A Galois connection for partial hyperfunctions

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Outline

Aim

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Give an approach for a Galois connection for partial hyperfunctions and show

- where it works,
- and where it fails.

Why?

The usual preservation of relations for partial functions does not suffice to describe all partial clones, i.e. does not lead to a Galois connection for partial clones

Hyperfunctions

Definition

Let A be a fixed finite set and $\mathcal{P}(A)$ the powerset of A. Let

$$\begin{aligned} \mathscr{H}_A^{(n)} &:= \{f : A^n \to \mathcal{P}(A)\}, \\ \mathscr{H}_A &:= \bigcup_{n \geq 1} \mathscr{H}_A^{(n)}. \end{aligned}$$

$\mathscr{H}_A$ is the set of all partial hyperfunctions on A.

The set of all total functions is
$\mathscr{O}_A = \{f^{(n)} \in \mathscr{H}_A \mid \forall x \in A^n : |f(\mathbf{x})| = 1\}$.
The set of all partial functions is
$\mathscr{P}_A = \{f^{(n)} \in \mathscr{H}_A \mid \forall x \in A^n : |f(\mathbf{x})| \leq 1\}$
with $\operatorname{dom} f := \{\mathbf{x} \mid f(\mathbf{x}) \neq \emptyset\}$.

Lifting

Definition

Let $f^{(n)} \in \mathscr{H}_A^{(n)}$. Then the lifting of f is the function $\hat{f} : \mathcal{P}(A)^n \to \mathcal{P}(A)$ defined by

$$\hat{f}(X_1, X_2, \ldots, X_n) := \bigcup_{(x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \cdots \times X_n} f(x_1, x_2, \ldots, x_n).$$

Clones

Definition

The set $C \subseteq \mathscr{H}_A$ is a clone iff it is closed und composition and contains all projections $e_i^{(n)}$.

The composition $f[g_1, \ldots, g_n] \in \mathscr{H}_A^{(m)}$ with $f \in \mathscr{H}_A^{(n)}$ and $g_1, \ldots, g_n \in \mathscr{H}_A^{(m)}$ is defined by

$$f[g_1, \ldots, g_n](\mathbf{x}) := \hat{f}(g_1(\mathbf{x}), \ldots, g_n(\mathbf{x})).$$

If additionally $C \subseteq \mathscr{O}_A$ or $C \subseteq \mathscr{P}_A$ then C is a (normal) clone of $\mathscr{O}_A$ resp. partial clone of $\mathscr{P}_A$.

Definition

Let

$$\mathscr{R}_A^{(h)} := \{\varrho \mid \varrho \subseteq \mathcal{P}(A)^h\}$$
$$\mathscr{R}_A := \bigcup_{h\geq 0} \mathscr{R}_A^{(h)}$$

be the set of all hyperrelations.

Elementary Operations on $\mathscr{R}_A$

Let $\varrho^{(h)}, \sigma^{(\mu)} \in \mathscr{R}_A$. Then let

$$
\begin{aligned}
\zeta\varrho &:= \{(x_2, x_3, \ldots, x_h, x_1) \mid (x_1, x_2, \ldots, x_h) \in \varrho\} \\
\tau\varrho &:= \{(x_2, x_1, x_3, \ldots, x_h) \mid (x_1, x_2, \ldots, x_h) \in \varrho\} \\
\operatorname{pr}\varrho &:= \{(x_2, x_3, \ldots, x_h) \mid \exists x_1 \in \mathcal{P}(A) : (x_1, x_2, \ldots, x_h) \in \varrho\} \\
\varrho \times \sigma &:= \{(x_1, \ldots, x_h, y_1, \ldots, y_\mu) \mid \\
&\qquad (x_1, \ldots, x_h) \in \varrho \wedge (y_1, \ldots, y_\mu) \in \sigma\} \\
\varrho \wedge \sigma &:= \varrho \cap \sigma \\
\delta^{(3)}_{\{1,2\}} &:= \{(x_1, x_2, x_3) \in \mathcal{P}(A)^3 \mid x_1 = x_2\}
\end{aligned}
$$

The set $Q \subseteq \mathscr{R}_A$ is a co-clone of $\mathscr{R}_A$ iff it is closed under $\zeta, \tau, \operatorname{pr}, \times, \wedge$ and contains $\delta^{(3)}_{\{1,2\}}$.

Definition

A function $f^{(n)} \in \mathscr{H}_A^{(n)}$ preserves a relation $\varrho^{(h)} \in \mathscr{R}_A^{(h)}$ iff

$$\hat{f}(r_1, \ldots, r_n) = \begin{pmatrix} \hat{f}(r_{11}, \ldots, r_{n1}) \\ \ldots \\ \hat{f}(r_{1h}, \ldots, r_{nh}) \end{pmatrix} \in \varrho$$

for all $r_i = (r_{i1}, \ldots, r_{ih}) \in \varrho$.
Written: $f \in \operatorname{Pol} \varrho$ and $\varrho \in \operatorname{Inv} f$.

Strict relations

Lemma

Let C be a clone and $\varrho^{(h)} \in \operatorname{Inv} C$. Let $r = (r_1, \ldots, r_h) \in \varrho$ and $I := \{i \in \{1, \ldots, h\} \mid r_i = \emptyset\}$.
Then for each $s = (s_1, \ldots, s_h) \in \varrho$ there is some $t = (t_1, \ldots, t_h) \in \varrho$ with $t_i = \begin{cases} \emptyset & \text{if } i \in I, \\ s_i & \text{otherwise.} \end{cases}$

Proof.

Take the projection $e_1^{(2)} \in C$, $r, s \in \operatorname{Inv} C$. Then $\hat{e}_1^{(2)}(s, r) = t$, i.e. $t \in \varrho$. □

Definition

Call $\varrho \in \mathscr{R}_A$ a strict relation iff $\varrho \in \operatorname{Inv} e_1^{(2)}$, i.e. is an invariant for some clone. Let $\mathscr{S}_A := \{\varrho \in \mathscr{R}_A \mid \varrho \text{ is strict}\}$.

Some operations on $\mathscr{R}_A$

Let $\varrho^{(h)} \in \mathscr{R}_A$ and $f \in \mathrm{Pol}\,\varrho$. Let

$$\begin{aligned}
\nu\varrho &:= \varrho \cup \{(\emptyset, \ldots, \emptyset) \in \mathcal{P}(A)^h\} \\
\hat{\nu}\varrho &:= \varrho \cup \{(x_1, \ldots, x_h) \mid \exists i : x_i = \emptyset\} \\
\kappa\varrho &:= \{((y_1 \cup \cdots \cup y_l), x_2, \ldots, x_h) \mid \\
&\quad\; l \geq 1, \forall j : (y_j, x_2, \ldots, x_h) \in \varrho\}
\end{aligned}$$

Lemma

Let $\varrho^{(h)} \in \mathscr{R}_A$ *and* $f \in \mathrm{Pol}\,\varrho$. *Then* $f \in \mathrm{Pol}\{\nu\varrho, \hat{\nu}\varrho, \kappa\varrho\}$.

A derivable operation is ν_l defined by

$$\nu_l\varrho := \varrho \cup \{(\emptyset, \ldots, \emptyset, x_{l+1}, \ldots, x_h) \mid \exists x_1, \ldots, x_l : (x_1, \ldots, x_h) \in \varrho\}.$$

Examples

Let $A = \{0, 1\}$ and $\varrho = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then

$$\begin{aligned} \nu\varrho &= \begin{pmatrix} 0 & \emptyset \\ 1 & \emptyset \end{pmatrix} \\ \nu_1\nu_2\varrho &= \begin{pmatrix} 0 & 0 & \emptyset & \emptyset \\ 1 & \emptyset & 1 & \emptyset \end{pmatrix} \\ \hat{\nu}\varrho &= \begin{pmatrix} 0 & 0 & 1 & \emptyset & \emptyset & \emptyset \\ 1 & \emptyset & \emptyset & 0 & 1 & \emptyset \end{pmatrix} \end{aligned}$$

A Galois connection for partial functions

Let $\varrho^{(h)} \in \mathscr{R}_A$. Then ϱ is a $\mathscr{P}_A$-relation iff

- ϱ is strict, i.e. $\varrho \in \mathscr{S}_A$, and
- $\forall (q_1, \ldots, q_h) \in \varrho \, \forall i : |q_i| \leq 1$.

Let $\mathscr{S}_{A,\mathscr{P}}$ be the set of all $\mathscr{P}_A$-relations.
Let $Q \subseteq \mathscr{R}_A$ be a $\mathscr{P}_A$-co-clone iff

- Q is a co-clone,
- Q is closed with respect to ν and $\hat{\nu}$,
- $Q \subseteq \mathscr{S}_{A,\mathscr{P}}$,
- $\{\{x\} \mid x \in A\} \cup \{\emptyset\} \in Q$.

A Galois connection for partial functions

Theorem

Let $\mathbb{L}(\mathscr{P}_A)$ *(or* $\mathbb{L}(\mathscr{S}_{A,\mathscr{P}})$*) be the set of all clones (or* $\mathscr{P}_A$*-co-clones) of* $\mathscr{P}_A$ *(or* $\mathscr{S}_{A,\mathscr{P}}$*) respectively. Then the mappings* Inv $: \mathbb{L}(\mathscr{P}_A) \to \mathbb{L}(\mathscr{S}_{A,\mathscr{P}})$ *and* Pol $: \mathbb{L}(\mathscr{S}_{A,\mathscr{P}}) \to \mathbb{L}(\mathscr{P}_A)$ *are bijective mappings, which reverse the partial order* $\subseteq$*.*
In other words:
The lattices $(\mathbb{L}(\mathscr{P}_A), \subseteq)$ *and* $(\mathbb{L}(\mathscr{S}_{A,\mathscr{P}}), \subseteq)$ *are antiisomorphic.*

Why does it fail for hyperfunctions? (I)

Let $C_l := \{e_i^{(n)} \mid 1 \leq i \leq n\} \cup \{f \in \mathscr{H}_A \mid \forall x : |f(x)| \geq l\}$. Then C_l is a clone. Let $l = 2$ and $A = \{0, 1, 2\}$. Then $\varrho \in \operatorname{Inv} C_l$ with

$$\varrho = \{(0, 1, 2)\} \cup \{(x_1, x_2, x_3) \in \mathcal{P}(A)^3 \mid |x_i| \geq 2\}.$$

But $\operatorname{Pol} \sigma$ with $\sigma := \operatorname{pr}_1 \varrho = \{\{0\}\} \cup \{x \in \mathcal{P}(A)^3 \mid |x| \geq 2\}$ is not a clone.

Why does it fail for hyperfunctions? (II)

Define $f^{(2)}$ by $f(0,0) = \{1,2\}$, $f(1,1) = f(2,2) = 1$ and $f(x,y) = A$ otherwise. Then $f \in \operatorname{Pol}\sigma$ with $\sigma := \operatorname{pr}_1 \varrho = \{\{0\}\} \cup \{x \in \mathcal{P}(A)^3 \mid |x| \geq 2\}$.
But $g := \Delta f \notin \operatorname{Pol}\sigma$ because $g(0) = \{1,2\}$, $g(1) = g(2) = 1$ and thus $g(\{1,2\}) = \{1\} \notin \sigma$. Thus $\operatorname{Pol}\sigma$ is not a clone.

Lemma

The projection pr *can not be used as operation on the relations.*

Some fix!?

Let $\varrho^{(h)} \in \mathscr{R}_A$. Then ϱ is $\subset$-strict iff for all $(x_1, \ldots, x_h) \in \varrho$ and $i \in \{1, \ldots, h\}$

$$\emptyset \neq y \subseteq x_i$$

implies

$$(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_h) \in \varrho)$$

With $\subset$-strict relations the operation pr works,
but some clones can not be described!

Conclusion

We found a Galois connection for partial functions, but it does not work for hyperfunctions.

The End

Thank you for your attention.