A Criterion for partial Sheffer functions in 4-valued logic

Karsten Schölzel

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Outline

1. Introduction
   - Definitions

2. Results

3. Minimal covering
Aim

- Criterion for partial Sheffer functions in 4-valued logic by determining a minimal covering of the maximal partial classes given by Haddad and Rosenberg
- Show that this minimal covering is unique
Definitions

Some sets

\[ E_k := \{0, 1, \ldots, k - 1\} \]
\[ \tilde{E}_k := E_k \cup \{\infty\} \]
\[ P_k := \{ f \mid f^{(n)} : E^n_k \rightarrow E_k, n \in \mathbb{N}_0 \} \]
\[ \tilde{P}_k := \{ f \mid f^{(n)} : E^n_k \rightarrow \tilde{E}_k, n \in \mathbb{N}_0 \} \]
We consider the algebra \( \tilde{P}_k; \zeta, \tau, \Delta, \nabla, \star \). For functions \( f^{(n)}, g^{(m)} \in \tilde{P}_k \) let

\[
(\zeta f)(x_1, \ldots, x_n) := f(x_2, x_3, \ldots, x_n, x_1)
\]

\[
(\tau f)(x_1, \ldots, x_n) := f(x_2, x_1, x_3, \ldots, x_n)
\]

\[
(\Delta f)(x_1, \ldots, x_{n-1}) := f(x_1, x_1, x_2, \ldots, x_{n-1}) \quad \text{if } n \geq 2
\]

\[
\zeta f = \tau f = \Delta f := f \quad \text{if } n = 1
\]

\[
(\nabla f)(x_1, \ldots, x_{n+1}) := f(x_2, \ldots, x_{n+1})
\]

\[
(f \star g)(x_1, \ldots, x_{m+n-1}) := \begin{cases} f(g(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1}) & \text{if } g(x_1, \ldots, x_m) \in E_k \\ \infty & \text{otherwise} \end{cases}
\]
A is called a class, if \( A = [A]_{\zeta, \tau, \Delta, \nabla, \star} \) holds. If \( J_k \subseteq A \) also holds, \( A \) is called a (partial) clone.

\[
e_i^n(x_1, \ldots, x_n) := x_i,
\]

\[
J_k := \{e_i^n | n \in \mathbb{N}, 1 \leq i \leq n\}.
\]
A (partial) function $f$ is called a (partial) Sheffer function, if

$$\tilde{P}_k = [\{f\}].$$
A class $A$ is called maximal, if

$$\forall A' \subset \tilde{P}_k : A \subset A' = [A'] \subset \tilde{P}_k.$$ 

Let $p\mathcal{M}_k$ be the set of all maximal partial classes.

Remark: every maximal class is a clone.
Because

$$\forall A \subset \tilde{P}_k, A = [A] \exists M_A \in p\mathcal{M}_k : A \subset M_A$$

it holds

$$f \text{ Sheffer} \iff \forall X \in p\mathcal{M}_k : f \notin X.$$
A (partial) function $f^{(n)} \in \widetilde{P}_k$ preserves the relation $\varrho^{(h)}$, if for all $r^1, \ldots, r^n$ with $r^i = (r_{1i}, \ldots, r_{hi})^T \in \varrho$ holds:

$$f(r^1, \ldots, r^n) := \left( \begin{array}{c} f(r_{11}, r_{12}, \ldots, r_{1n}) \\
 f(r_{21}, r_{22}, \ldots, r_{2n}) \\
 \vdots \\
 f(r_{h1}, r_{h2}, \ldots, r_{hn}) \end{array} \right) \in \varrho.$$ 

Short: $f \in pPol_k \varrho$. 
For $\varrho \subseteq E^h_k$ define

$$pPOL_k \varrho := pPol_k \left( \varrho \cup \left( \widetilde{E}^h_k \setminus E^h_k \right) \right).$$
**Haddad-Rosenberg Theorem [Haddad, Rosenberg, 1989, 1992]**

If $C$ is a maximal partial clone of $\tilde{P}_k$, i.e. $C \in p\mathcal{M}_k$, then

$$C = P_k \cup \{c_\infty\} = P_k \cup \{f \in \tilde{P}_k | \text{dom}(f) = \emptyset\}$$

or

$$C = p\text{POL}_k \varrho$$

with a coherent relation $\varrho$.

The definition of *coherent relation* is complex, so just some examples:

- non-trivial unary relations
- totally symmetric, totally reflexive relations
- non-trivial partial orders
- equivalence relations

A coherent relation is at most $\max(k, 4)$–ary.
### Number of maximal (partial) clones

| $k$ | $|\mathcal{M}_k|$ | $|p\mathcal{M}_k|$ | References |
|-----|------------------|-----------------|-------------|
| 2   | 5                | 8               | [Freivald 1966] |
| 3   | 18               | 58              | [Lau 1977], [Romov 1980] |
| 4   | 82               | **1 102**       |             |
| 5   | 643              | $>16 487$       |             |
| 6   | 15 182           | ?               |             |
| 7   | 7 848 984        | ?               |             |
| 8   | 549 758 283 980  | ?               |             |

Haddad and Simons determined the maximal partial clones for $k = 4$ in 2002 and gave $|p\mathcal{M}_4| = 1 235$. This was due to errors while counting so we gave a full list and determined $|p\mathcal{M}_4|$. 
## Unary relations

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### Binary asymmetric areflexive relations

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# Binary antisymmetric reflexive relations

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Ternary relations with $\delta = \delta^3_{\{0,1\}}$ and $G_\sigma = \{\text{id}\}$

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Ternary totally reflexive, totally symmetric relations

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## Quartary areflexive relations

<table>
<thead>
<tr>
<th>Nr.</th>
<th>( \delta )</th>
<th>( \sigma )</th>
<th>iso</th>
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<tbody>
<tr>
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### Quartary relations with $\delta = \delta^4_{\{0,1,2,3\}}$

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Quartary relations with $\delta = \delta^4_{\{0,1,2\}}$

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<td>119</td>
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Quartary relations with $\delta = \delta_4^{\{0,1\}}$

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<td>0123, 0132</td>
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</tr>
<tr>
<td>123</td>
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<td>0123, 1032</td>
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<tr>
<td>124</td>
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Quartary relations with $\delta = \delta_4^{\{0,1\},\{2,3\}}$

<table>
<thead>
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## Special quartary relations

<table>
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<td>137</td>
<td>( \iota_2 )</td>
<td>0123, 1230, 2301, 3012, 2103, 3210, 0321, 1032</td>
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</table>
Definition

A subset $\mathcal{X} \subset \mathcal{P}_k$ is a minimal covering if

$$\forall f \in \tilde{P}_k : ([\{f\}] = \tilde{P}_k \iff \forall A \in \mathcal{X} : f \notin A)$$

(1)

$$\forall A \in \mathcal{X} \exists f \in A \forall B \in \mathcal{X} \setminus \{A\} : f \notin B$$

(2)
# Minimal covering for $k = 4$

**Theorem**

*There is exactly one minimal covering of $\mathcal{P}_4$ and it has 449 elements.*

| $k$ | $|\mathcal{P}_k|$ | $|\mathcal{X}|$     | Source                  |
|-----|-----------------|-----------------|------------------------|
| 2   | 8               | 4               | [Haddad, Rosenberg, 1991] |
| 3   | 58              | 26              | [Haddad, Lau, 2006]    |
| 4   | 1102            | 449             |                        |
A partial order on $pM_k$

**Definition**

Define $\alpha : pM_k \rightarrow \mathbb{N}$ by

$$\alpha(X) := \begin{cases} 
1 & \text{if } X = P_k \cup \{\{c_\infty\}\}, \\
\ h & \text{if } X = pPOL_k \varrho \text{ and } \varrho \text{ is an } h\text{-ary coherent relation.}
\end{cases}$$

**Lemma**

Let $X \in pM_k \setminus \{P_k \cup \{c_\infty\}\}$ and $\varrho$ a coherent relation with

$$X = pPOL_k \varrho.$$

Then $\varrho$ is unique except for permutation of coordinates. Thus $\alpha$ is well-defined.
A partial order on $p\mathcal{M}_k$

**Lemma**

Let $X, Y \in p\mathcal{M}_k$. Then $\prec$ given by

$$X \prec Y \iff \alpha(X) < \alpha(Y)$$

is a partial order on $p\mathcal{M}_k$. 
Determine a minimal covering

**Theorem**

Let $O : p\mathcal{M}_k \rightarrow 2^{p\mathcal{M}_k}$ with

- $O(X) = \emptyset$, if $X \in p\mathcal{M}_k$ belongs to every minimal covering, i.e.
  \[ \exists f \in X \forall Y \in p\mathcal{M}_k \setminus \{X\} : f \notin Y, \]
- and
  \[ \forall X \in p\mathcal{M}_k \forall Y \in O(X) : Y \prec X. \]

Then

\[ X := \{X \in p\mathcal{M}_k | O(X) = \emptyset\} \]

is the unique minimal covering of $p\mathcal{M}_k$. 
Some elements of every minimal covering of \( pM_k \)

Let \( X = pPOL_k \varrho \in pM_k \) and \( \varrho \) an \( h \)-ary coherent relation. Then \( X \) is in every minimal covering, if

- \( \varrho = \sigma_1 \cup \sigma_2 \) and there is \( A \subset E_k \), \( A \neq \emptyset \) with \( \sigma_1 \subset A^h \) and \( \sigma_2 \subset E_k^h \setminus A^h \),
- \( \varrho \in \{ \iota^3_k, \varrho_1, \varrho_2 \} \) with
  \[
  \iota^h_k = \left\{ (x_1, \ldots, x_h) \in E_k^h \left| |\{x_1, \ldots, x_h\}| \leq h - 1 \right. \right\},
  \]
  \[
  R_1 = \left\{ (a, a, b, b), (a, b, a, b), (a, b, b, a) \mid a, b \in E_k \right\},
  \]
  \[
  R_2 = \left\{ (a, a, b, b), (a, b, a, b) \mid a, b \in E_k \right\}.
  \]

...
Maximal clones not in every minimal covering of $p\mathcal{M}_k$

Let $X = p\text{POL}_{k\varrho} \in p\mathcal{M}_k$ and $\varrho$ an $h$-ary coherent relation. Then $X$ is not every minimal covering, if

- $h = 2$ and the transitive closure of $\varrho$ is a partial order with a central element, i.e.
  \[
  \exists c \in E_k \forall x \in E_k : (x, c) \in \varrho \lor (c, x) \in \varrho
  \]

- $\sigma \cup \iota^h_k$ for $h \geq 4$
  \[
  \iota^h_k := \left\{ (x_0, \ldots, x_{h-1}) \in E^h_k \left| \left| \{x_0, \ldots, x_{h-1}\} \right| \leq h - 1 \right. \right\}
  \]

- $\ldots$

I.e. if there is only one minimal covering, then these clones are not in this minimal covering.
A binary partial Sheffer function for $k \geq 3$

**Lemma**

The function $f_k$ defined by

$$f_k(x, y) := \begin{cases} 
  x + 1 \mod k & \text{if } x = y, \\
  x + 2 \mod k & \text{if } x \neq 0 \text{ and } y = 0, \\
  0 & \text{if } x = 0 \text{ and } y \neq 0, \\
  0 & \text{if } x \neq k - 1 \text{ and } y = x + 1, \\
  \infty & \text{otherwise}
\end{cases}$$

is a partial Sheffer function for $k$ with $k \geq 3$. 
What to do next?

- generalize the results for all $k \geq 3$
- a formula for the number of maximal partial clones for a given $k$
- determine generating sets (or the cardinality of these) for maximal partial clones
- ...
Thank you for your attention.