A Criterion for partial Sheffer functions in 4-valued logic

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2007-04-15

Outline

1. **Introduction**
   - Definitions
2. **Results**
3. **Minimal covering**

Aim

- Criterion for partial Sheffer functions in 4-valued logic by determining a minimal covering of the maximal partial classes given by Haddad and Rosenberg
- Show that this minimal covering is unique

Some sets

$$
\begin{aligned}
E_k &:= \{0, 1, \ldots, k-1\} \\
\widetilde{E}_k &:= E_k \cup \{\infty\} \\
P_k &:= \left\{ f \middle| f^{(n)} : E_k^n \to E_k, n \in \mathbb{N}_0 \right\} \\
\widetilde{P}_k &:= \left\{ f \middle| f^{(n)} : E_k^n \to \widetilde{E}_k, n \in \mathbb{N}_0 \right\}
\end{aligned}
$$

Mal’tsev operations

We consider the algebra $(\widetilde{P}_k; \zeta, \tau, \Delta, \nabla, \star)$. For functions $f^{(n)}, g^{(m)} \in \widetilde{P}_k$ let

$$\begin{array}{rcl}
(\zeta f)(x_1, \ldots, x_n) & := & f(x_2, x_3, \ldots, x_n, x_1) \\
(\tau f)(x_1, \ldots, x_n) & := & f(x_2, x_1, x_3, \ldots, x_n) \\
(\Delta f)(x_1, \ldots, x_{n-1}) & := & f(x_1, x_1, x_2, \ldots, x_{n-1}) \quad \text{if } n \geq 2 \\
\zeta f = \tau f = \Delta f & := & f \quad \text{if } n = 1 \\
(\nabla f)(x_1, \ldots, x_{n+1}) & := & f(x_2, \ldots, x_{n+1}) \\
(f \star g)(x_1, \ldots, x_{m+n-1}) & := & \begin{cases} f(g(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1}) & \\ \qquad \text{if } g(x_1, \ldots, x_m) \in E_k & \\ \infty \quad \text{otherwise} & \end{cases}
\end{array}$$

Classes, (partial) clones

A is called a class, if $A = [A]_{\zeta,\tau,\Delta,\nabla,\star}$ holds.
If $J_k \subseteq A$ also holds, A is called a (partial) clone.

$$\begin{aligned} e_i^n(x_1,\ldots,x_n) &:= x_i \\ J_k &:= \{e_i^n | n \in \mathbb{N}, 1 \leq i \leq n\}. \end{aligned}$$

Sheffer function

Definition

A (partial) function f is called a (partial) Sheffer function, if

$$\widetilde{P}_k = [\{f\}].$$

Maximal partial clones

A class A is called maximal, if

$$\nexists A' \subset \widetilde{P}_k : A \subset A' = [A'] \subset \widetilde{P}_k.$$

Let $p\mathscr{M}_k$ be the set of all maximal partial classes.
Remark: every maximal class is a clone.
Because

$$\forall A \subset \widetilde{P}_k, A = [A]\ \exists M_A \in p\mathscr{M}_k : A \subset M_A$$

it holds

$$f \text{ Sheffer} \iff \forall X \in p\mathscr{M}_k : f \notin X.$$

Preservation of relations

A (partial) function $f^{(n)} \in \widetilde{P}_k$ *preserves* the relation $\varrho^{(h)}$, if for all $\mathbf{r}^1, \ldots, \mathbf{r}^n$ with $\mathbf{r}^i = (r_{1i}, \ldots, r_{hi})^{\mathrm{T}} \in \varrho$ holds:

$$f(\mathbf{r}^1, \ldots, \mathbf{r}^n) := \begin{pmatrix} f(r_{11}, r_{12}, \ldots, r_{1n}) \\ f(r_{21}, r_{22}, \ldots, r_{2n}) \\ \vdots \\ f(r_{h1}, r_{h2}, \ldots, r_{hn}) \end{pmatrix} \in \varrho.$$

Short: $f \in pPol_k\,\varrho$.
For $\varrho \subseteq E_k^h$ define

$$pPOL_k\,\varrho := pPol_k\left(\varrho \cup \left(\widetilde{E}_k^h \setminus E_k^h\right)\right).$$

Haddad-Rosenberg Theorem [Haddad, Rosenberg, 1989, 1992]

If C is a maximal partial clone of $\widetilde{P}_k$, i.e. $C \in p\mathscr{M}_k$, then

$$C = P_k \cup [\{c_\infty\}] = P_k \cup \{f \in \widetilde{P}_k | \operatorname{dom}(f) = \emptyset\}$$

or

$$C = pPOL_k \varrho$$

with a coherent relation ϱ.
The definition of *coherent relation* is complex, so just some examples:

- non-trivial unary relations
- totally symmetric, totally reflexive relations
- non-trivial partial orders
- equivalence relations

A coherent relation is at most $\max(k, 4)$–ary.

Number of maximal (partial) clones

k	$\lvert\mathcal{M}_k\rvert$	$\lvert p\mathscr{M}_k\rvert$	
2	5	8	[Freivald 1966]
3	18	58	[Lau 1977], [Romov 1980]
4	82	1 102	
5	643	$>$ 16 487	
6	15 182	?	
7	7 848 984	?	
8	549 758 283 980	?	

Haddad and Simons determined the maximal partial clones for $k = 4$ in 2002 and gave $\lvert p\mathscr{M}_4\rvert = 1\,235$. This was due to errors while counting so we gave a full list and determined $\lvert p\mathscr{M}_4\rvert$.

Unary relations

Nr.	δ	σ	iso
1	$\emptyset$	0	4
2	$\emptyset$	$0, 1$	6
3	$\emptyset$	$0, 1, 2$	4

Binary asymmetric areflexive relations

Nr.	δ	σ	iso
4	$\emptyset$	01	6
5	$\emptyset$	$01, 02$	12
6	$\emptyset$	$01, 23$	6
7	$\emptyset$	$01, 02, 03$	4
8	$\emptyset$	$01, 02, 32$	12
9	$\emptyset$	$01, 02, 31, 32$	3

Binary symmetric areflexive relations

Nr.	δ	σ	iso
10	$\emptyset$	$01, 10$	6
11	$\emptyset$	$01, 10, 02, 20$	12
12	$\emptyset$	$01, 10, 23, 32$	3
13	$\emptyset$	$01, 10, 02, 20, 03, 30$	4
14	$\emptyset$	$01, 10, 12, 21, 23, 32$	12
15	$\emptyset$	$01, 10, 02, 20, 13, 31, 23, 32$	3

Binary antisymmetric reflexive relations

Nr.	δ	σ	iso
16	$\delta^2_{\{0,1\}}$	01	6
17	$\delta^2_{\{0,1\}}$	01, 02	12
18	$\delta^2_{\{0,1\}}$	01, 12	12
19	$\delta^2_{\{0,1\}}$	01, 23	6
20	$\delta^2_{\{0,1\}}$	01, 02, 03	4
21	$\delta^2_{\{0,1\}}$	01, 02, 30	12
22	$\delta^2_{\{0,1\}}$	01, 02, 12	12
23	$\delta^2_{\{0,1\}}$	01, 02, 13	24
24	$\delta^2_{\{0,1\}}$	01, 02, 32	12
25	$\delta^2_{\{0,1\}}$	01, 12, 23	12

Binary antisymmetric reflexive relations

Nr.	δ	σ	iso
26	$\delta^2_{\{0,1\}}$	01, 02, 03, 12	24
27	$\delta^2_{\{0,1\}}$	01, 02, 30, 32	24
28	$\delta^2_{\{0,1\}}$	01, 02, 31, 32	3
29	$\delta^2_{\{0,1\}}$	01, 02, 12, 23	24
30	$\delta^2_{\{0,1\}}$	01, 03, 12, 23	12
31	$\delta^2_{\{0,1\}}$	01, 02, 13, 23	6
32	$\delta^2_{\{0,1\}}$	01, 02, 03, 12, 13	12
33	$\delta^2_{\{0,1\}}$	01, 02, 03, 12, 31	24
34	$\delta^2_{\{0,1\}}$	01, 02, 03, 13, 23	6
35	$\delta^2_{\{0,1\}}$	01, 02, 12, 13, 23	12
36	$\delta^2_{\{0,1\}}$	01, 02, 03, 12, 13, 23	12

Binary symmetric reflexive relations

Nr.	δ	σ	iso
37	$\delta^2_{\{0,1\}}$	$01, 10$	6
38	$\delta^2_{\{0,1\}}$	$01, 10, 02, 20$	12
39	$\delta^2_{\{0,1\}}$	$01, 10, 23, 32$	3
40	$\delta^2_{\{0,1\}}$	$01, 10, 02, 20, 12, 21$	4
41	$\delta^2_{\{0,1\}}$	$01, 10, 02, 20, 03, 30$	4
42	$\delta^2_{\{0,1\}}$	$01, 10, 12, 21, 23, 32$	12
43	$\delta^2_{\{0,1\}}$	$01, 10, 02, 20, 03, 30, 12, 21$	12
44	$\delta^2_{\{0,1\}}$	$01, 10, 02, 20, 13, 31, 23, 32$	3
45	$\delta^2_{\{0,1\}}$	$01, 10, 02, 20, 03, 30, 12, 21, 13, 31$	6

Ternary areflexive relations

Nr.	δ	σ	iso
46	$\emptyset$	012	4
47	$\emptyset$	$012, 021$	12
48	$\emptyset$	$012, 120, 201$	4
49	$\emptyset$	$012(S_3)$	4
50	$\emptyset$	$012, 013$	6
51	$\emptyset$	$012, 013, 102, 103$	6
52	$\emptyset$	$012, 013, 021, 031$	12
53	$\emptyset$	$012, 120, 201, 013, 130, 301$	6
54	$\emptyset$	$012(S_3), 013(S_3)$	6

Ternary relations with $\delta = \delta^3_{\{0,1,2\}}$

Nr.	δ	σ	iso
55	$\delta^3_{\{0,1,2\}}$	012	4
56	$\delta^3_{\{0,1,2\}}$	012, 021	12
57	$\delta^3_{\{0,1,2\}}$	012, 120, 201	4
58	$\delta^3_{\{0,1,2\}}$	$012(S_3)$	4
59	$\delta^3_{\{0,1,2\}}$	012, 013	6
60	$\delta^3_{\{0,1,2\}}$	012, 013, 102, 103	6
61	$\delta^3_{\{0,1,2\}}$	012, 013, 021, 023	12
62	$\delta^3_{\{0,1,2\}}$	012, 120, 201, 013, 130, 301	6
63	$\delta^3_{\{0,1,2\}}$	$012(S_3), 013(S_3)$	6

Ternary relations with $\delta = \delta^3_{\{0,1\}}$ and $G_\sigma = \{\mathrm{id}\}$

Nr.	δ	σ	iso
64	$\delta^3_{\{0,1\}}$	012	12
65	$\delta^3_{\{0,1\}}$	012, 013	6
66	$\delta^3_{\{0,1\}}$	012, 023	24
67	$\delta^3_{\{0,1\}}$	012, 032	12
68	$\delta^3_{\{0,1\}}$	012, 123	24
69	$\delta^3_{\{0,1\}}$	012, 132	12
70	$\delta^3_{\{0,1\}}$	012, 230	12
71	$\delta^3_{\{0,1\}}$	012, 231	12

Ternary relations with $\delta = \delta^3_{\{0,1\}}$ and $G_\sigma = \{\mathrm{id}\}$

Nr.	δ	σ	iso
72	$\delta^3_{\{0,1\}}$	012, 031, 032	24
73	$\delta^3_{\{0,1\}}$	012, 023, 123	24
74	$\delta^3_{\{0,1\}}$	012, 032, 312	12
75	$\delta^3_{\{0,1\}}$	012, 230, 231	24
76	$\delta^3_{\{0,1\}}$	012, 130, 132	24
77	$\delta^3_{\{0,1\}}$	012, 013, 023, 123	24
78	$\delta^3_{\{0,1\}}$	012, 013, 230, 231	6

Ternary relations with $\delta = \delta^3_{\{0,1\}}$ and $G_\sigma = \{\mathrm{id}, (0\,1)\}$

Nr.	δ	σ	iso
79	$\delta^3_{\{0,1\}}$	012, 102	12
80	$\delta^3_{\{0,1\}}$	012, 102, 013, 103	6
81	$\delta^3_{\{0,1\}}$	012, 102, 023, 203	24
82	$\delta^3_{\{0,1\}}$	012, 102, 032, 302	12
83	$\delta^3_{\{0,1\}}$	012, 102, 230, 320	12
84	$\delta^3_{\{0,1\}}$	012, 102, 031, 301, 032, 302	24
85	$\delta^3_{\{0,1\}}$	012, 102, 023, 203, 123, 213	12
86	$\delta^3_{\{0,1\}}$	012, 102, 032, 302, 312, 132	4
87	$\delta^3_{\{0,1\}}$	012, 102, 230, 320, 231, 321	12
88	$\delta^3_{\{0,1\}}$	012, 102, 013, 103, 023, 203, 123, 213	12
89	$\delta^3_{\{0,1\}}$	012, 102, 013, 103, 230, 320, 231, 321	3

Ternary totally reflexive, totally symmetric relations

Nr.	δ	σ	iso
90	ι_4^3	$\emptyset$	1
91	ι_4^3	$012(S_3)$	4
92	ι_4^3	$012(S_3), 013(S_3)$	6
93	ι_4^3	$012(S_3), 013(S_3), 023(S_3)$	4

Quartary areflexive relations

Nr.	δ	σ	iso
94	$\emptyset$	0123	1
95	$\emptyset$	0123, 1023	6
96	$\emptyset$	0123, 1032	3
97	$\emptyset$	0123, 1203, 2013	4
98	$\emptyset$	0123, 1230, 2301, 3012	3
99	$\emptyset$	0123, 0132, 1023, 1032	3
100	$\emptyset$	0123, 1032, 3210, 2301	1
101	$\emptyset$	0123, 0132, 0213, 0231, 0312, 0321	4
102	$\emptyset$	0123, 1230, 2301, 3012, 2103, 3210, 0321, 1032	3
103	$\emptyset$	$0123(A_4)$	1
104	$\emptyset$	$0123(S_4)$	1

Quartary relations with $\delta = \delta^4_{\{0,1,2,3\}}$

Nr.	δ	σ	iso
105	$\delta^4_{\{0,1,2,3\}}$	0123	1
106	$\delta^4_{\{0,1,2,3\}}$	0123, 1023	6
107	$\delta^4_{\{0,1,2,3\}}$	0123, 1032	3
108	$\delta^4_{\{0,1,2,3\}}$	0123, 1203, 2013	4
109	$\delta^4_{\{0,1,2,3\}}$	0123, 1230, 2301, 3012	3
110	$\delta^4_{\{0,1,2,3\}}$	0123, 0132, 1023, 1032	3
111	$\delta^4_{\{0,1,2,3\}}$	0123, 1032, 3210, 2301	1
112	$\delta^4_{\{0,1,2,3\}}$	0123, 0132, 0213, 0231, 0312, 0321	4
113	$\delta^4_{\{0,1,2,3\}}$	0123, 1230, 2301, 3012, 2103, 3210, 0321, 1032	3
114	$\delta^4_{\{0,1,2,3\}}$	0123(A_4)	1
115	$\delta^4_{\{0,1,2,3\}}$	0123(S_4)	1

Quartary relations with $\delta = \delta^4_{\{0,1,2\}}$

Nr.	δ	σ	iso
116	$\delta^4_{\{0,1,2\}}$	0123	4
117	$\delta^4_{\{0,1,2\}}$	0123, 1023	12
118	$\delta^4_{\{0,1,2\}}$	0123, 1203, 2013	4
119	$\delta^4_{\{0,1,2\}}$	0123, 0213, 1023, 1203, 2013, 2103	4

Quartary relations with $\delta = \delta^4_{\{0,1\}}$

Nr.	δ	σ	iso
120	$\delta^4_{\{0,1\}}$	0123	6
121	$\delta^4_{\{0,1\}}$	0123, 1023	6
122	$\delta^4_{\{0,1\}}$	0123, 0132	6
123	$\delta^4_{\{0,1\}}$	0123, 1032	6
124	$\delta^4_{\{0,1\}}$	0123, 0132, 1023, 1032	6

Quartary relations with $\delta = \delta^4_{\{0,1\},\{2,3\}}$

Nr.	δ	σ	iso
125	$\delta^4_{\{0,1\},\{2,3\}}$	0123	3
126	$\delta^4_{\{0,1\},\{2,3\}}$	0123, 1023	6
127	$\delta^4_{\{0,1\},\{2,3\}}$	0123, 1032	3
128	$\delta^4_{\{0,1\},\{2,3\}}$	0123, 2301	6
129	$\delta^4_{\{0,1\},\{2,3\}}$	0123, 0132, 1023, 1032	3
130	$\delta^4_{\{0,1\},\{2,3\}}$	0123, 1032, 2310, 3201	3
131	$\delta^4_{\{0,1\},\{2,3\}}$	0123, 1032, 2301, 3210	3
132	$\delta^4_{\{0,1\},\{2,3\}}$	0123, 0132, 1023, 1032, 2301, 2310, 3201, 3210	3

Special quartary relations

Nr.	δ	σ	iso
133	ι_4^4	$\emptyset$	1
134	ϱ_1	$\emptyset$	1
135	ϱ_2	$\emptyset$	1
136	ϱ_1	$0123(S_4)$	1
137	ϱ_2	$0123, 1230, 2301, 3012, 2103, 3210, 0321, 1032$	3

Minimal coverings

Definition

A subset $\mathcal{X} \subset p\mathscr{M}_k$ is a minimal covering if

$$\forall f \in \widetilde{P}_k : ([\{f\}] = \widetilde{P}_k \Longleftrightarrow \forall A \in \mathcal{X} : f \notin A) \tag{1}$$

$$\forall A \in \mathcal{X}\ \exists f \in A\ \forall B \in \mathcal{X} \setminus \{A\} : f \notin B \tag{2}$$

Minimal covering for $k = 4$

Theorem

There is exactly one minimal covering of $p\mathscr{M}_4$ *and it has* 449 *elements.*

k	$\lvert p\mathscr{M}_k \rvert$	$\lvert \mathcal{X} \rvert$	
2	8	4	[Haddad, Rosenberg, 1991]
3	58	26	[Haddad, Lau, 2006]
4	1 102	449	

A partial order on $p\mathscr{M}_k$

Definition

Define $\alpha : p\mathscr{M}_k \to \mathbb{N}$ by

$$\alpha(X) := \begin{cases} 1 & \text{if } X = P_k \cup [\{c_\infty\}], \\ h & \text{if } X = pPOL_k\,\varrho \text{ and } \varrho \text{ is an } h\text{-ary coherent relation.} \end{cases}$$

Lemma

Let $X \in p\mathscr{M}_k \setminus \{P_k \cup [c_\infty]\}$ and ϱ a coherent relation with

$$X = pPOL_k\,\varrho.$$

Then ϱ is unique except for permutation of coordinates. Thus α is well-defined.

A partial order on $p\mathscr{M}_k$

Lemma

Let $X, Y \in p\mathscr{M}_k$. Then $\prec$ given by

$$X \prec Y \iff \alpha(X) < \alpha(Y)$$

is a partial order on $p\mathscr{M}_k$.

Determine a minimal covering

Theorem

Let $O : p\mathscr{M}_k \to 2^{p\mathscr{M}_k}$ *with*

- $O(X) = \emptyset$, *if* $X \in p\mathscr{M}_k$ *belongs to every minimal covering, i.e.*
$$\exists f \in X \; \forall Y \in p\mathscr{M}_k \setminus \{X\} : f \notin Y,$$
- *and*
$$\forall X \in p\mathscr{M}_k \; \forall Y \in O(X) : Y \prec X.$$

Then

$$\mathscr{X} := \{X \in p\mathscr{M}_k | O(X) = \emptyset\}$$

is the unique minimal covering of $p\mathscr{M}_k$.

Some elements of every minimal covering of $p\mathscr{M}_k$

Let $X = pPOL_k\varrho \in p\mathscr{M}_k$ and ϱ an h-ary coherent relation. Then X is in every minimal covering, if

- $\varrho = \sigma_1 \cup \sigma_2$ and there is $A \subset E_k$, $A \neq \emptyset$ with $\sigma_1 \subset A^h$ and $\sigma_2 \subset E_k^h \setminus A^h$,
- $\varrho \in \left\{\iota_k^3, \varrho_1, \varrho_2\right\}$ with

$$\begin{aligned}
\iota_k^h &= \left\{(x_1, \dots, x_h) \in E_k^h \;\middle|\; |\{x_1, \dots, x_h\}| \leq h-1\right\}, \\
R_1 &= \{(a,a,b,b), (a,b,a,b), (a,b,b,a) \mid a,b \in E_k\}, \\
R_2 &= \{(a,a,b,b), (a,b,a,b) \mid a,b \in E_k\}.
\end{aligned}$$

- $\ldots$

Maximal clones not in every minimal covering of $p\mathscr{M}_k$

Let $X = pPOL_k\varrho \in p\mathscr{M}_k$ and ϱ an h-ary coherent relation. Then X is not every minimal covering, if

- $h = 2$ and the transitive closure of ϱ is a partial order with a central element, i.e.

$$\exists c \in E_k \, \forall x \in E_k : (x, c) \in \varrho \vee (c, x) \in \varrho$$

- $\sigma \cup \iota_k^h$ for $h \geq 4$

$$\iota_k^h := \left\{ (x_0, \ldots, x_{h-1}) \in E_k^h \middle| |\{x_0, \ldots, x_{h-1}\}| \leq h-1 \right\}$$

- $\ldots$

I.e. if there is only one minimal covering, then these clones are not in this minimal covering.

A binary partial Sheffer function for $k \geq 3$

Lemma

The function f_k defined by

$$f_k(x, y) := \begin{cases} x + 1 \bmod k & \textit{if } x = y, \\ x + 2 \bmod k & \textit{if } x \neq 0 \textit{ and } y = 0, \\ 0 & \textit{if } x = 0 \textit{ and } y \neq 0, \\ 0 & \textit{if } x \neq k - 1 \textit{ and } y = x + 1, \\ \infty & \textit{otherwise} \end{cases}$$

is a partial Sheffer function for k with $k \geq 3$.

What to do next?

- generalize the results for all $k \geq 3$
- a formula for the number of maximal partial clones for a given k
- determine generating sets (or the cardinality of these) for maximal partial clones
- …

The End

Thank you for your attention.