Universität Rostock Traditio et Innovatio

Number of Maximal Partial Clones

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Aim

We want to determine all maximal partial clones on four, five and six element sets using a computer program.

Some sets

Definition

$$E_k := \{0, 1, \ldots, k-1\}$$
$$P_k := \left\{ f \,\middle|\, f^{(n)} : E_k^n \to E_k, n \geq 1 \right\}$$

Let $D \subseteq E_k^n$, $n \geq 1$ and $f^{(n)} : D \to E_k$. Then f is called a n-ary partial function on E_k with domain D. We also write $\mathrm{dom}(f) = D$. Let $\widetilde{P}_k$ be the set of all n-ary partial functions on E_k with $n \geq 1$.

Partial clones

Definition

The set $A \subseteq \widetilde{P}_k$ is a partial clone iff it is closed under composition and contains all projections.

The composition $f(g_1, \ldots, g_n) \in \widetilde{P}_k^{(m)}$ with $f \in \widetilde{P}_k^{(n)}$ and $g_1, \ldots, g_n \in \widetilde{P}_k^{(m)}$ is defined by

$$f(g_1, \ldots, g_n)(\mathbf{x}) := \begin{cases} f(g_1(\mathbf{x}), \ldots, g_n(\mathbf{x})) & \text{if } \mathbf{x} \in \bigcap_{i=1}^{n} \mathrm{dom}(g_i) \text{ and} \\ & (g_1(\mathbf{x}), \ldots, g_n(\mathbf{x})) \in \mathrm{dom}(f), \\ \text{not defined} & \text{otherwise.} \end{cases}$$

Maximal partial clones

Definition

A clone $A \neq \widetilde{P}_k$ is called maximal, if there is no clone A' with

$$A \subset A' \subset \widetilde{P}_k.$$

Let $p\mathscr{M}_k$ be the set of all maximal partial clones.

Preservation of relations

A (partial) function $f^{(n)} \in \widetilde{P}_k$ *preserves* the relation $\varrho \subseteq E_k^h$, if for all $\mathbf{r}_{*1}, \ldots, \mathbf{r}_{*n}$ with $\mathbf{r}_{*j} = (r_{1j}, \ldots, r_{hj})^{\mathrm{T}} \in \varrho$ and $\mathbf{r}_{i*} = (r_{i1}, \ldots, r_{in}) \in \mathrm{dom}(f)$ holds:

$$f(\mathbf{r}_{*1}, \ldots, \mathbf{r}_{*n}) := \begin{pmatrix} f(r_{11}, r_{12}, \ldots, r_{1n}) \\ f(r_{21}, r_{22}, \ldots, r_{2n}) \\ \vdots \\ f(r_{h1}, r_{h2}, \ldots, r_{hn}) \end{pmatrix} \in \varrho.$$

Short: $f \in pPOL_k\varrho$.

Haddad-Rosenberg Theorem (1989, 1992)

Theorem

If C is a maximal partial clone of $\widetilde{P}_k$, then

$$C = P_k \cup \{f \in \widetilde{P}_k \mid \mathrm{dom}(f) = \emptyset\}$$

or

$$C = pPOL_k\varrho$$

for some relation $\varrho \in \widetilde{R}_k^{\max}$.

The relations in $\widetilde{R}_k^{\max}$ describe only maximal partial clones. The description of these given by Haddad and Rosenberg is quite complex and not needed here.

What to do?

First try

Idea

Determine all relations in $\widetilde{R}_k^{max}$ and count how many different maximal partial clones exist.

Problem

- *The list of coherent relations $\widetilde{R}_k^{max}$ contains many different relations which describe the same maximal partial clone.*
  Can we avoid listing them twice?
- *There are maximal partial clones C, C' which are isomorphic, i.e., there is some bijection φ on E_k with $C' = \varphi(C)$.*
  Do we need to list them seperately?
- *The search tree involved in the generation is HUGE!*

What to do?

Second try

Idea

- *Combine isomorphic clones together into one class and generate the canonical representatives*
- *Use a backtracking algorithm on the search tree with a good test to eliminate many subtrees which can not contain canonical representatives*
- *The number of different clones in one class is generated efficiently on the way*

Canonical form of relations

A partial order on $\widetilde{R}_k^{\max}$

Define a partial order $\prec$ on $\widetilde{R}_k^{\max}$ as the lexicographical order on relations.

For example

$$\begin{pmatrix} 0 & 4 \\ 1 & 5 \\ 2 & 6 \end{pmatrix} \prec \begin{pmatrix} 0 & 4 \\ 1 & 6 \\ 2 & 5 \end{pmatrix} \prec \begin{pmatrix} 0 & 4 \\ 1 & 5 \\ 3 & 6 \end{pmatrix} \prec \begin{pmatrix} 0 & 4 & 4 \\ 1 & 5 & 6 \\ 3 & 6 & 5 \end{pmatrix}$$

Canonical form of relations

Relation-Class and Quasi-Relation-Class

Definition

The quasi-relation-class $\mathrm{qclass}(\varrho)$ is the set of relations generated by ϱ through mapping by all bijections on E_k.
The relation-class $\mathrm{class}(\varrho)$ is the set of relations generated by ϱ through permutation of rows and mapping by all bijections on E_k.

Definition

The relation ϱ is quasi-canonical if $\varrho = \min_{\prec} \mathrm{qclass}(\varrho)$ and ϱ is canonical if $\varrho = \min_{\prec} \mathrm{class}(\varrho)$.

Canonical form of relations

Example

Let

$$\varrho_0 := \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \quad \varrho_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 2 \end{pmatrix} \quad \varrho_2 := \begin{pmatrix} 0 & 0 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}$$

Then ϱ_1 is the quasi-canonical form of ϱ_0 and ϱ_2 is the canonical form of both ϱ_0 and ϱ_1.

Canonical form of relations

Different relations – Different clones

Theorem

Let $\varrho, \sigma \in \widetilde{R}_k^{\max}$ *with* $pPOL_k\varrho = pPOL_k\sigma$.
Then $\sigma = \varrho'$ *where* ϱ' *is generated from* ϱ *by permuting rows.*

Lemma

Let $\varrho, \sigma \in \widetilde{R}_k^{\max}$ *different canonical relations.*
Then $\mathrm{pPOL}_k\, \varrho$ *is not isomorphic to* $\mathrm{pPOL}_k\, \sigma$.

Canonical form of relations

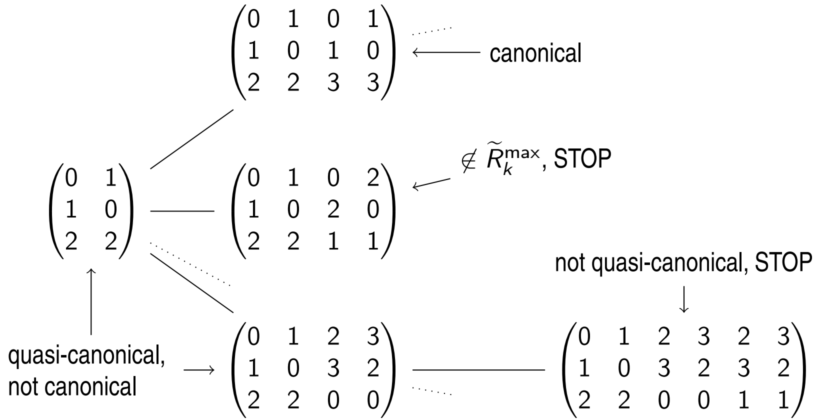
Lemma

Every canonical relation is also quasi-canonical.

Lemma

Every h-ary canonical relation ϱ includes the tuple $(0, 1, \ldots, h-1)$.

Example of a search tree



Number of maximal (partial) clones

k	$\lvert\mathscr{M}_k\rvert$	$\lvert p\mathscr{M}_k\rvert$	$\lvert p\mathscr{M}_k^{\mathrm{C}}\rvert$	$\frac{\lvert p\mathscr{M}_k\rvert}{\lvert p\mathscr{M}_k^{\mathrm{C}}\rvert \cdot k!}$
2	5	8	7	0.57
3	18	58	26	0.37
4	82	1 102	138	0.33
5	643	325 722	3 287	0.82
6	15 182	5 242 621 816	7 322 017	0.99
7	7 848 984	?	?	$>$ 0.99?
8	549 761 933 169	?	?	$>$ 0.99?

$\lvert\mathscr{M}_k\rvert$	Number of maximal clones
$\lvert p\mathscr{M}_k\rvert$	Number of maximal partial clones
$\lvert p\mathscr{M}_k^{\mathrm{C}}\rvert$	Number of relation classes

Open questions

- Is there a formula for the number of maximal partial clones?
- The maximal partial clones are used in testing for completeness. Are there other criteria for completeness which do not use the complete list of maximal partial clones?
- For $k \geq 5$ certain quasi-diagonal relations describe most maximal partial clones. Is there a way to describe them more efficiently?

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