Uniqueness of minimal coverings of maximal partial clones

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Outline

Aim

Show that there is a unique minimal covering of the maximal partial clones for each k-valued logic with $k \geq 2$.

- $k = 2$ solved by Haddad and Rosenberg in 1991
  (4 maximal clones in the minimal covering out of 8)
- $k = 3$ solved by Haddad and Lau in 2006
  (26 out of 58)
- $k = 4$ solved by the speaker in 2008
  (449 out of 1102)

Some sets

Definition

$$\begin{aligned} E_k &:= \{0, 1, \ldots, k-1\} \\ P_k &:= \left\{ f \middle| f^{(n)} : E_k^n \to E_k, n \geq 1 \right\} \end{aligned}$$

Let $D \subseteq E_k^n$, $n \geq 1$ and $f^{(n)} : D \to E_k$. Then f is called a n-ary partial function on E_k with domain D. We also write $\mathrm{dom}(f) = D$. Let $\widetilde{P}_k$ be the set of all n-ary partial functions on E_k with $n \geq 1$.

Partial clones

Definition

The set $A \subseteq \widetilde{P}_k$ is a partial clone iff it is closed under composition and contains all projections.

The composition $f[g_1, \ldots, g_n] \in \widetilde{P}_k^{(m)}$ with $f \in \widetilde{P}_k^{(n)}$ and $g_1, \ldots, g_n \in \widetilde{P}_k^{(m)}$ is defined by

$$f[g_1, \ldots, g_n](\mathbf{x}) := \begin{cases} f(g_1(\mathbf{x}), \ldots, g_n(\mathbf{x})) & \\ & \text{if } \mathbf{x} \in \bigcap_{i=1}^{n} \mathrm{dom}(g_i), \\ \text{not defined} & \text{otherwise.} \end{cases}$$

Sheffer function

Definition

A partial function f is called a partial Sheffer function, if

$$[\{f\}] = \widetilde{P}_k.$$

Maximal partial clones

Definition

A clone $A \neq \widetilde{P}_k$ is called maximal, if there is no clone A' with

$$A \subset A' \subset \widetilde{P}_k.$$

Let $p\mathscr{M}_k$ be the set of all maximal partial clones.

Because for every partial clone $A \subset \widetilde{P}_k$ there is some maximal partial clone M_A with $A \subseteq M_A$ it holds

$$f \text{ is Sheffer} \iff \forall X \in p\mathscr{M}_k : f \notin X.$$

Preservation of relations

A (partial) function $f^{(n)} \in \widetilde{P}_k$ *preserves* the relation $\varrho \subseteq E_k^h$, if for all $\mathbf{r}_{*1}, \ldots, \mathbf{r}_{*n}$ with $\mathbf{r}_{*j} = (r_{1j}, \ldots, r_{hj})^{\mathrm{T}} \in \varrho$ and $\mathbf{r}_{i*} = (r_{i1}, \ldots, r_{in}) \in \mathrm{dom}(f)$ holds:

$$f(\mathbf{r}_{*1}, \ldots, \mathbf{r}_{*n}) := \begin{pmatrix} f(r_{11}, r_{12}, \ldots, r_{1n}) \\ f(r_{21}, r_{22}, \ldots, r_{2n}) \\ \vdots \\ f(r_{h1}, r_{h2}, \ldots, r_{hn}) \end{pmatrix} \in \varrho.$$

Short: $f \in pPOL_k\varrho$.

Haddad-Rosenberg Theorem [Haddad, Rosenberg, 1989, 1992]

Theorem

If C is a maximal partial clone of $\widetilde{P}_k$, then

$$C = P_k \cup \{f \in \widetilde{P}_k \mid \mathrm{dom}(f) = \emptyset\}$$

or

$$C = pPOL_k \varrho$$

for some relation $\varrho \in \widetilde{R}_k^{\mathrm{max}}$.

The relations in $\widetilde{R}_k^{\mathrm{max}}$ describe only maximal partial clones. The description of these given by Haddad and Rosenberg is quite complex and not needed here.

Partition of $\widetilde{R}_k^{\max}$ with a coarse description

$\widetilde{R}_k^{\max} = \mathcal{U} \cup \mathcal{A} \cup \mathcal{Q} \cup \mathcal{S} \cup \mathcal{L}$

- $\mathcal{U}$: unary relations ($\varrho \in \mathcal{U} \iff \emptyset \subset \varrho \subset E_k$),
- $\mathcal{A}$: areflexive relations,
- $\mathcal{Q}$: quasi-diagonal relations, i.e. if $\varrho \in \mathcal{Q}$ then $\varrho = \sigma \cup \delta_\varepsilon$ with σ areflexive and ε a non-trivial equivalence relation.
    - $\mathcal{Q}_0$: ε has no singular equivalence classes,
    - $\mathcal{Q}_1$: ε has at least one singular equivalence class,
- $\mathcal{S}$: totally reflexive, totally symmetric relations,
- $\mathcal{L}$: quaternary relations.

Let $\hat{\mathcal{U}} := \{pPOL_k\varrho \mid \varrho \in \mathcal{U}\}, \ldots$

$$\delta_\varepsilon^{(h)} := \{(x_1, \ldots, x_h) \in E_k^h \mid (i,j) \in \varepsilon \Longrightarrow x_i = x_j\}$$

Minimal covering

Definition

A subset $\hat{\mathscr{X}} \subseteq p\mathscr{M}_k$ is a minimal covering if

$$\forall f \in \widetilde{P}_k : ([\{f\}] = \widetilde{P}_k \iff \forall A \in \hat{\mathscr{X}} : f \notin A)$$

$$\forall A \in \hat{\mathscr{X}} \; \exists f \in A \; \forall B \in \hat{\mathscr{X}} \setminus \{A\} : f \notin B$$

Lemma

Let $X \in p\mathscr{M}_k$. Then X is in every minimal covering of $p\mathscr{M}_k$ iff there is some $f \in X$ with

$$\forall Z \in p\mathscr{M}_k \setminus \{X\} : f \notin Z.$$

Definition

Let $X, Y \in p\mathscr{M}_k$. Then $X \ll Y$ iff for every $f \in X$ there is some $g \in X$ with $g \notin Y$ and

$$\forall Z \in p\mathscr{M}_k : (f \notin Z \implies g \notin Z).$$

Let $\hat{\mathscr{X}}, \hat{\mathscr{Y}} \subseteq p\mathscr{M}_k$. Then $\hat{\mathscr{X}} \ll \hat{\mathscr{Y}}$ iff $X \ll Y$ for all $X \in \hat{\mathscr{X}}, Y \in \hat{\mathscr{Y}}, X \neq Y$.

Example for $\hat{\mathcal{A}} \ll \hat{\mathcal{Q}}_0$

$$k = 2$$

$$\varrho = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathcal{A} \text{ (areflexive)}$$

$$f \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\chi = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \in \mathcal{Q}_0 \text{ (quasi-diagonal)}$$

$$g \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{array} \right) = \left(\begin{array}{c} 0 \\ 1 \\ 1 \\ \hline 1 \\ 0 \end{array} \right)$$

The areflexive relations

Theorem

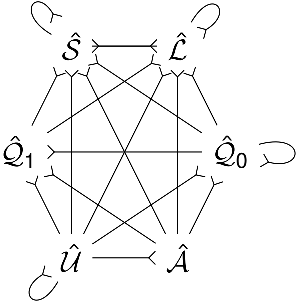
Let $\varrho^{(h)} \in \mathcal{A}$. Then $pPOL_k\varrho$ is in every minimal covering of $p\mathscr{M}_k$ if and only if there is some $\varphi \in \mathrm{Pol}_k^{(1)}\varrho$ with

$$\begin{aligned}&\forall 0 < l < h\, \forall D \subseteq E_k^l \setminus \iota_k^l\, \forall v \in E_k^{h-l} \setminus \iota_k^{h-l}\, \forall \pi \in S_h\\ &\exists m \geq 0 : D \times \{\varphi^m(v)\} \not\subseteq \varrho^{(\pi)}.\end{aligned} \tag{1}$$

If (1) *is false, then $pPOL_k\varrho$ is in no minimal covering of $p\mathscr{M}_k$.*

- $\iota_k^h := \{(v_1, v_2, \ldots, v_h) \in E_k^h \mid |\{v_1, v_2, \ldots, v_h\}| \leq h-1\}$,
- S_h the set of all permutations on E_h,
- $\varphi^0 = \mathrm{id}$, $\varphi^m(v) = \varphi(\varphi^{m-1}(v))$ for $m \geq 1$,
- $\varrho^{(\pi)} = \{(x_{\pi(0)}, \ldots, x_{\pi(h-1)}) \mid (x_0, \ldots, x_{h-1}) \in \varrho\}$.

	$\hat{\mathcal{U}}$	$\hat{\mathcal{A}}$	$\hat{\mathcal{Q}}_0$	$\hat{\mathcal{Q}}_1$	$\hat{\mathcal{S}}$	$\hat{\mathcal{L}}$
$\hat{\mathcal{U}}$	$\ll$	$\ll$	$\ll$	$\ll$	$\ll$	$\ll$
$\hat{\mathcal{A}}$			$\ll$	$\ll$	$\ll$	$\ll$
$\hat{\mathcal{Q}}_0$			$\ll$	$\ll$	$\ll$	$\ll$
$\hat{\mathcal{Q}}_1$					$\ll$	$\ll$
$\hat{\mathcal{S}}$					$\ll$	$\ll$
$\hat{\mathcal{L}}$					$\ll$	$\ll$



The preceding theorem and the table imply

Lemma

Let $\hat{\mathcal{X}}, \hat{\mathcal{Y}} \subseteq p\mathscr{M}_k$ be minimal coverings of $p\mathscr{M}_k$. Then

$$\hat{\mathcal{X}} \cap (p\mathscr{M}_k \setminus \hat{\mathcal{Q}}_1) = \hat{\mathcal{Y}} \cap (p\mathscr{M}_k \setminus \hat{\mathcal{Q}}_1).$$

Tackling $\hat{\mathscr{Q}}_1$

- define partial order $\prec$ on $\hat{\mathscr{Q}}_1$ similar to $\ll$
- $\prec$ implies equivalence relation $\sim$ such that for any minimal coverings $\hat{\mathscr{X}}, \hat{\mathscr{Y}}$ holds

$$\forall C \in \hat{\mathscr{X}} \cap \hat{\mathscr{Q}}_1 \, \forall f \in C \, \exists C' \in \hat{\mathscr{Y}} \cap \hat{\mathscr{Q}}_1 : f \in C' \wedge C \sim C'$$

- the equivalence classes can now be checked individually

Lemma

Every $C \in \hat{\mathscr{Q}}_1$ is either in every minimal covering of $p\mathscr{M}_k$ or none of them.

Uniqueness

Theorem

Let $\hat{\mathscr{X}}, \hat{\mathscr{Y}} \subseteq p\mathscr{M}_k$ be minimal coverings of $p\mathscr{M}_k$. Then

$$\hat{\mathscr{X}} = \hat{\mathscr{Y}},$$

i.e. there is a unique minimal covering of $p\mathscr{M}_k$ for each $k \geq 2$.

The End

Thank you for your attention.