begin{smallmatrix} \infty \\ \infty \end{smallmatrix}\right)}(\infty, \infty).$$

This result concludes the search of the finite intervals in \tilde{P}_2 . An interesting question persists, if the intervals $\mathcal{I}_2(C)$ with

$$C \in \{M \cap S\} \cup \{T_{a,\mu}, T_{a,\mu} \cap M, T_{a,\mu} \cap T, T_{a,\mu} \cap M \cap T \mid a \in \{0, 1\}, \mu \in \mathbb{N}, \mu \geq 2\}$$

are countably infinite or have the cardinality of the continuum.

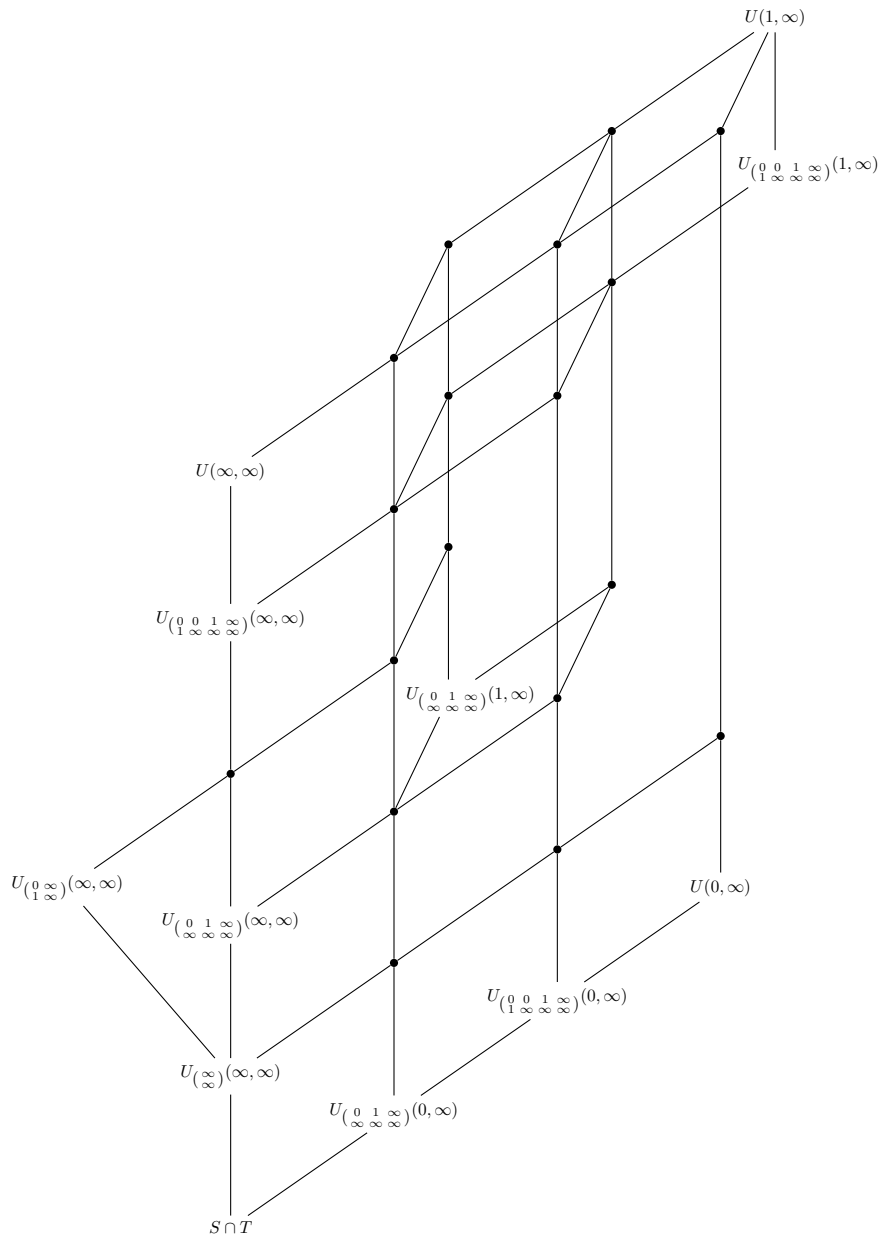


Figure 8.2: The clones C with $S \cap T \subseteq C \subseteq \bigcup_{c \in \tilde{E}_2} U(c, \infty)$

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Nomenclature

- E_k := $\{0, 1, \dots, k-1\}$, page 6
 \tilde{E}_k := $E_k \cup \{\infty\}$, page 6
 $f^{(n)}$ n -ary function f , page 6
 \tilde{P}_k all partial functions on E_k , page 6
 P_k all total functions on E_k , page 6
 $\text{dom } f$ domain of the function f , page 6
 $c_\infty^{(n)}$ n -ary partial function with empty domain, page 6
 C_∞ all partial functions with empty domain, page 6
 $(f^\infty)^{(n)}$ the function extended from $f^{(n)} : E_k^n \rightarrow \tilde{E}_k$ to $(f^\infty)^{(n)} : \tilde{E}_k^n \rightarrow \tilde{E}_k$, page 7
 $(\tilde{P}_k; \zeta, \tau, \Delta, \nabla, \star, e_1^{(2)})$ full partial function algebra, page 8
 $c_{i\star}$ i -th row of the matrix c , page 11
 $c_{\star j}$ j -th column of the matrix c , page 11
 $\hat{\mathcal{R}}_k$ set of all relations on \tilde{E}_k , page 11
 \mathcal{R}_k set of all relations on E_k , page 11
 $\text{pPol}_k \varrho$ set of partial functions which preserve ϱ , page 11
 $\text{pPol}_k Q$ set of partial functions which preserve all relations in Q , page 11
 $\text{pInv}_k f$ set of invariants of the function f , page 11
 $\text{pInv}_k A$ set of invariants of all functions in A , page 11
 $\text{Pol}_k Q$ set of total functions which preserve all relations in Q , page 12
 $\text{Inv}_k A$ set of invariants (on E_k) of all functions in A , page 12
 $\tilde{\mathcal{R}}_k$ set of all ∞ -strict relations on \tilde{E}_k , page 12
 $\tilde{\delta}_\varepsilon^{(h)}$ h -ary diagonal relation on \tilde{E}_k , page 12
 $\delta_\varepsilon^{(h)}$ h -ary diagonal relation on E_k , page 12
 $\zeta \varrho$ cyclical exchange of the rows, page 13
 $\tau \varrho$ exchange of the first two rows, page 13

- $\text{pr } \varrho$ projection onto the rows $2, \dots, h$, page 13
 $\varrho \times \sigma$ cartesian product of ϱ and σ , page 13
 $\varrho \wedge \sigma$ intersection of ϱ and σ , page 13
 $\hat{\kappa}\varrho$ primitive ∞ extension of ϱ , page 13
 $\mathcal{P}(A)$ powerset of A , page 14
 $I_\infty(\varrho)$ sets of ∞ -indices of tuples in ϱ , page 14
 $\varrho^{[\pi]}$ permutation of coordinates of ϱ by the permutation π , page 14
 pr_A projection on a set of variables, page 14
 Δ_I identification of coordinates, page 14
 ν_i doubling of coordinates, page 15
 ∇ adding of fictious coordinates, page 15
 unpr_J^h adding of fictious coordinates, page 15
 \circ_t general composition (relation product), page 15
 κ_I general ∞ -extension, page 15
 κ full ∞ -extension, page 15
 $\bar{\kappa}$ partial ∞ -extension, page 15
 κ_I^* general ∞ -subextension, page 15
 κ^* full ∞ -subextension, page 16
 $g \leq f$ g is a subfunction of f , page 16
 $\text{pPOL}_k \varrho = \text{pPol}_k(\kappa\varrho)$, page 17
 $(\tilde{\mathcal{R}}_k; \tilde{\delta}_{k; \{1,2\}}^{(3)}, \zeta, \tau, \text{pr}, \wedge, \times, \hat{\kappa})$ full partial relation algebra, page 17
 $(\mathcal{R}_k; \delta_{k; \{1,2\}}^{(3)}, \zeta, \tau, \text{pr}, \wedge, \times)$ full total relation algebra, page 17
 $\chi_{k;n}$ all- n -tuple relation on E_k , page 19
 $G_n(A)$ n -th graphic of $A \subseteq \tilde{P}_k$, page 19
 $\mathbb{L}(\tilde{P}_k)$ set of all partial clones of \tilde{P}_k , page 23
 $\mathbb{L}(\tilde{\mathcal{R}}_k)$ set of all partial co-clones of $\tilde{\mathcal{R}}_k$, page 23
 $\mathbb{L}(P_k)$ set of all clones of P_k , page 23
 $\mathbb{L}(\mathcal{R}_k)$ set of all co-clones of \mathcal{R}_k , page 23
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 $\lambda_{A \Rightarrow B}$ ∞ -forcing relation from A to B , page 27
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 Γ_σ symmetry group of the relation σ , page 43
 $M(\varrho)$ model of the relation ϱ , page 43
 $\widetilde{\mathcal{R}}_k^{\max}$ set of all coherent relations, page 43
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Appendix A

Coherent relations

A.1 Program to generate the list of coherent relations

```
{-# OPTIONS -fglasgow-exts #-}

module Main
where

import IO
import List
import Maybe
import Data.Set (Set)
import qualified Data.Set as Set
import System
import Monad (foldM, msum)
```

Permutations are written as lists of integers on the set E_k where the i -th entry determines where to map i . For example, $[0,2,3,1,5]$ symbolizes a permutation π on E_5 with $\pi = (1\ 2\ 3)$. The function `allperm` generates all permutations on the set E_h where `allperm'` is a helper function which works on any set of integers.

```
type Perm = [Int]
type SPerm = Set Perm

allperm' :: [Int] -> [Perm]
allperm' [x] = [[x]]
allperm' xs = concat [ map (\ys -> x:ys) (allperm' (xs \\< [x])) | x <- xs ]

allperm :: Int -> [Perm]
allperm h = allperm' $ [0..h-1]
```

In the program we need all subgroups of S_h at one point. To generate all these groups we need to multiply two permutations $a, b : E_h \rightarrow E_h$ and this can be done by `multperm`. If $c = \text{multperm } a \ b$ then c is a permutation with $c(x) = b(a(x))$ for all $x \in E_h$. Given two sets of permutations $p_a, p_b \subseteq S_h$ we can form the set of all binary products

$$p_{ab} := \text{prodperm } p_a \ p_b = \{\text{multperm } a \ b \mid a \in p_a, b \in p_b\}.$$

```
multperm :: Perm -> Perm -> Perm
multperm a b = map (b !!) a

prodperm :: SPerm -> SPerm -> SPerm
prodperm pa pb = Set.fromList . concat $ map (\b -> map (\a -> multperm a b) (Set.toList pa)) (Set.toList pb)
```

We generate a group $G = [A]$ from a set $A \subseteq S_h$ recursively. Let $A_0 := A$ and $A_{n+1} := A \cup (\text{prodperm } A \ A_n)$ for $n \geq 0$. Since $G \leq S_h$ is a finite group we get $A_{n+1} = A_n$ for some $n \in \mathbb{N}$ and this process terminates. The function `closure'` implements the recursion and `closureS` is the interface for sets of permutations and `closureL` is the interface for lists of permutations.

```
closure' :: SPerm -> SPerm -> SPerm
closure' p0 ps = if ps == px then ps else closure' p0 px
  where
    px = Set.union ps (prodperm p0 ps)

closureS :: SPerm -> SPerm
closureS ps = closure' ps ps

closureL :: [Perm] -> [Perm]
closureL = Set.toAscList . closureS . Set.fromList
```

The following code generates all subgroups of S_h . The algorithm is not very efficient but works. Starting with $P_0 := \{X_1, \dots, X_l\} := \{\{x\} \mid x \in S_h\}$ we generate $P_n := P_{n-1} \cup \{[X_n \cup X] \mid X \in P_{n-1}\}$ for $n \in \{1, \dots, l\}$.

```
allgroups' :: [SPerm] -> Set SPerm -> Set SPerm
allgroups' [] pss = pss
allgroups' (x:xs) pss = allgroups' xs $! Set.union pss pn
  where
    pn = Set.map (closureS . Set.union x) . Set.filter (not . Set.isSubsetOf x) $
      pss

allgroups :: Int -> Set SPerm
allgroups h = allgroups' ps pss
  where
    ps = nub . map (closureS . Set.singleton) $ allperm h
    pss = Set.fromList $! ps
```

The following functions deal with partitions representing the equivalence relations ε used in the description of coherent relations in the terms δ_ε . Since we can permute the rows of relations without affecting the corresponding clone we sort each partition such that the longest equivalence class comes first. The function `allinpart` generates all such partitions on E_h .

The functions `vecinpart` and `vecinmultipart` test if a given vector v belongs to δ_{pa} and $\bigcup_{pa \in pas} \delta_{pa}$, respectively.

```
lencomp :: [a] -> [a] -> Ordering
lencomp a b = compare (length a) (length b)

longest :: [ [a] ] -> [a]
longest = maximumBy lencomp

type Partition = [[Int]]

allinpart' :: [Int] -> [Partition]
allinpart' [i] = [ [[i]] ]
allinpart' (i:is) = concat [ [[i]:(c0:cs), (i:c0):cs] | (c0:cs) <- allinpart' is ]

allinpart :: Int -> [Partition]
allinpart h = sort
  . filter (\p -> (reverse . sortBy lencomp . reverse $ p) == p )
  . allinpart' $ [0..h-1]

vecinpart :: Partition -> [Int] -> Bool
vecinpart [] _ = False
```

```

vecinpart pa v = all (\(i1:is) -> and [ (v !! i1) == (v !! i2) | i2 <- is ] ) pa
vecinmultipart :: [Partition] -> [Int] -> Bool
vecinmultipart pas v = any (\pa -> vecinpart pa v) pas

```

The function `filtergroupprevpart gs pa` gets all groups `g` from `gs` such that all permutations $\pi \in g$ preserve the partition `pa`, i.e., $\delta_{pa}^{[\pi]} = \delta_{pa}$. Additionally `map Set.toAscList` enforces that the identity permutation $[0, 1, \dots, h-1]$ is the first element in the list presentation of a group.

```

applypermtopart :: Perm -> Partition -> Partition
applypermtopart pi pa = sort . map (sort . map (\x -> pi !! x)) $ pa

piprevpart :: Perm -> Partition -> Bool
piprevpart pi pa = (pa == (applypermtopart pi pa))

groupprevpart :: SPerm -> Partition -> Bool
groupprevpart g pa = all (\pi -> piprevpart pi pa) $ Set.toList g

filtergroupprevpart :: Set SPerm -> Partition -> [[Perm]]
filtergroupprevpart gs pa = sortBy lencomp . map Set.toAscList . Set.toList . Set.
  filter (\g -> groupprevpart g pa) $ gs

```

The function `applyperm` takes a permutation `p` and applies it to a vector `x`.

```

applyperm :: Perm -> [Int] -> [Int]
applyperm p x = map (x !!) p

applygroup :: [Perm] -> [Int] -> [ [Int] ]
applygroup g x = [ applyperm p x | p <- g ]

listmodgroup :: [Perm] -> [ [Int] ] -> [ [Int] ]
listmodgroup _ [] = []
listmodgroup _ [a] = [a]
listmodgroup g (a:as) = a : (listmodgroup g (as \ \ (applygroup g a)))

```

We need to have all tuples from $E_k^h \setminus \iota_k^h$ which are generated by `allareflvec`. The tuples in ι_k^h are handled differently as the Theorem of Haddad and Rosenberg suggests.

```

allareflvec' :: [Int] -> Int -> [ [Int] ]
allareflvec' 0 = [ [] ]
allareflvec' [] = [ [] ]
allareflvec' xs h = concat [ map (x:) (allareflvec' (xs \ \ [x]) (h-1)) | x <- xs ]

allareflvec :: Int -> Int -> [ [Int] ]
allareflvec k h = sort $ allareflvec' [0..k-1] h

```

The definition of coherent relations includes the use of relational homomorphisms from ϱ to $M(\varrho)$. Candidates for these relational homomorphisms are all surjective functions φ from E_k to E_h since there shall be an $v \in \varrho$ with $\varphi(v) = \eta_h$. The function `allsurjfunc` generates all of them.

```

allfunc' :: [Int] -> [Int] -> [ [Int] ]
allfunc' [] ys = [ [y] | y <- ys ]
allfunc' (_:xs) ys = [ y:r | y <- ys, r <- allfunc' xs ys ]

allsurjfunc' :: Int -> Int -> [ [Int] ]
allsurjfunc' k h = filter (\f -> h == length (nub f)) (allfunc' [0..k-1] [0..h-1])

transfunc :: Int -> [Int] -> (Int -> Int)
transfunc _ fi x = fi !! x

allsurjfunc :: Int -> Int -> [ Int -> Int ]

```

```
allsurjfunc k h = map (transfunc k) ( allsurjfunc ' k h )
```

The function `testhom` checks if a given function `f` maps the tuple `r` into the model of ϱ and additionally returns information if the tuple is not in ι_h^h . If the last bit is true and `f` is a relational homomorphism, then there is some `f'` with `f'(r) = \eta_h`. The relation $\varrho = \sigma \cup \delta(\varrho)$ is encoded in `gsigma` = G_σ and the function `patest` representing $\delta(\varrho)$. By the way we represent relations and permutations we have `gsigma` = $M(\varrho) \setminus \iota_k^h$. Additionally `patest` does not change when switching to the model of ϱ .

The function `ishomwitheta` then checks if `f` is a relational homomorphism such that some $r \in \varrho$ exists with `f(r) = \eta_h`. `existrelhom` tests all surjective functions if they are relational homomorphisms, and `iscoherent` does this test for all subsets of $\sigma(\varrho)$, i.e., checking if ϱ is coherent, since ϱ is constructed such that the condition on G_σ is fulfilled.

```
testhom :: [Perm] -> ([Int] -> Bool) -> (Int -> Int) -> [Int] -> (Bool, Maybe [Int])
testhom gsigma patest f r = case ( fr 'elem' msigma, patest fr ) of
  (True, _) -> (True, Just r)
  (_, True) -> (True, Nothing)
  (_, _) -> (False, Nothing)
  where
    fr = map f r
    msigma = gsigma -- model of sigma equals G_sigma because of representation of gsigma

ishomwitheta :: [ [Int] ] -> [Perm] -> ([Int] -> Bool) -> (Int -> Int) -> Maybe [ [Int] ]
ishomwitheta sigma gsigma patest f = case (and inMrho, catMaybes mappedtoeta) of
  (True, []) -> Nothing
  (True, ls) -> Just ls
  (False, _) -> Nothing
  where
    (inMrho, mappedtoeta) = unzip . map (testhom gsigma patest f) $ sigma

existrelhom :: [ Int -> Int ] -> [ [Int] ] -> ([Int] -> Bool) -> [Perm] -> Maybe ([ [Int] ])
existrelhom ksurjh sigma patest gsigma = msum . map (ishomwitheta sigma gsigma patest) $ ksurjh

iscoherent :: [ Int -> Int ] -> [ [Int] ] -> ([Int] -> Bool) -> [Perm] -> Bool
iscoherent _ [] _ _ = True
iscoherent _ [_] _ _ = True
iscoherent ksurjh sigma patest g = value mrs
  where
    mrs = existrelhom ksurjh sigma patest g -- vectors mapped into Model(sigma)
    value :: Maybe ([ [Int] ]) -> Bool
    value Nothing = False
    value (Just rs) = iscoherent ksurjh (sigma \\ rs) patest g
```

The type `RelInfo` stores all kinds of information and is used and updated in the recursive search. `rel` stores only the tuples such that

$$\text{fullrel} = \sigma(\varrho) = \bigcup_{\pi \in \text{gsigma}} \text{rel}^{\pi}$$

with `gsigma` = G_σ . This is used in the function `iscoherent` to reduce the number of checks at that point.

The entry `pa` takes the partition corresponding to the equivalence relation ε for areflexive and quasi-diagonal relations. The function `patest` is the test used in `iscoherent` and `parep` is the entry which will be displayed in the list of relations representing $\delta(\varrho)$.

The function `isgood` is used to decide if the given relation is non-trivial, i.e., check that $\text{pPOL}_k \varrho \neq \tilde{P}_k$.

The entries `ksurjh`, `allbij`, `k` and `h` are parameters of the relation and added to this record to avoid recalculation of them at every step, i.e., reducing computations.

Let G' be the set of permutations that preserve $\delta(\varrho)$. Then $\text{gsigma} \leq G'$ and there are cosets of gsigma in G' . The list `coreps` contains the coset representatives of the left cosets of gsigma in G' ; see also `cosetreprs`. It is useful to use `coreps` instead of G' since $\varrho^{[\pi']} = \varrho^{[\pi]}$ if $\pi' \in \pi \text{gsigma}$, i.e., if π' and π are in the same coset of gsigma .

Let `allbij` = $[f_1, \dots, f_l]$ and `coreps` = $[\pi_1, \dots, \pi_p]$. The entry `xclass` is a list of lists of relations, where

- `xclass` = $[L_1, \dots, L_l]$, indexed by the functions in `allbij`;
- $L_i = [r_{i1}, \dots, r_{ip}]$, where $i \in \{1, \dots, l\}$;
- $r_{ij} = \{f_i(v) \mid v \in \text{fullrel}^{[\pi_j]}\}$, i.e., the relation generated from `fullrel` = $\sigma(\varrho)$ by applying the permutation $\pi_j \in \text{coreps}$ and then mapping by the bijection $f_i \in \text{allbij}$.

Let $r, r' \in L_i$ for some $i \in \{1, \dots, l\}$. Then $\text{pPOL}_k r = \text{pPOL}_k r'$ since we can write

$$r' = f_i(\text{fullrel}^{[\pi']}) = f_i(\text{fullrel})^{[\pi']} = f_i(\text{fullrel})^{[\pi\pi^{-1}\pi']} = r^{[\pi^{-1}\pi']}.$$

Thus every L_i describes exactly one maximal partial clone. Since the first element in `coreps` is the identity permutation (see `filtergrouppreupart`) the first element in L_i is $f_i(\text{fullrel})$. Therefore `quasiclass` generates the quasi-relation-class `qclass`(ϱ). The relation-class `relclass` yields a list of the minimal representatives for a given maximal clone. These coincide if and only if the corresponding maximal clones are equal. Thus we can count the maximal partial clones with a given minimal relation by counting the different relations in `relclass`.

```

data RelInfo a = RelInfo {
  rel      :: [[Int]],
  fullrel  :: [[Int]],
  gsigma   :: [Perm],
  pa       :: [[Int]],
  patest   :: [Int] -> Bool,
  parep    :: [a],
  coreps   :: [Perm],
  xclass   :: [[ [Int] ]],
  ksurjh   :: [ Int -> Int ],
  allbij   :: [ Int -> Int ],
  k        :: Int,
  h        :: Int,
  isgood   :: RelInfo a -> Bool
}

quasiclass :: RelInfo a -> [ [[Int]] ]
quasiclass = map head . xclass

relclass   :: RelInfo a -> [ [[Int]] ]
relclass   = map minimum . xclass

checkQuasiMin :: RelInfo a -> Bool
checkQuasiMin ri = all ( >= (fullrel ri) ) (quasiclass ri)

checkMin     :: RelInfo a -> Bool
checkMin     ri = all ( >= (fullrel ri) ) (relclass ri)

```

The function `preInitRelInfo` initializes a `RelInfo` structure for an empty relation with given `gsigma` and

```

cosetreprs :: [Perm] -> [Perm] -> [Perm]
cosetreprs g subg = nub . map (\pi -> minimum . map (multperm pi) $ subg) $ g

```

```

preInitRelInfo :: Int -> Int -> [ Int -> Int ] -> [ Int -> Int ] -> (b,[Perm],[Perm]
,[ [Int] ]) -> RelInfo a
preInitRelInfo k' h' allbij' allksurjh' (_,gofpa',g',_) = RelInfo {
  rel = [], fullrel = [],
  gsigma = g',
  coreps = coreps',
  xclass = [ [ [] | pi <- coreps' ] | f <- allbij' ],
  ksurjh = allksurjh', allbij = allbij', k = k', h = h',
  pa = [], patest = \_ -> True, parep = [], isgood = \_ -> True
}
where
  coreps' = cosetreps gofpa' g'

```

The representative of the relation class and the size of the relation class are used for the output of coherent relations with `computeRelRep` and `printGetSize`. `printGetSize` only prints output and returns non-zero class size, if the relation is really a coherent relation (i.e., `isgood` is true) and the relation is a minimal relation. The check `iscoherent` done before, does not check these criteria.

The output consists of a tuple (`length,parep,rel`) where

- `length` is the size of relation class;
- `parep` is
 - `[]` if the relation is areflexive,
 - the partition `pa` as a list representing ε if the relation $\rho = \sigma \cup \delta_\varepsilon$ is quasi-diagonal,
 - `iota_k~h` if the relation is totally symmetric and totally reflexive,
 - `rho_1` or `rho_2` if $\rho_1 \subseteq \rho$ or $\rho_2 \subseteq \rho$ for the relation ρ ;
- `rel` are the tuples of $\sigma(\rho)$ for the relation class representative ρ .

The function `checkCoherent` lifts the `iscoherent` function to the `RelInfo` type.

```

checkCoherent :: RelInfo a -> Bool
checkCoherent ri = iscoherent (ksurjh ri) (rel ri) (patest ri) (gsigma ri)

computeRelRep :: Show a => RelInfo a -> (Int, [a], [[Int]])
computeRelRep ri = (length rc, parep ri, head rc)
where
  rc = map head . group . sort . relclass $ ri

printGetSize :: Show a => RelInfo a -> IO Integer
printGetSize ri =
  if isgood ri ri &&& checkMin ri
  then (print $! relRep) >> return sizeRep
  else return 0
where
  relRep = computeRelRep ri
  sizeRep = toInteger $ (\(s,_,_) -> s) relRep

```

The function `insertNew` inserts an element `v` into an ordered list if it is not included yet.

The function `relInfoAddVec v ri` updates the structure `ri` by adding `v` to `rel`, `fullrel` and `xclass`. The term $gv = \{v^{[\pi]} \mid \pi \in g\sigma\}$ is used for `fr` and `xclass`. The new `xclass` is generated from the old one by replacing r_{ij} by $r_{ij} \cup \{w^{[\pi]} \mid \pi \in g\sigma\}$ where $w := f_i(v)^{[\pi_j]}$.

```

insertNew :: Ord a => a -> [a] -> [a]
insertNew v [] = [v]
insertNew v (r:rs) = case compare v r of
  LT -> v:r:rs
  EQ -> r:rs

```

```

GT -> r:(insertNew v rs)

relInfoAddVec :: [Int] -> RelInfo a -> RelInfo a
relInfoAddVec v ri = ri { rel = rn, fullrel = fr, xclass = xc }
  where
    rn = insertNew v (rel ri)
    fr = foldr insertNew (fullrel ri) gv
    xc = map yc $ zip (xclass ri) (allbij ri)
    yc (c,f) = map (\(d,pi) -> foldr (insertNew . map f . applyperm pi) d gv) $ zip
              c (coreps ri)
    gv = map (\pi -> applyperm pi v) (gsigma ri)

```

`coherentSubsetsWith` is the central function traversing the tree of coherent relations for given initialisation with the root at $\sigma(\varrho) = \emptyset$. If the current relation is not quasi-minimal then the rest of this subtree can be omitted. Thus the function returns 0. The sub-nodes are created in `aav'` and then checked to be coherent. If the relation is not coherent for some `v` then we can not add `v` anywhere in this subtree. Therefore `aavn` are the tuples from `aav` which are allowed in the subtrees. Since we only want to generate a given relation once we allow in the subtree to `v` only tuples greater than `v`, see `aavngtv` in `recurse`.

The `foldM` line traverses all pairs (v, rj) and sums up the number of maximal partial clones.

```

coherentSubsetsWith :: Show a => RelInfo a -> [ [Int] ] -> IO Integer
coherentSubsetsWith ri aav =
  if not (checkQuasiMin ri)
  then return 0
  else do
    size <- printGetSize ri
    let aav' = map (\v -> (v, relInfoAddVec v ri)) (aav \\ (fullrel ri))
        vrjs = filter (\(_,rj) -> checkCoherent rj) $ aav'
        aavn = map fst vrjs
    foldM (recurse aavn) size vrjs

recurse :: Show a => [[Int]] -> Integer -> ([Int], RelInfo a) -> IO Integer
recurse aavn s (v,rj) = coherentSubsetsWith rj aavngtv >>= strictAddToS
  where
    strictAddToS x = return $! (s + x)
    aavngtv = filter (>v) aavn

```

The function `isEkh` is used for areflexive and quasi-diagonal relations to check if $\varrho = E_k^h$. There are the following cases

- $h = 1$; then $\varrho = E_k^h$ if $|\sigma(\varrho)| = k$;
- $h = 2$; then $\varrho = E_k^h$ if $\delta(\varrho) = \delta_{\{0,1\}}^{(2)} = \iota_k^2$ and $|\sigma(\varrho)| = \binom{k}{2} = k \cdot (k - 1)$;
- for every $h > 2$ we have $\varrho \neq E_k^h$.

A relation is a valid candidate for a maximal partial clone if $\varrho \neq E_k^h$ and $\sigma(\varrho) \neq \emptyset$ (see `isgood` definition in `findcoherent`).

`findcoherent` takes a tuple `i` consisting of the partition `pa'`, the group preserving `pa'`, the group `gsigma'` and a list of tuples `av`. The tuples in `av` represent all tuples in $\sigma(E_k^h)$, but reduced with respect to `gsigma'`. Let $\{V_1, \dots, V_n\} := \{\{v^{[\pi]} \mid \pi \in \text{gsigma}'\} \mid v \in \sigma(E_k^h)\}$. Then `av` = $\{v_1, \dots, v_n\}$ where $v_i \in V_i$ for all $i \in \{1, \dots, n\}$. This is done in `getpgvs` or more specifically in `listmodgroup`.

`getpgvs` takes all groups preserving `pa` (denoted by `ga0`) and emits all the tuples for `findcoherent`. `printcoherent` just sums the number of different maximal partial clones over all different partitions for a given h and returns this sum.

```

isEkh :: Int -> Int -> Partition -> [ [Int] ] -> Bool
isEkh k 1 _ rel = length (rel) == k
isEkh k 2 pa rel = pa == [[0,1]] && length (rel) == k*(k-1)
isEkh _ h _ _ | h > 2 = False

findcoherent :: Int -> Int -> [ Int -> Int ] -> [ Int -> Int ] -> (Partition,[Perm]
],[Perm],[ [Int] ]) -> IO Integer
findcoherent k' h' allbij' allksurjh' i@(pa',gofpa',gsigma',av) = do
  let ri0 = (preInitRelInfo k' h' allbij' allksurjh' i) {
        pa = pa', patest = vecinpart pa', parep = pa',
        isgood = \ri -> not (isEkh (k ri) (h ri) (pa ri) (fullrel ri)) && fullrel
        ri /= []
      }
    coherentSubsetsWith ri0 av >>= (\x -> return $! x)

getpgvs :: Set SPerm -> [[Int]] -> Partition -> [ (Partition,[Perm],[Perm],[ [Int]
]) ]
getpgvs gs aav pa = map (\g -> (pa,longest ga0,g,listmodgroup g aav)) ga0
  where
    ga0 = filtergroupprevpart gs pa

printcoherent :: Int -> Int -> IO (Integer)
printcoherent k h = do
  let pts = [ ] : ((allinpart h) \ \ [[x] | x <- [0..h-1]])
        allbij = allsurjfunc k k
        allksurjh = allsurjfunc k h
        gs = allgroups h
        aav = allareflvec k h
        pgvs = concatMap (getpgvs gs aav) pts
    mapM (findcoherent k h allbij allksurjh) pgvs >>= (return . sum)

```

The totally reflexive, totally symmetric relation ϱ is not E_k^h if $|\sigma(\varrho)| \neq |\sigma(E_k^h)| = k \cdot k - 1 \cdot \dots \cdot k - h + 1$, the number of h -tuples with pairwise different entries. Thus `isgood` is defined that way. Any tuple not in $\sigma(E_k^h)$ is in ν_k^h and thus in ϱ . Therefore `patest` is always true.

By the Theorem of Haddad and Rosenberg `gsigma` = `gofpa` = S_h .

```

prod :: [Int] -> Int
prod xs = foldl (*) 1 xs

findIota :: Int -> Int -> [ Int -> Int ] -> [ Int -> Int ] -> (a,[Perm],[Perm],[ [
Int ] ]) -> IO Integer
findIota k' h' allbij' allksurjh' i@(_,_,_,av) = do
  let ri0 = (preInitRelInfo k' h' allbij' allksurjh' i) {
        pa = [], patest = (\_ -> True),
        parep = "iota_" ++ (show k') ++ "^" ++ (show h'),
        isgood = \ri -> length (fullrel ri) /= prod [k'-h'+1..k']
      }
    coherentSubsetsWith ri0 av >>= (\x -> return $! x)

printIotakh :: Int -> Int -> IO (Integer)
printIotakh k h = do
  let g = allperm h
        aav = allareflvec k h
        pg0 = ([],g,g,listmodgroup g aav)
        allbij = allsurjfunc k k
        allksurjh = allsurjfunc k h
    findIota k h allbij allksurjh pg0

```

The quaternary relations with $\varrho_i \subseteq \varrho$ for some $i \in \{1, 2\}$ are generated next where `rhoiparam` gives the group G_σ , the set of partitions of ϱ_i , and the output string. Since any such relation (which is generated in this program) is a valid coherent relation, we have `isgood` always be true.

```

rhoiparam :: Int -> ([Perm],[Partition],String)
rhoiparam 1 = (closureL [[2,0,3,1],[3,1,2,0]], [[ [0,1],[2,3] ], [ [0,2],[1,3] ] ],
  "rho_1")
rhoiparam 2 = (allperm 4
  [[ [0,1],[2,3] ], [ [0,2],[1,3] ], [ [0,3],[1,2] ] ], "rho_2")

```

```

printRhoi :: Int -> Int -> IO (Integer)
printRhoi k i = do
  let h = 4
      (g, pas, delta) = rhoiparam i
      aav = allareflvec k h
      pg0 = (pas, g, g, listmodgroup g aav)
      allbij = allsurjfunc k k
      allksurjh = allsurjfunc k h
      findRhoi k h allbij allksurjh pg0 delta

findRhoi :: Int -> Int -> [ Int -> Int ] -> [ Int -> Int ] -> ([Partition],[Perm],[
  Perm],[ [Int] ]) -> String -> IO Integer
findRhoi k' h' allbij' allksurjh' i@(pas',_,_,av) parep' = do
  let ri0 = (preInitRelInfo k' h' allbij' allksurjh' i) {
      pa = [], patest = vecinmultipart pas', parep = parep',
      isgood = \_ -> True
    }
      coherentSubsetsWith ri0 av >>= (\x -> return $! x)

```

Print the list of coherent relations in the following order:

- the reflexive and quasi-diagonal relations,
- the totally symmetric, totally reflexive relations with $h \geq 3$,
- the quaternary relations with $\varrho_i \subseteq \varrho$ for some $i \in \{1, 2\}$,
- $P_k \cup C_\infty$.

In the end the numbers are printed grouped as above, and a grand total.

The k for which to generate the list of coherent relations is given as a command line parameter.

```

runPrint :: Int -> IO ()
runPrint k = do
  sizes <- mapM (printcoherent k) [1..k]
  sizes2 <- mapM (printlotakh k) [3..k]
  sizes3 <- mapM (printRhoi k) [1,2]
  putStrLn ("(1,P_ " ++ (show k) ++ " \cup C_\infty)")
  sizes4 <- return [1] -- P_k \cup C_\infty
  print (sizes, sizes2, sizes3, sizes4, sum (sizes ++ sizes2 ++ sizes3 ++ sizes4))

main = do
  x <- getArgs
  let k :: Int = read (head x)
      runPrint k
  return 0

```

A.2 Lists of coherent relations for $k \in \{2, 3\}$

```
(2, [], [[0]])
(1, [], [[0, 1]])
(1, [], [[0, 1], [1, 0]])
(1, [[0, 1]], [[0, 1]])
(1, "rho_1", [])
(1, "rho_2", [])
(1, P_2 \cup C_\infty)
([2, 3], [], [1, 1], [1], 8)
```

Listing A.1: Coherent relations for $k = 2$

```
(3, [], [[0]])
(3, [], [[0], [1]])
(3, [], [[0, 1]])
(3, [], [[0, 1], [0, 2]])
(3, [], [[0, 1], [1, 0]])
(3, [], [[0, 1], [0, 2], [1, 0], [2, 0]])
(3, [[0, 1]], [[0, 1]])
(3, [[0, 1]], [[0, 1], [0, 2]])
(3, [[0, 1]], [[0, 1], [0, 2], [1, 2]])
(3, [[0, 1]], [[0, 1], [1, 2]])
(3, [[0, 1]], [[0, 1], [1, 0]])
(3, [[0, 1]], [[0, 1], [0, 2], [1, 0], [2, 0]])
(1, [], [[0, 1, 2]])
(3, [], [[0, 1, 2], [0, 2, 1]])
(1, [], [[0, 1, 2], [1, 2, 0], [2, 0, 1]])
(1, [], [[0, 1, 2], [0, 2, 1], [1, 0, 2], [1, 2, 0], [2, 0, 1], [2, 1, 0]])
(3, [[0, 1], [2]], [[0, 1, 2]])
(3, [[0, 1], [2]], [[0, 1, 2], [1, 0, 2]])
(1, [[0, 1, 2]], [[0, 1, 2]])
(3, [[0, 1, 2]], [[0, 1, 2], [0, 2, 1]])
(1, [[0, 1, 2]], [[0, 1, 2], [1, 2, 0], [2, 0, 1]])
(1, [[0, 1, 2]], [[0, 1, 2], [0, 2, 1], [1, 0, 2], [1, 2, 0], [2, 0, 1], [2, 1, 0]])
(1, "iota_3^3", [])
(1, "rho_1", [])
(1, "rho_2", [])
(1, P_3 \cup C_\infty)
([6, 30, 18], [1], [1, 1], [1], 58)
```

Listing A.2: Coherent relations for $k = 3$

Appendix B

The interval $\mathcal{I}_2(S \cap T)$

B.1 Program to generate the interval $\mathcal{I}_2(S \cap T)$

```
import IO
import Data.List ((\), intersect, elemIndex, findIndex, inits, union)
import Data.Maybe (fromJust)
```

The type `C` represents the constants in $\tilde{E}_2 = \{0, 1, \infty\}$ and the type `Chi` represents the sets

$$(0\ 1), \dots, \begin{pmatrix} 0 & 0 \\ 1 & \infty \end{pmatrix}, E_2^2.$$

Finally the type `F a b chi` represents $U_{\text{chi}}(a, b)$.

```
data C = C0 | C1 | CE deriving Eq
data Chi = F01 | FEE | F01EE | F0E1EEE | F010E1EEE | U deriving Eq
data F = F C C Chi deriving Eq

instance Show C where
  show C0 = "0"
  show C1 = "1"
  show CE = "\\notdef"

instance Show Chi where
  show F01      = "\\ROI"
  show FEE      = "\\REE"
  show F01EE    = "\\ROIEE"
  show F0E1EEE  = "\\ROE"
  show F010E1EEE = "\\ROIOE"
  show U        = "\\pE_2^2"

instance Show F where
  show (F C0 C1 F01) = "S \\cap T"
  show (F a0 a1 U) = "U(" ++ show a ++ ", " ++ show b ++ ")"
  show (F a b r) = "U_{ " ++ (show r) ++ " } (" ++ show a ++ ", " ++ show b ++ ")"
```

`inlist` lists all possible combinations for the sets $U_R(a, b)$.

```
inlist =
  [ F C0 C1 r | r <- [F01, F01EE, F010E1EEE] ] ++
  [ F a0 a1 r | (a0, a1) <- [(C0, CE), (C1, CE), (CE, C1), (CE, C0)], r <- [F0E1EEE,
    F010E1EEE, U] ] ++
  [ F CE CE r | r <- [FEE, F01EE, F0E1EEE, F010E1EEE, U] ]
```



```

"& \\multicolumn{3}{|c|}{\\$U(1,\\notdef)$}" ++
"& \\multicolumn{3}{|c|}{\\$U(\\notdef,1)$}" ++
"& \\multicolumn{3}{|c|}{\\$U(\\notdef,0)$}" ++
"& \\multicolumn{5}{|c|}{\\$U(\\notdef,\\notdef)$}" ++
"\\\\\\") >>
putStrLn (
" & $R_1$ & $R_2$ & $R_3$ " ++
" & $R_4$ & $R_3$ & $\\pE_2^2$ " ++
" & $R_4$ & $R_3$ & $\\pE_2^2$ " ++
" & $R_4$ & $R_3$ & $\\pE_2^2$ " ++
" & $R_4$ & $R_3$ & $\\pE_2^2$ " ++
" & $R_5$ & $R_2$ & $R_4$ & $R_3$ & $\\pE_2^2$ " ++
"\\\\\\") >>
putStrLn "\\hline \\endhead" >>
putStrLn (
"\\hline \\multicolumn{21}{|r|}{{" ++
"$R_1 = \\ROIS$, $R_2 = \\ROIEES$, $R_3 = \\ROIOES$, $R_4 = \\ROES$, $R_5 = \\REES$"}
"++
"}} \\\\") >>
putStrLn "\\multicolumn{21}{|r|}{\\Continued on next page}} \\\\ \\hline" >>
putStrLn "\\endfoot" >>
putStrLn (
"\\hline \\multicolumn{21}{|r|}{{" ++
"$R_1 = \\ROIS$, $R_2 = \\ROIEES$, $R_3 = \\ROIOES$, $R_4 = \\ROES$, $R_5 = \\REES$"}
"++
"}} \\\\") >>
putStrLn "\\hline" >>
putStrLn "\\endlastfoot"

printTableFoot :: IO ()
printTableFoot =
  putStrLn "\\end{longtable}" >>
  putStrLn "\\end{center}"

removeSubsets :: [F] -> [F]
removeSubsets s = s \\ (concatMap subFs s)

showElemsTic :: (Int,[F]) -> String
showElemsTic (i,x) = show i ++ ft ++ "\\\\\\")
  where
    xn = removeSubsets x
    ft = concatMap ft' inlist
    ft' e | e 'elem' xn = " & $\\times$"
          | e 'elem' x  = " & $\\cdot$"
          | True       = " & "

printTable :: [[F]] -> IO ()
printTable gs = do
  printTableHead
  mapM (putStrLn . showElemsTic) $ zip [1..] gs
  printTableFoot

```

Generic functions working on lists. `subsets` generates all sublists of a given list. Since the input list is repetition-free we call `subsets`.

```

isSubsetOf :: Eq a => [a] -> [a] -> Bool
isSubsetOf u l = all (\x -> x 'elem' l) u

subsets :: [a] -> [[a]]
subsets [] = [[]]
subsets (first : rest) =
  let partial = subsets rest
  in map (first:) partial ++ partial

```

Take all subsets of `inlist`, check if they are

- closed with respect to subsets and include $F \ C0 \ C1 \ F01 = S \cap T$, and
- closed with respect to \star .

```
goodsets :: [[F]]
goodsets = filter isstarclosed . filter subsetlclosed . subsets $ inlist

main = do
  let gs = goodsets
  printTable gs
```

B.2 List of elements of $\mathcal{I}_2(S \cap T)$

The following list includes all partial clones in the interval $\mathcal{I}_2(S \cap T)$. The crosses \times determine a partial completely by the union of the sets thus marked. A dot \cdot marks a set which is included because it is a subset of some set marked with a cross.

For example let C be the partial clone with the number 38. Then

$$C = U_{R_3}(0, 1) \cup U_{R_3}(0, \infty) \cup U_{R_4}(1, \infty) \cup U_{R_4}(\infty, \infty)$$

holds and $U_{R_1}(0, 1) = S \cap T$, $U_{R_2}(0, 1)$, $U_{R_4}(0, \infty)$ and $U_{R_5}(\infty, \infty)$ are all subsets of C .

Nr.	$U(0, 1)$ $R_1 R_2 R_3$	$U(0, \infty)$ $R_4 R_3 \tilde{E}_2^2$	$U(1, \infty)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty, 1)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty, 0)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty, \infty)$ $R_5 R_2 R_4 R_3 \tilde{E}_2^2$
1	$\cdot \cdot \times$	$\cdot \cdot \times$	$\cdot \cdot \times$	$\cdot \cdot \times$	$\cdot \cdot \times$	$\cdot \cdot \cdot \cdot \times$
2	$\cdot \cdot \times$	$\cdot \cdot \times$	$\cdot \cdot \times$	$\cdot \cdot \times$		$\cdot \cdot \cdot \cdot \times$
3	$\cdot \cdot \times$	$\cdot \cdot \times$	$\cdot \cdot \times$	$\cdot \times$	\times	$\cdot \cdot \cdot \cdot \times$
4	$\cdot \cdot \times$	$\cdot \cdot \times$	$\cdot \cdot \times$	$\cdot \times$		$\cdot \cdot \cdot \cdot \times$
5	$\cdot \cdot \times$	$\cdot \cdot \times$	$\cdot \cdot \times$	\times	\times	$\cdot \cdot \cdot \cdot \times$
6	$\cdot \cdot \times$	$\cdot \cdot \times$	$\cdot \cdot \times$	\times		$\cdot \cdot \cdot \cdot \times$
7	$\cdot \cdot \times$	$\cdot \cdot \times$	$\cdot \cdot \times$			$\cdot \cdot \cdot \cdot \times$
8	$\cdot \cdot \times$	$\cdot \cdot \times$		$\cdot \cdot \times$	$\cdot \cdot \times$	$\cdot \cdot \cdot \cdot \times$
9	$\cdot \cdot \times$	$\cdot \cdot \times$		$\cdot \cdot \times$		$\cdot \cdot \cdot \cdot \times$
10	$\cdot \cdot \times$	$\cdot \cdot \times$		$\cdot \times$	\times	$\cdot \cdot \cdot \cdot \times$
11	$\cdot \cdot \times$	$\cdot \cdot \times$		$\cdot \times$		$\cdot \cdot \cdot \cdot \times$
12	$\cdot \cdot \times$	$\cdot \cdot \times$		\times	\times	$\cdot \cdot \cdot \cdot \times$
13	$\cdot \cdot \times$	$\cdot \cdot \times$		\times		$\cdot \cdot \cdot \cdot \times$
14	$\cdot \cdot \times$	$\cdot \cdot \times$				$\cdot \cdot \cdot \cdot \times$
15	$\cdot \cdot \times$	$\cdot \cdot \times$				\times
16	$\cdot \cdot \times$	$\cdot \cdot \times$				
17	$\cdot \cdot \times$	$\cdot \times$	$\cdot \times$	$\cdot \times$	$\cdot \times$	$\cdot \cdot \cdot \cdot \times$
18	$\cdot \cdot \times$	$\cdot \times$	$\cdot \times$	$\cdot \times$	\times	$\cdot \cdot \cdot \cdot \times$
19	$\cdot \cdot \times$	$\cdot \times$	$\cdot \times$	$\cdot \times$		$\cdot \cdot \cdot \cdot \times$
20	$\cdot \cdot \times$	$\cdot \times$	$\cdot \times$	\times	\times	$\cdot \cdot \cdot \cdot \times$
21	$\cdot \cdot \times$	$\cdot \times$	$\cdot \times$	\times		$\cdot \cdot \cdot \cdot \times$
22	$\cdot \cdot \times$	$\cdot \times$	$\cdot \times$			$\cdot \cdot \cdot \cdot \times$
23	$\cdot \cdot \times$	$\cdot \times$	\times	$\cdot \cdot \times$	$\cdot \cdot \times$	$\cdot \cdot \cdot \cdot \times$
24	$\cdot \cdot \times$	$\cdot \times$	\times	$\cdot \cdot \times$		$\cdot \cdot \cdot \cdot \times$
25	$\cdot \cdot \times$	$\cdot \times$	\times	$\cdot \times$	$\cdot \times$	$\cdot \cdot \cdot \cdot \times$
26	$\cdot \cdot \times$	$\cdot \times$	\times	$\cdot \times$	\times	$\cdot \cdot \cdot \cdot \times$
27	$\cdot \cdot \times$	$\cdot \times$	\times	$\cdot \times$	\times	$\cdot \cdot \cdot \cdot \times$
28	$\cdot \cdot \times$	$\cdot \times$	\times	$\cdot \times$		$\cdot \cdot \cdot \cdot \times$
29	$\cdot \cdot \times$	$\cdot \times$	\times	$\cdot \times$		$\cdot \cdot \cdot \cdot \times$
30	$\cdot \cdot \times$	$\cdot \times$	\times	\times	\times	$\cdot \cdot \cdot \cdot \times$
31	$\cdot \cdot \times$	$\cdot \times$	\times	\times	\times	$\cdot \cdot \cdot \cdot \times$
32	$\cdot \cdot \times$	$\cdot \times$	\times	\times	\times	$\cdot \cdot \cdot \cdot \times$
33	$\cdot \cdot \times$	$\cdot \times$	\times	\times		$\cdot \cdot \cdot \cdot \times$
34	$\cdot \cdot \times$	$\cdot \times$	\times	\times		$\cdot \cdot \cdot \cdot \times$
35	$\cdot \cdot \times$	$\cdot \times$	\times	\times		$\cdot \cdot \cdot \cdot \times$
36	$\cdot \cdot \times$	$\cdot \times$	\times			$\cdot \cdot \cdot \cdot \times$
37	$\cdot \cdot \times$	$\cdot \times$	\times			$\cdot \cdot \cdot \cdot \times$
38	$\cdot \cdot \times$	$\cdot \times$	\times			$\cdot \cdot \cdot \cdot \times$

$$R_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & \infty \\ 1 & \infty \end{pmatrix}, R_3 = \begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}, R_4 = \begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}, R_5 = \begin{pmatrix} \infty \\ \infty \end{pmatrix}$$

Continued on next page

Nr.	$U(0,1)$ $R_1 R_2 R_3$	$U(0,\infty)$ $R_4 R_3 \tilde{E}_2^2$	$U(1,\infty)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,1)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,0)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,\infty)$ $R_5 R_2 R_4 R_3 \tilde{E}_2^2$
39	. . x	. x		. . x	. . x x
40	. . x	. x		. . x	 x
41	. . x	. x		. x	. x x
42	. . x	. x		. x	x x
43	. . x	. x		. x	x x
44	. . x	. x		. x	 x
45	. . x	. x		. x	 x
46	. . x	. x		x	x x
47	. . x	. x		x	x x
48	. . x	. x		x	x	. x
49	. . x	. x		x	 x
50	. . x	. x		x	 x
51	. . x	. x		x		. x
52	. . x	. x			 x
53	. . x	. x			 x
54	. . x	. x				. x
55	. . x	. x				x
56	. . x	. x				
57	. . x	x	x	. . x	. . x x
58	. . x	x	x	. . x	 x
59	. . x	x	x	. x	. x x
60	. . x	x	x	. x	x x
61	. . x	x	x	. x	x x
62	. . x	x	x	. x	x	. x
63	. . x	x	x	. x	 x
64	. . x	x	x	. x	 x
65	. . x	x	x	. x		. x
66	. . x	x	x	x	x x
67	. . x	x	x	x	x x
68	. . x	x	x	x	x	. x
69	. . x	x	x	x	 x
70	. . x	x	x	x	 x
71	. . x	x	x	x		. x
72	. . x	x	x		 x
73	. . x	x	x		 x
74	. . x	x	x			. x
75	. . x	x		. . x	. . x x
76	. . x	x		. . x	 x
77	. . x	x		. x	. x x
78	. . x	x		. x	x x
79	. . x	x		. x	x x
80	. . x	x		. x	x	. x
81	. . x	x		. x	 x
82	. . x	x		. x	 x
83	. . x	x		. x		. x
84	. . x	x		x	x x
85	. . x	x		x	x x
86	. . x	x		x	x	. x
87	. . x	x		x	 x

$R_1 = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & \infty \\ & 1 \end{pmatrix}$, $R_3 = \begin{pmatrix} 0 & 0 & 1 & \infty \\ & 1 & \infty & \infty \\ & & \infty & \infty \\ & & & \infty \end{pmatrix}$, $R_4 = \begin{pmatrix} 0 & 1 & \infty \\ & \infty & \infty & \infty \\ & & \infty & \infty \end{pmatrix}$, $R_5 = \begin{pmatrix} \infty \\ & \infty \\ & & \infty \\ & & & \infty \end{pmatrix}$
 Continued on next page

Nr.	$U(0,1)$ $R_1 R_2 R_3$	$U(0,\infty)$ $R_4 R_3 \tilde{E}_2^2$	$U(1,\infty)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,1)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,0)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,\infty)$ $R_5 R_2 R_4 R_3 \tilde{E}_2^2$
88	. . x	x		x		. . . x
89	. . x	x		x		. x
90	. . x	x			 x
91	. . x	x				. . . x
92	. . x	x				. x
93	. . x	x				x
94	. . x	x				
95	. . x			. . x	. . x x
96	. . x			. . x	 x
97	. . x			. . x		x
98	. . x			. . x		
99	. . x			. x	. x	. . . x
100	. . x			. x	x x
101	. . x			. x	x	. . . x
102	. . x			. x	x	. x
103	. . x			. x	 x
104	. . x			. x		. . . x
105	. . x			. x		. x
106	. . x			. x		x
107	. . x			. x		
108	. . x			x	x x
109	. . x			x	x	. . . x
110	. . x			x	x	. x
111	. . x			x	 x
112	. . x			x		. . . x
113	. . x			x		. x
114	. . x			x		x
115	. . x			x		
116	. . x				 x
117	. . x					. . . x
118	. . x					. x
119	. . x					x
120	. . x					
121	. x	. . x	. . x	. . x	. . x x
122	. x	. . x	. . x	. . x	 x
123	. x	. . x	. . x	. x	x x
124	. x	. . x	. . x	. x	 x
125	. x	. . x	. . x	x	x x
126	. x	. . x	. . x	x	 x
127	. x	. . x	. . x		 x
128	. x	. . x		. . x	. . x x
129	. x	. . x		. . x	 x
130	. x	. . x		. x	x x
131	. x	. . x		. x	 x
132	. x	. . x		x	x x
133	. x	. . x		x	 x
134	. x	. . x			 x
135	. x	. . x				x
136	. x	. . x				

$R_1 = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & \infty \\ & 1 \end{pmatrix}$, $R_3 = \begin{pmatrix} 0 & 0 & 1 & \infty \\ & 1 & \infty & \infty \end{pmatrix}$, $R_4 = \begin{pmatrix} 0 & 1 & \infty \\ & \infty & \infty \end{pmatrix}$, $R_5 = \begin{pmatrix} \infty \\ & \infty \end{pmatrix}$
 Continued on next page

Nr.	$U(0,1)$ $R_1 R_2 R_3$	$U(0,\infty)$ $R_4 R_3 \tilde{E}_2^2$	$U(1,\infty)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,1)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,0)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,\infty)$ $R_5 R_2 R_4 R_3 \tilde{E}_2^2$
137	· ×	· ×	· ×	· ×	· ×	· · · ×
138	· ×	· ×	· ×	· ×	×	· · · ×
139	· ×	· ×	· ×	· ×		· · · ×
140	· ×	· ×	· ×	×	×	· · · ×
141	· ×	· ×	· ×	×		· · · ×
142	· ×	· ×	· ×			· · · ×
143	· ×	· ×	×	· · ×	· · ×	· · · · ×
144	· ×	· ×	×	· · ×		· · · · ×
145	· ×	· ×	×	· ×	· ×	· · · ×
146	· ×	· ×	×	· ×	×	· · · · ×
147	· ×	· ×	×	· ×	×	· · · ×
148	· ×	· ×	×	· ×		· · · · ×
149	· ×	· ×	×	· ×		· · · ×
150	· ×	· ×	×	×	×	· · · · ×
151	· ×	· ×	×	×	×	· · · ×
152	· ×	· ×	×	×	×	· ×
153	· ×	· ×	×	×		· · · · ×
154	· ×	· ×	×	×		· · · ×
155	· ×	· ×	×	×		· ×
156	· ×	· ×	×			· · · · ×
157	· ×	· ×	×			· · · ×
158	· ×	· ×	×			· ×
159	· ×	· ×		· · ×	· · ×	· · · · ×
160	· ×	· ×		· · ×		· · · · ×
161	· ×	· ×		· ×	· ×	· · · ×
162	· ×	· ×		· ×	×	· · · · ×
163	· ×	· ×		· ×	×	· · · ×
164	· ×	· ×		· ×		· · · · ×
165	· ×	· ×		· ×		· · · ×
166	· ×	· ×		×	×	· · · · ×
167	· ×	· ×		×	×	· · · ×
168	· ×	· ×		×	×	· ×
169	· ×	· ×		×		· · · · ×
170	· ×	· ×		×		· · · ×
171	· ×	· ×		×		· ×
172	· ×	· ×				· · · · ×
173	· ×	· ×				· · · ×
174	· ×	· ×				· ×
175	· ×	· ×				×
176	· ×	· ×				
177	· ×	×	×	· · ×	· · ×	· · · · ×
178	· ×	×	×	· · ×		· · · · ×
179	· ×	×	×	· ×	· ×	· · · ×
180	· ×	×	×	· ×	×	· · · · ×
181	· ×	×	×	· ×	×	· · · ×
182	· ×	×	×	· ×	×	· ×
183	· ×	×	×	· ×		· · · · ×
184	· ×	×	×	· ×		· · · ×
185	· ×	×	×	· ×		· ×

$R_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & \infty \\ 1 & \infty \end{pmatrix}$, $R_3 = \begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}$, $R_4 = \begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}$, $R_5 = \begin{pmatrix} \infty \\ \infty \end{pmatrix}$
 Continued on next page

Nr.	$U(0,1)$ $R_1 R_2 R_3$	$U(0,\infty)$ $R_4 R_3 \tilde{E}_2^2$	$U(1,\infty)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,1)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,0)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,\infty)$ $R_5 R_2 R_4 R_3 \tilde{E}_2^2$
186	· ×	×	×	×	×	· · · · ×
187	· ×	×	×	×	×	· · · ×
188	· ×	×	×	×	×	· × ×
189	· ×	×	×	×	×	· ×
190	· ×	×	×	×		· · · · ×
191	· ×	×	×	×		· · · ×
192	· ×	×	×	×		· × ×
193	· ×	×	×	×		· ×
194	· ×	×	×			· · · · ×
195	· ×	×	×			· · · ×
196	· ×	×	×			· × ×
197	· ×	×	×			· ×
198	· ×	×		· · ×	· · ×	· · · · ×
199	· ×	×		· · ×		· · · · ×
200	· ×	×		· ×	· ×	· · · ×
201	· ×	×		· ×	×	· · · · ×
202	· ×	×		· ×	×	· · · ×
203	· ×	×		· ×	×	· ×
204	· ×	×		· ×		· · · · ×
205	· ×	×		· ×		· · · ×
206	· ×	×		· ×		· ×
207	· ×	×		×	×	· · · · ×
208	· ×	×		×	×	· · · ×
209	· ×	×		×	×	· × ×
210	· ×	×		×	×	· ×
211	· ×	×		×		· · · · ×
212	· ×	×		×		· · · ×
213	· ×	×		×		· × ×
214	· ×	×		×		· ×
215	· ×	×				· · · · ×
216	· ×	×				· · · ×
217	· ×	×				· × ×
218	· ×	×				· ×
219	· ×	×				×
220	· ×	×				
221	· ×			· · ×	· · ×	· · · · ×
222	· ×			· · ×		· · · · ×
223	· ×			· · ×		×
224	· ×			· · ×		
225	· ×			· ×	· ×	· · · ×
226	· ×			· ×	×	· · · · ×
227	· ×			· ×	×	· · · ×
228	· ×			· ×	×	· ×
229	· ×			· ×		· · · · ×
230	· ×			· ×		· · · ×
231	· ×			· ×		· ×
232	· ×			· ×		×
233	· ×			· ×		
234	· ×			×	×	· · · · ×

$R_1 = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & \infty \\ & 1 \end{pmatrix}$, $R_3 = \begin{pmatrix} 0 & 0 & 1 & \infty \\ & 1 & \infty & \infty \end{pmatrix}$, $R_4 = \begin{pmatrix} 0 & 1 & \infty \\ & \infty & \infty \end{pmatrix}$, $R_5 = \begin{pmatrix} \infty \\ & \infty \end{pmatrix}$
 Continued on next page

Nr.	$U(0,1)$ $R_1 R_2 R_3$	$U(0,\infty)$ $R_4 R_3 \tilde{E}_2^2$	$U(1,\infty)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,1)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,0)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,\infty)$ $R_5 R_2 R_4 R_3 \tilde{E}_2^2$
235	· ×			×	×	· · · ×
236	· ×			×	×	· × ×
237	· ×			×	×	· ×
238	· ×			×		· · · · ×
239	· ×			×		· · · ×
240	· ×			×		· × ×
241	· ×			×		· ×
242	· ×			×		×
243	· ×			×		
244	· ×					· · · · ×
245	· ×					· · · ×
246	· ×					· × ×
247	· ×					· ×
248	· ×					· ×
249	· ×					×
250	· ×					
251	×	· · ×	· · ×	· · ×	· · ×	· · · · ×
252	×	· · ×	· · ×	· · ×		· · · · ×
253	×	· · ×	· · ×	· ×	×	· · · · ×
254	×	· · ×	· · ×	· ×		· · · · ×
255	×	· · ×	· · ×	×	×	· · · · ×
256	×	· · ×	· · ×	×		· · · · ×
257	×	· · ×	· · ×			· · · · ×
258	×	· · ×		· · ×	· · ×	· · · · ×
259	×	· · ×		· · ×		· · · · ×
260	×	· · ×		· ×	×	· · · · ×
261	×	· · ×		· ×		· · · · ×
262	×	· · ×		×	×	· · · · ×
263	×	· · ×		×		· · · · ×
264	×	· · ×				· · · · ×
265	×	· · ×				×
266	×	· · ×				
267	×	· ×	· ×	· ×	· ×	· · · ×
268	×	· ×	· ×	· ×	×	· · · ×
269	×	· ×	· ×	· ×		· · · ×
270	×	· ×	· ×	×	×	· · · ×
271	×	· ×	· ×	×		· · · ×
272	×	· ×	· ×			· · · ×
273	×	· ×	×	· · ×	· · ×	· · · · ×
274	×	· ×	×	· · ×		· · · · ×
275	×	· ×	×	· ×	· ×	· · · ×
276	×	· ×	×	· ×	×	· · · · ×
277	×	· ×	×	· ×	×	· · · ×
278	×	· ×	×	· ×		· · · · ×
279	×	· ×	×	· ×		· · · ×
280	×	· ×	×	×	×	· · · · ×
281	×	· ×	×	×	×	· · · ×
282	×	· ×	×	×	×	· ×
283	×	· ×	×	×		· · · · ×

$R_1 = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & \infty \\ & 1 \end{pmatrix}$, $R_3 = \begin{pmatrix} 0 & 0 & 1 & \infty \\ & 1 & \infty & \infty \end{pmatrix}$, $R_4 = \begin{pmatrix} 0 & 1 & \infty \\ & \infty & \infty \end{pmatrix}$, $R_5 = \begin{pmatrix} \infty \\ & \infty \end{pmatrix}$
 Continued on next page

Nr.	$U(0,1)$ $R_1 R_2 R_3$	$U(0,\infty)$ $R_4 R_3 \tilde{E}_2^2$	$U(1,\infty)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,1)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,0)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,\infty)$ $R_5 R_2 R_4 R_3 \tilde{E}_2^2$
284	×	· ×	×	×		· · · ×
285	×	· ×	×	×		· ×
286	×	· ×	×			· · · · ×
287	×	· ×	×			· · · ×
288	×	· ×	×			· ×
289	×	· ×		· · ×	· · ×	· · · · ×
290	×	· ×		· · ×		· · · · ×
291	×	· ×		· ×	· ×	· · · ×
292	×	· ×		· ×	×	· · · · ×
293	×	· ×		· ×	×	· · · ×
294	×	· ×		· ×		· · · · ×
295	×	· ×		· ×		· · · ×
296	×	· ×		×	×	· · · · ×
297	×	· ×		×	×	· · · ×
298	×	· ×		×	×	· ×
299	×	· ×		×		· · · · ×
300	×	· ×		×		· · · ×
301	×	· ×		×		· ×
302	×	· ×				· · · · ×
303	×	· ×				· · · ×
304	×	· ×				· ×
305	×	· ×				×
306	×	· ×				
307	×	×	×	· · ×	· · ×	· · · · ×
308	×	×	×	· · ×		· · · · ×
309	×	×	×	· ×	· ×	· · · ×
310	×	×	×	· ×	×	· · · · ×
311	×	×	×	· ×	×	· · · ×
312	×	×	×	· ×	×	· ×
313	×	×	×	· ×		· · · · ×
314	×	×	×	· ×		· · · ×
315	×	×	×	· ×		· ×
316	×	×	×	×	×	· · · · ×
317	×	×	×	×	×	· · · ×
318	×	×	×	×	×	· × ×
319	×	×	×	×	×	· ×
320	×	×	×	×		· · · · ×
321	×	×	×	×		· · · ×
322	×	×	×	×		· × ×
323	×	×	×	×		· ×
324	×	×	×			· · · · ×
325	×	×	×			· · · ×
326	×	×	×			· × ×
327	×	×	×			· ×
328	×	×		· · ×	· · ×	· · · · ×
329	×	×		· · ×		· · · · ×
330	×	×		· ×	· ×	· · · ×
331	×	×		· ×	×	· · · · ×
332	×	×		· ×	×	· · · ×

$R_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $R_2 = \begin{pmatrix} 0 & \infty \\ 1 & \infty \end{pmatrix}$, $R_3 = \begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}$, $R_4 = \begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}$, $R_5 = \begin{pmatrix} \infty \\ \infty \end{pmatrix}$
 Continued on next page

Nr.	$U(0,1)$ $R_1 R_2 R_3$	$U(0,\infty)$ $R_4 R_3 \tilde{E}_2^2$	$U(1,\infty)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,1)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,0)$ $R_4 R_3 \tilde{E}_2^2$	$U(\infty,\infty)$ $R_5 R_2 R_4 R_3 \tilde{E}_2^2$
333	×	×		· ×	×	· ×
334	×	×		· ×		· · · ×
335	×	×		· ×		· · · ×
336	×	×		· ×		· ×
337	×	×		×	×	· · · ×
338	×	×		×	×	· · · ×
339	×	×		×	×	· × ×
340	×	×		×	×	· ×
341	×	×		×		· · · ×
342	×	×		×		· · · ×
343	×	×		×		· × ×
344	×	×		×		· ×
345	×	×				· · · ×
346	×	×				· · · ×
347	×	×				· × ×
348	×	×				· ×
349	×	×				×
350	×	×				
351	×			· · ×	· · ×	· · · ×
352	×			· · ×		· · · ×
353	×			· · ×		×
354	×			· · ×		
355	×			· ×	· ×	· · · ×
356	×			· ×	×	· · · ×
357	×			· ×	×	· · · ×
358	×			· ×	×	· ×
359	×			· ×		· · · ×
360	×			· ×		· · · ×
361	×			· ×		· ×
362	×			· ×		×
363	×			· ×		
364	×			×	×	· · · ×
365	×			×	×	· · · ×
366	×			×	×	· × ×
367	×			×	×	· ×
368	×			×		· · · ×
369	×			×		· · · ×
370	×			×		· × ×
371	×			×		· ×
372	×			×		×
373	×			×		
374	×					· · · ×
375	×					· · · ×
376	×					· × ×
377	×					· ×
378	×					· ×
379	×					×
380	×					

$R_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & \infty \\ 1 & \infty \end{pmatrix}, R_3 = \begin{pmatrix} 0 & 0 & 1 & \infty \\ 1 & \infty & \infty & \infty \end{pmatrix}, R_4 = \begin{pmatrix} 0 & 1 & \infty \\ \infty & \infty & \infty \end{pmatrix}, R_5 = \begin{pmatrix} \infty \\ \infty \end{pmatrix}$

Selbständigkeitserklärung nach §4 Abs. 1 der Promotionsordnung

Ich versichere hiermit an Eides statt, dass ich die vorliegende Arbeit selbständig angefertigt und ohne fremde Hilfe verfasst habe, keine außer den von mir angegebenen Hilfsmitteln und Quellen dazu verwendet habe und die den benutzten Werken inhaltlich und wörtlich entnommenen Stellen als solche kenntlich gemacht habe.

Rostock, den 26. Juli 2010

Karsten Schölzel

Lebenslauf

08.06.1982	geboren in Grevesmühlen
1989 – 2002	Schulbesuch, Abschluss Abitur am Gymnasium Demmin
Mai 2001	3. Preis Bundesrunde Mathematikolympiade, Magdeburg
2002 – 2003	Zivildienst an der Schule zur Individuellen Lebensbewältigung, Demmin
2003 – 2008	Diplomstudium Mathematik mit Nebenfach Physik, Universität Rostock
2004 – 2005	1. Vorsitzender RHO-Verein
2005 – 2006	Auslandsjahr, Luleå Technical University, Schweden
28. Mai 2008	Diplom der Mathematik bei Prof. Lau an der Universität Rostock “Ein Kriterium für partielle Sheffer-Funktionen in 4-wertiger Logik”
seit 01.06.2008	wissenschaftlicher Mitarbeiter und Promotionsstudent, Universität Rostock

Zusammenfassung

deutsch

In der Theorie der partiellen Funktionen auf endlichen Mengen ist die Kenntnis der partiellen Klone, der abgeschlossenen Mengen, und deren Verhältnis zueinander von grundlegender Bedeutung. Es wird eine neue Galois-Verbindung zwischen partiellen Klonen und Relationen-Algebren erarbeitet. Mit deren Hilfe werden Aussagen zur Bestimmung der maximalen partiellen Klone neu bewiesen. Danach werden alle maximalen partiellen Klone für höchstens 6-elementige Mengen bestimmt. Hauptergebnis der Arbeit ist der Nachweis, dass bezüglich der maximalen partiellen Klone genau ein optimales Vollständigkeitskriterium für Sheffer-Funktionen existiert. Zum Abschluß werden mit einem Program alle 380 partiellen Klone des letzten endlichen Intervalls auf einer 2-elementigen Menge bestimmt.

englisch

In the theory of partial functions the knowledge of the closed sets, called clones, on finite sets and the relations between them play a very important role. A new Galois connection between clones and relation algebras is established. This is used to proof statements for the determination of the maximal partial clones in a new way. Then all maximal partial clones for at most 6-element sets are determined. Main result of the thesis is the proof that only one optimal completeness criterion for Sheffer-functions exists with respect to the maximal partial clones. Finally a program determines all 380 partial clones of the last finite interval for a 2-element set.