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Galois theory for partial clones on finite sets

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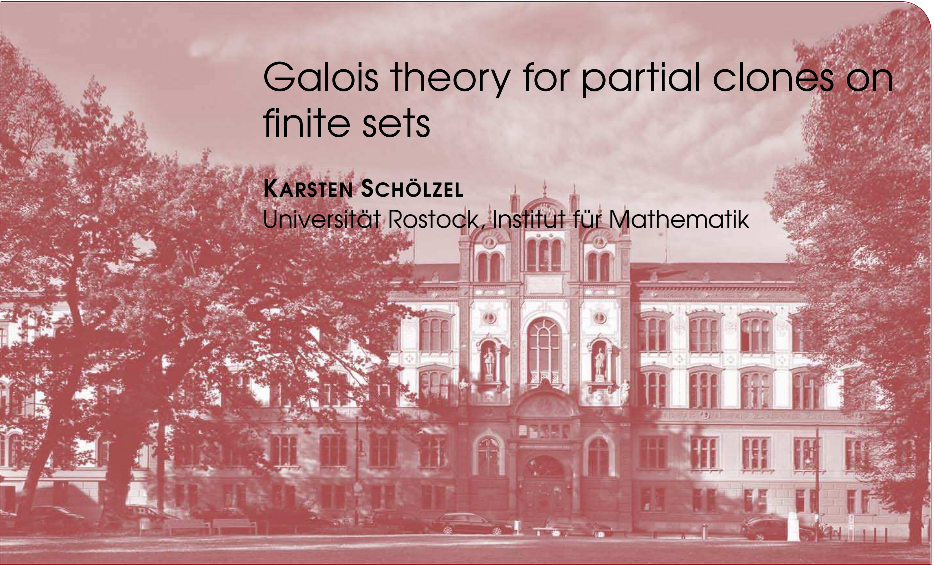

Aim

- Find a Galois connection between partial clones on finite sets and specific sets of relations (partial co-clones)
- Characterize the partial co-clones without use of the Galois connection
- Use the techniques known from the case of (total) clones and (total) co-clones

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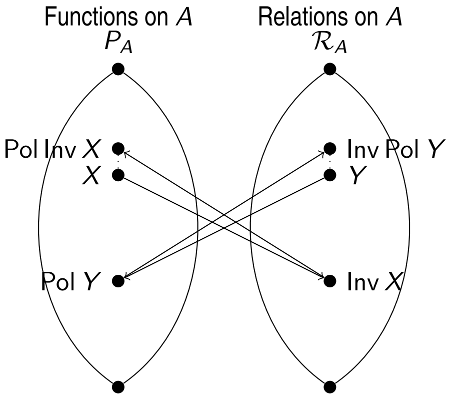
Total clones and co-clones



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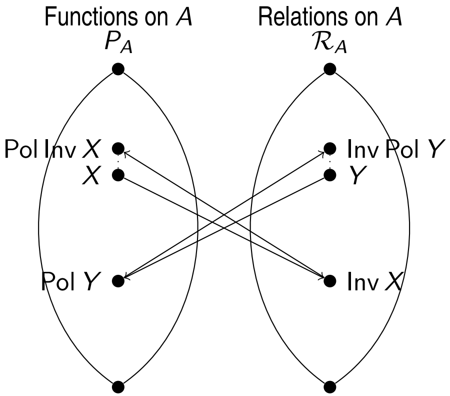

$\mathrm{Clone}(X) = \mathrm{Pol}\,\mathrm{Inv}\,X$ and $\mathrm{CoClone}(Y) = \mathrm{Inv}\,\mathrm{Pol}\,Y$

Partial clones I

Let A be a finite set and ∞ a symbol with $\infty \notin A$
Let $\widetilde{A} := A \cup \{\infty\}$
$f:\ A^n \to \widetilde{A}$ — n-ary partial function on A
$\widetilde{P}_A^{(n)}$ — set of all n-ary partial functions on A
$\widetilde{P}_A := \bigcup_{n\geq 1} \widetilde{P}_A^{(n)}$

Partial clones II

Definition (partial clone)

A set $C \subseteq \widetilde{P}_A$ which is closed with respect to the composition and contains the projections is called a partial clone.

projection: $e_i^{(n)}(x_1, \ldots, x_n) := x_i$
composition:

$$f(g_1, \ldots, g_n)(x_1, \ldots, x_m) := \begin{cases} f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m)) \\ \quad \text{if } g_i(x_1, \ldots, x_m) \neq \infty \text{ for all } i, \\ \infty \\ \quad \text{otherwise.} \end{cases}$$

Relations

$\varrho \subseteq \widetilde{A}^h$ — h-ary relation on $\widetilde{A}$
${\mathcal{R}^\star}_A^{(h)}$ — set of all h-ary relations on $\widetilde{A}$
${\mathcal{R}^\star}_A := \bigcup_{h \geq 1} {\mathcal{R}^\star}_A^{(h)}$

Preserving of relations

$f \in \widetilde{P}_A^{(n)}$ — n-ary function on A
$\varrho \in \mathcal{R}^{\star}{}_A^{(h)}$ — h-ary relation on $\widetilde{A}$

Definition ($f \rhd \varrho$ — f preserves ϱ)

$f \rhd \varrho$ holds iff for all tuples $r_i = (r_{1i}, r_{2i}, \ldots, r_{hi}) \in \varrho$ with $i \in \{1, 2, \ldots, n\}$ the statement

$$f(r_1, \ldots, r_n) := \begin{pmatrix} f(r_{11}, \ldots, r_{1n}) \\ f(r_{21}, \ldots, r_{2n}) \\ \ldots \\ f(r_{h1}, \ldots, r_{hn}) \end{pmatrix} \in \varrho$$

holds.

pPol — pInv

$X \subseteq \widetilde{P}_A$, $Y \subseteq \mathcal{R}^{\star}{}_A$

$\mathrm{pPol}_A\, Y := \{f \in \widetilde{P}_A \mid \forall \varrho \in Y : f \rhd \varrho\}$

$\mathrm{pInv}_A\, X := \{\varrho \in \mathcal{R}^{\star}{}_A \mid \forall f \in X : f \rhd \varrho\}$

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Problem: $\mathrm{pPol}_A\; Y$ may be no partial clone.

Example

Let $\varrho = \begin{pmatrix} 0 & 2 \\ 1 & \infty \end{pmatrix}$. Because

$$e_1^{(2)} \begin{pmatrix} 0 & 2 \\ 1 & \infty \end{pmatrix} = \begin{pmatrix} e_1^{(2)}(0,2) \\ e_1^{(2)}(1,\infty) \end{pmatrix} = \begin{pmatrix} 0 \\ \infty \end{pmatrix} \notin \varrho$$

we have $e_1^{(2)} \notin \mathrm{pPol}_A\{\varrho\}$.

Solution I

Restrict the set of relations: $\widetilde{\mathcal{R}}_A := \mathrm{pInv}_A\{e_1^{(2)}\}$.
Redefine pPol and pInv:
$X \subseteq \widetilde{P}_A$, $Y \subseteq \widetilde{\mathcal{R}}_A$
$\mathrm{pPol}_A\, Y := \{f \in \widetilde{P}_A \mid \forall \varrho \in Y : f \rhd \varrho\}$
$\mathrm{pInv}_A\, X := \{\varrho \in \widetilde{\mathcal{R}}_A \mid \forall f \in X : f \rhd \varrho\}$

Lemma

Let $Y \subseteq \widetilde{\mathcal{R}}_A$. Then $\mathrm{pPol}_A\, Y$ is a partial clone.
For every partial clone X the statement $X = \mathrm{pPol}_A\, \mathrm{pInv}_A\, X$ holds.

Total co-clones

A set $R \subseteq \mathcal{R}_A$ which is closed with respect to the operations ζ, τ, pr, $\times$, $\wedge$, and contains δ is called (total) co-clone

$$\begin{aligned}
\zeta\varrho &:= \{(a_2, a_3, \ldots, a_h, a_1) \mid (a_1, a_2, \ldots, a_h) \in \varrho\} \\
\tau\varrho &:= \{(a_2, a_1, a_3, \ldots, a_h) \mid (a_1, a_2, \ldots, a_h) \in \varrho\} \\
\mathrm{pr}\,\varrho &:= \{(a_2, \ldots, a_h) \mid \exists a_1 \in A : (a_1, a_2, \ldots, a_h) \in \varrho\} \\
\varrho \times \sigma &:= \{(a_1, \ldots, a_h, b_1, \ldots, b_\mu) \mid (a_1, \ldots, a_h) \in \varrho, (b_1, \ldots, b_\mu) \in \sigma\} \\
\varrho \wedge \sigma &:= \varrho \cap \sigma \\
\delta &:= \{(a_1, a_2, a_3) \in A^3 \mid a_1 = a_2\}
\end{aligned}$$

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\end{aligned}$$

Problem for partial co-clones: There a $Y \subset \widetilde{\mathcal{R}}_A$ which are closed with respect to the operations ζ, τ, pr, $\times$, $\wedge$, and contain δ, but for which $Y \neq \mathrm{pInv}_A\,\mathrm{pPol}_A\,Y$ holds.

Partial co-clones — the new operation $\hat{\kappa}$

A set $R \subseteq \widetilde{\mathcal{R}}_A$ which is closed with respect to the operations ζ, τ, pr, $\times$, $\wedge$, $\hat{\kappa}$ and contains $\widetilde{\widetilde{\delta}}$ is called partial co-clone

$$\hat{\kappa}\varrho \quad := \quad \begin{cases} \varrho \cup \{\infty\}, \\ \qquad \text{if } \varrho = \emptyset \text{ or } \varrho \text{ is unary,} \\ \varrho \cup \{(a_1, a_2, a_3, \ldots, a_h) \in \widetilde{A}^h \mid a_1 = a_2 = \infty\}, \\ \qquad \text{if } \varrho \text{ is } h\text{-ary with } h \geq 2. \end{cases}$$

$$\widetilde{\delta} \quad := \quad \{(a_1, a_2, a_3) \in \widetilde{A}^3 \mid a_1 = a_2\}$$

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$$\widetilde{\delta} := \{(a_1, a_2, a_3) \in \widetilde{A}^3 \mid a_1 = a_2\}$$

Lemma

Let $X \subseteq \widetilde{P}_A$. Then $\mathrm{pInv}_A X$ is a partial co-clone.
For every partial co-clone Y the statement $Y = \mathrm{pPol}_A \, \mathrm{pInv}_A \, Y$ holds.

Galois connection

Theorem

Let $\mathbb{L}(\widetilde{P}_A)$ *be the set of all clones of* $\widetilde{P}_A$ *and let* $\mathbb{L}(\widetilde{\mathcal{R}}_A)$ *be the set of all co-clones of* $\widetilde{\mathcal{R}}_A$*. Then the mappings*

$$\mathrm{pInv}_A : \mathbb{L}(\widetilde{P}_A) \longrightarrow \mathbb{L}(\widetilde{\mathcal{R}}_A), X \mapsto \mathrm{pInv}_A X$$

and

$$\mathrm{pPol}_A : \mathbb{L}(\widetilde{\mathcal{R}}_A) \longrightarrow \mathbb{L}(\widetilde{P}_A), Y \mapsto \mathrm{pPol}_A Y$$

are bijective mappings, which reverse the partial order $\subseteq$.

Possible applications

- New, extended and more readable proof of the Definability Lemma of Romov (1981), which is essential for the classification of the maximal partial clones by Haddad and Rosenberg.
- Determining some intervals of boolean partial clones.
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Thank you for your attention.