Uniqueness of minimal coverings of maximal partial clones

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May 21, 2009

Outline

Coverings of finite family of sets

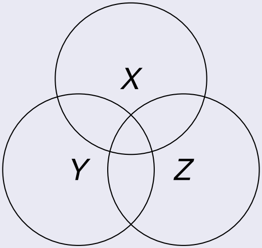
Definition

Let $\hat{\mathcal{C}} := \{C_1, \ldots, C_n\}$ with $C_i \subseteq P$ for some set P.
$\hat{\mathcal{X}} \subseteq \hat{\mathcal{C}}$ is a covering of $\hat{\mathcal{C}}$ if $\bigcup \hat{\mathcal{X}} = \bigcup \hat{\mathcal{C}} = \bigcup_{i=1}^{n} C_i$.
A covering $\hat{\mathcal{X}}$ is minimal if $\bigcup \hat{\mathcal{Y}} \subset \bigcup \hat{\mathcal{X}}$ for all $\hat{\mathcal{Y}} \subset \hat{\mathcal{X}}$.

Example

$\hat{\mathcal{C}} := \{X \setminus Z, Y \setminus Z, (X \cup Z) \setminus Y, (Y \cup Z) \setminus X\}$
$\hat{\mathcal{X}} := \{X \setminus Z, Y \setminus Z, (X \cup Z) \setminus Y\}$

$\hat{\mathcal{X}}$ is a minimal covering of $\hat{\mathcal{C}}$ because $\hat{\mathcal{X}} \subseteq \hat{\mathcal{C}}$ and $\bigcup \hat{\mathcal{X}} = X \cup Y \cup Z = \bigcup \hat{\mathcal{C}}$



The BIG Question

Is there a unique minimal covering for the set of maximal partial clones in k-valued logic?

Why is the BIG Question interesting?

- better criterion for Sheffer partial functions
- learn more about the structure of maximal partial clones

Maximal clones of boolean functions

An n-ary boolean function $f^{(n)}$ is a function $f^{(n)} : \{0,1\}^n \to \{0,1\}$. Let P_2 be the set of all finitary boolean functions. $C \subseteq P_2$ is a boolean clone if it is closed under composition and contains all projections.

The set $\hat{\mathcal{C}} = \{T_0, T_1, M, S, L\}$ is the set of maximal clones of P_2 with

- $T_0 := \{f \in P_2 \mid f(0,\ldots,0) = 0\}$,
- $T_1 := \{f \in P_2 \mid f(1,\ldots,1) = 1\}$,
- M are all monotone functions of P_2,
- S are all selfdual functions of P_2,
- L are all linear functions of P_2.

Maximal clones of boolean functions

Lemma (Minimal covering of boolean maximal clones)

$\hat{\mathcal{X}} = \{T_0, T_1, S\}$ *is the unique minimal covering of* $\hat{\mathcal{C}}$.

Covering.

If $f \notin \bigcup \hat{\mathcal{X}}$, then $f(0, \ldots, 0) = 1$, $f(1, \ldots, 1) = 0$ and there are $a_1, \ldots, a_n \in E_2$ with $f(a_1, \ldots, a_n) = f(\bar{a}_1, \ldots, \bar{a}_n)$. This implies $f \notin M$ and $f \notin L$ and thus $\bigcup \hat{\mathcal{X}} = \bigcup \hat{\mathcal{C}}$, i.e. $\hat{\mathcal{X}}$ is a covering of $\hat{\mathcal{C}}$.

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Minimal and unique.

Define f_0, f_1, f_S by

x	y	$f_0(x,y)$	$f_1(x,y)$
0	0	0	1
0	1	0	1
1	0	1	0
1	1	0	1

$$f_S(x,y,z) := \begin{cases} 0 & \text{if } (x,y,z) \text{ has at least 2 entries equal to 1,} \\ 1 & \text{otherwise.} \end{cases}$$

Then $f_0 \in T_0 \setminus (T_1 \cup M \cup S \cup L)$, $f_1 \in T_1 \setminus (T_0 \cup M \cup S \cup L)$, and $f_S \in S \setminus (T_0 \cup T_1 \cup M \cup L)$, i.e. $\hat{\mathcal{X}}$ is minimal and the only minimal covering of $\hat{\mathcal{C}}$. □

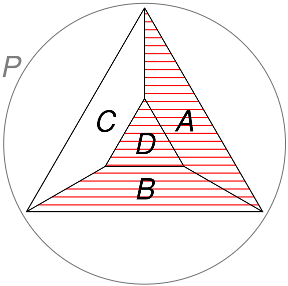
Aim

The BIG Question

Is there a unique minimal covering for the set of maximal partial clones in k-valued logic?

- $k = 2$: YES given by Haddad and Rosenberg in 1991 (4 maximal clones in the minimal covering out of 8)
- $k = 3$: YES given by Haddad and Lau in 2006 (26 out of 58)
- $k = 4$: YES given by Schölzel in 2008 (449 out of 1102)
- $k \geq 5$: *Left to show, but we get to it right away*

Why do we have to show uniqueness?



$X = A \cup B \cup D$
$Y = B \cup C \cup D$
$Z = C \cup A \cup D$
$\hat{\mathcal{C}} = \{X, Y, Z\}$
Minimal coverings of $\hat{\mathcal{C}}$:
$\{X, Y\}, \{X, Z\}, \{Y, Z\}$

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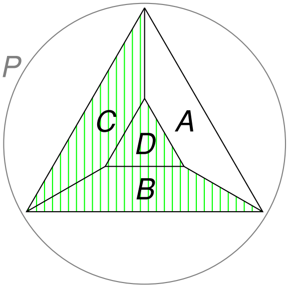

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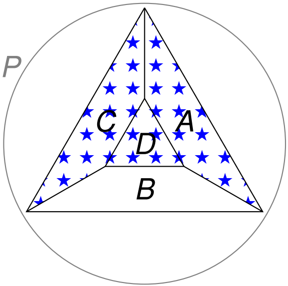

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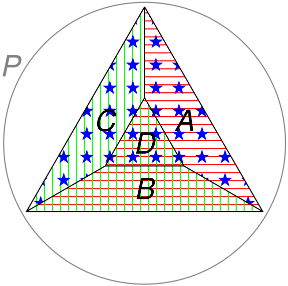

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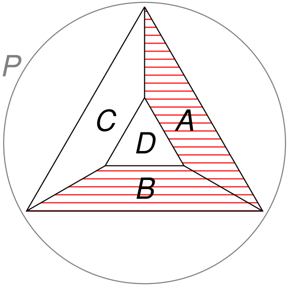

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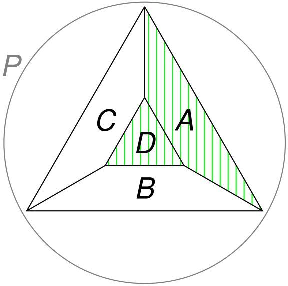

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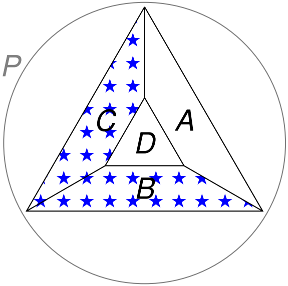

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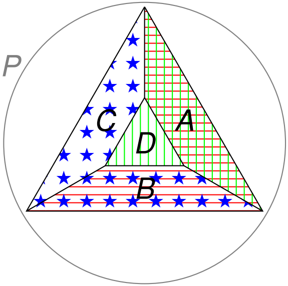

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Some sets

Definition

$$
\begin{aligned}
E_k &:= \{0, 1, \ldots, k-1\} \\
P_k &:= \left\{ f^{(n)} \middle| f^{(n)} : E_k^n \to E_k, n \geq 1 \right\}
\end{aligned}
$$

Let $D \subseteq E_k^n$, $n \geq 1$ and $f^{(n)} : D \to E_k$. Then f is called a n-ary partial function on E_k with domain D. We also write $\mathrm{dom}(f) = D$. Let $\widetilde{P}_k$ be the set of all n-ary partial functions on E_k with $n \geq 1$.

Uniqueness of minimal coverings of maximal partial clones

Schölzel

Introduction
The question for boolean functions
History for the maximal partial clones

Definitions

Uniqueness of minimal coverings

Result

Partial clones

Definition

The set $A \subseteq \widetilde{P}_k$ is a partial clone iff it is closed under composition and contains all projections.

The composition $f[g_1, \ldots, g_n] \in \widetilde{P}_k^{(m)}$ with $f \in \widetilde{P}_k^{(n)}$ and $g_1, \ldots, g_n \in \widetilde{P}_k^{(m)}$ is defined by

$$f[g_1, \ldots, g_n](\mathbf{x}) := \begin{cases} f(g_1(\mathbf{x}), \ldots, g_n(\mathbf{x})) & \text{if } \mathbf{x} \in \bigcap_{i=1}^{n} \mathrm{dom}(g_i), \\ \text{not defined} & \text{otherwise.} \end{cases}$$

Maximal partial clones

Definition

A clone $A \neq \widetilde{P}_k$ is called maximal, if there is no clone A' with

$$A \subset A' \subset \widetilde{P}_k.$$

Let $p\mathscr{M}_k$ be the set of all maximal partial clones.

For further investigations $p\mathscr{M}_k$ is split into

$$p\mathscr{M}_k = \hat{\mathcal{U}} \cup \hat{\mathcal{A}} \cup \hat{\mathcal{Q}}_0 \cup \hat{\mathcal{Q}}_1 \cup \hat{\mathcal{S}} \cup \hat{\mathcal{L}}.$$

Idea

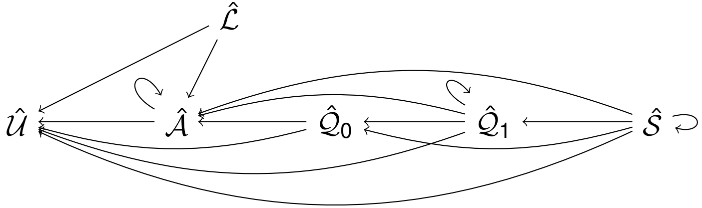
- Define directed graph $G = (V, E)$ with
  - $V = p\mathscr{M}_k$
  - $E \subseteq V^2$
  - $XY \notin E$ iff for every $f \in X$ there is some $g \in X$ with $g \notin Y$ and

$$\forall Z \in p\mathscr{M}_k \, (f \notin Z \Longrightarrow g \notin Z).$$

- show that G is acyclic
- if $X \in p\mathscr{M}_k$ is a sink in G then X is in every minimal covering of $p\mathscr{M}_k$
- X is covered by its successors in G, i.e. $X \subseteq \bigcup S(X)$ with $S(X) := \{Y \in p\mathscr{M}_k \mid XY \in E\}$

General shape of G

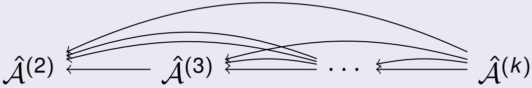
G is a subgraph of the following graph



$\hat{\mathcal{A}}$, $\hat{\mathcal{S}}$

Lemma

$G \cap \hat{\mathcal{A}} = (\hat{\mathcal{A}}, E_{\hat{\mathcal{A}}})$ with $E_{\hat{\mathcal{A}}} = \{XY \in E \mid X, Y \in \hat{\mathcal{A}}\}$ is a subgraph of



where $\hat{\mathcal{A}}^{(h)} := \{X \in \hat{\mathcal{A}} \mid \mathrm{ar}(X) = h\}$ for some function ar (arity of a correspnding relation given by the Theorem of Haddad and Rosenberg [1989,1992]).

Lemma

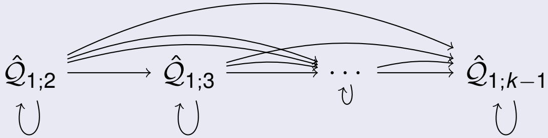
$G \cap \hat{\mathcal{S}}$ looks like $G \cap \hat{\mathcal{A}}$.

The beast $\hat{\mathcal{Q}}_1$ (Act 1)

Let $\operatorname{pp} X$ be a certain relation corresponding to $X \in \hat{\mathcal{Q}}_1$ and $\operatorname{ar}_{\hat{\mathcal{Q}}} X$ the arity of $\operatorname{pp} X$.

Lemma

$G \cap \hat{\mathcal{Q}}_1$ is a subgraph of



where $\hat{\mathcal{Q}}_{1;h} := \{X \in \hat{\mathcal{Q}}_1 \mid \operatorname{ar}_{\hat{\mathcal{Q}}}(X) = h\}$.

Lemma

$G \cap \hat{\mathcal{Q}}_{1;h}$ is a subgraph of $(\hat{\mathcal{Q}}_{1;h}, E')$ *with* $E' := \{XY \in (\hat{\mathcal{Q}}_{1;h})^2 \mid \operatorname{pp} Y \subseteq \operatorname{pp} X\}$.

The beast $\hat{\mathcal{Q}}_1$ (Act 2)

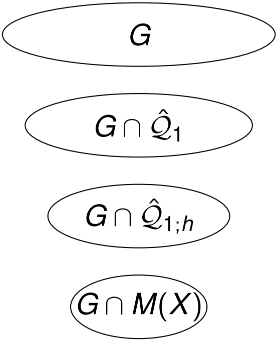
Let $X \in \hat{\mathcal{Q}}_1$ be arbitrary and $M(X) := \{Y \in \hat{\mathcal{Q}}_1 \mid \mathrm{pp}\, Y = \mathrm{pp}\, X\}$.

Lemma

$G \cap M(X)$ is acyclic.

Theorem (Conclusion)

$G \cap \hat{\mathcal{Q}}_1$ is acyclic.



Answering the BIG Question

The BIG Question

Is there a unique minimal covering for the set of maximal partial clones in k-valued logic?

Theorem (YES, it is unique!)

The sinks of G are in every minimal covering of $p\mathscr{M}_k$. The graph G is acyclic and therefore there is a unique minimal covering of $p\mathscr{M}_k$ for each $k \geq 2$.

BUT

The proof for uniqueness does not give all members of the unique minimal covering of $p\mathscr{M}_k$.

The End

Thank you for your attention.