The minimal covering of maximal partial clones in 4-valued logic

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Outline

Some sets

Definition

$$E_k := \{0, 1, \ldots, k-1\}$$

$$P_k := \left\{ f^{(n)} \middle| f^{(n)} : E_k^n \to E_k, n \geq 1 \right\}$$

Let $D \subseteq E_k^n$, $n \geq 1$ and $f^{(n)} : D \to E_k$. Then f is called a n-ary partial function on E_k with domain D. We also write $\mathrm{dom}(f) = D$. Let $\widetilde{P}_k$ be the set of all n-ary partial functions on E_k with $n \geq 1$.

Partial clones

Definition

The set $A \subseteq \widetilde{P}_k$ is a partial clone iff it is closed under composition and contains all projections.

The composition $f[g_1, \ldots, g_n] \in \widetilde{P}_k^{(m)}$ with $f \in \widetilde{P}_k^{(n)}$ and $g_1, \ldots, g_n \in \widetilde{P}_k^{(m)}$ is defined by

$$f[g_1, \ldots, g_n](\mathbf{x}) := \begin{cases} f(g_1(\mathbf{x}), \ldots, g_n(\mathbf{x})) & \\ & \text{if } \mathbf{x} \in \bigcap_{i=1}^{n} \mathrm{dom}(g_i), \\ \text{not defined} & \text{otherwise.} \end{cases}$$

Maximal partial clones

Definition

A clone $A \neq \widetilde{P}_k$ is called maximal, if there is no clone A' with

$$A \subset A' \subset \widetilde{P}_k.$$

Let $p\mathscr{M}_k$ be the set of all maximal partial clones.

Coverings of maximal partial clones

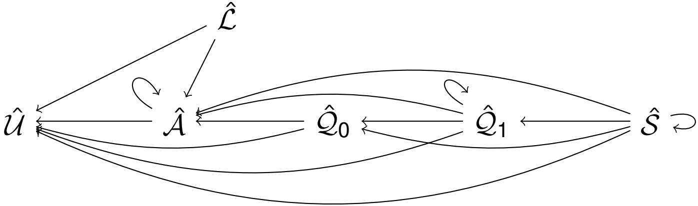
Let $\hat{\mathcal{C}} := \{C_i \mid i \in I\}$ with $C_i \subseteq P$ for some set P.
$\hat{\mathcal{X}} \subseteq \hat{\mathcal{C}}$ is a covering of $\hat{\mathcal{C}}$ if $\bigcup \hat{\mathcal{X}} = \bigcup \hat{\mathcal{C}}$.
A covering $\hat{\mathcal{X}}$ is minimal if $\bigcup \hat{\mathcal{Y}} \subset \bigcup \hat{\mathcal{X}}$ for all $\hat{\mathcal{Y}} \subset \hat{\mathcal{X}}$.

Theorem

There is a unique minimal covering of $p\mathscr{M}_k$ for each $k \geq 2$.

Aim

Show which clones represent sinks in the graph and which do not



- $k = 2$: determined by Haddad and Rosenberg in 1991 (4 maximal clones in the minimal covering out of 8)
- $k = 3$: determined by Haddad and Lau in 2006 (26 out of 58)

Preservation of relations

A (partial) function $f^{(n)} \in \widetilde{P}_k$ *preserves* the relation $\varrho \subseteq E_k^h$, if for all $\mathbf{r}_{*1}, \ldots, \mathbf{r}_{*n}$ with $\mathbf{r}_{*j} = (r_{1j}, \ldots, r_{hj})^{\mathrm{T}} \in \varrho$ and $\mathbf{r}_{i*} = (r_{i1}, \ldots, r_{in}) \in \mathrm{dom}(f)$ holds:

$$f(\mathbf{r}_{*1}, \ldots, \mathbf{r}_{*n}) := \begin{pmatrix} f(r_{11}, r_{12}, \ldots, r_{1n}) \\ f(r_{21}, r_{22}, \ldots, r_{2n}) \\ \vdots \\ f(r_{h1}, r_{h2}, \ldots, r_{hn}) \end{pmatrix} \in \varrho.$$

Short: $f \in pPOL_k\varrho$.

Haddad-Rosenberg Theorem [Haddad, Rosenberg, 1989, 1992]

Theorem

If C is a maximal partial clone of $\widetilde{P}_k$, then

$$C = P_k \cup \{f \in \widetilde{P}_k \mid \mathrm{dom}(f) = \emptyset\}$$

or

$$C = pPOL_k \varrho$$

for some relation $\varrho \in \widetilde{R}_k^{max}$.

The relations in $\widetilde{R}_k^{max}$ describe only maximal partial clones. The description of these given by Haddad and Rosenberg is quite complex and not needed here.

Partition of $\widetilde{R}_k^{\max}$ with a coarse description

$\widetilde{R}_k^{\max} = \mathcal{U} \cup \mathcal{A} \cup \mathcal{Q} \cup \mathcal{S} \cup \mathcal{L}$

- $\mathcal{U}$: unary relations ($\varrho \in \mathcal{U} \iff \emptyset \subset \varrho \subset E_k$),
- $\mathcal{A}$: areflexive relations,
- $\mathcal{Q}$: quasi-diagonal relations, i.e. if $\varrho \in \mathcal{Q}$ then $\varrho = \sigma \cup \delta_\varepsilon$ with σ areflexive and ε a non-trivial equivalence relation.
  - $\mathcal{Q}_0$: ε has no singular equivalence classes,
  - $\mathcal{Q}_1$: ε has at least one singular equivalence class,
- $\mathcal{S}$: non-trivial totally reflexive, totally symmetric relations,
- $\mathcal{L}$: special quaternary relations.

Let $\hat{\mathcal{U}} := \{pPOL_k\varrho \mid \varrho \in \mathcal{U}\}, \ldots$

A member of the minimal covering $\hat{\mathcal{X}}_4$ of $p\mathscr{M}_4$

Lemma

Let $\varrho := \begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} \in \mathcal{A}$*. Then* $C := pPOL_4\varrho \in \hat{\mathcal{X}}_4$*.*

Proof.

- show there is some $f \in C$ with $f \notin X$ for all $X \in p\mathscr{M}_4 \setminus \{C\}$,
- do it step by step:
  - find partial functions $f_0, \dots, f_l \in C$ such that for each $X \in p\mathscr{M}_4 \setminus \{C\}$ there is some f_i with $f_i \notin X$
  - generate f from a good combination of $f_0, \dots, f_l$
- start with a unary function f_0.

Doing it step by step — a product of partial functions

- $f^{(n)}, g^{(m)} \in \widetilde{P}_k$ and $D := \mathrm{dom}\, f$, $E := \mathrm{dom}\, g$
- D' is a matrix representing D (every row of D' is an element of D)
- E' is analogous, i.e. if $m = 3$, $E = \{(1,2,3),(2,3,0)\}$ then $E' = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \end{pmatrix}$
- $f \otimes g$ is an nm-ary function with $A := \mathrm{dom}(f \otimes g)$, $A' := \begin{pmatrix} D'_1 & \dots & D'_m \\ E' & \dots & E' \end{pmatrix}$ (D'_i is the i-th column of D') and $(f \otimes g)\begin{pmatrix} D'_1 & \dots & D'_m \\ E' & \dots & E' \end{pmatrix} = \begin{pmatrix} f(D') \\ g(E') \end{pmatrix}$

Doing it step by step — a product of partial functions

- $f \otimes g$ is an nm-ary function with $A := \mathrm{dom}(f \otimes g)$,
  $A' := \begin{pmatrix} D'_1 & \dots & D'_m \\ E' & \dots & E' \end{pmatrix}$ (D'_i is the i-th column of D')
  and $(f \otimes g) \begin{pmatrix} D'_1 & \dots & D'_m \\ E' & \dots & E' \end{pmatrix} = \begin{pmatrix} f(D') \\ g(E') \end{pmatrix}$
- $D' := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $f(D') = f \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} := \begin{pmatrix} 1 \\ 2 \end{pmatrix}$
- $E' := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \end{pmatrix}$, $f(E') := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
- $(f \otimes g) \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ \hline 1 & 2 & 3 & 1 & 2 & 3 \\ 2 & 3 & 0 & 2 & 3 & 0 \end{array}\right) = \left(\begin{array}{c} 1 \\ 2 \\ \hline 0 \\ 1 \end{array}\right)$

The function f_0

Proof.

The unary function f_0 should be defined such that $f_0 \notin pPOL_4\{x\}$ for all $x \in E_4$.

Define f_0 by

x	$f_0(x)$
0	2
1	3
2	0
3	1

Then $f_0 \in C$ and $f_0 \notin pPOL_4\{a\}$ for all $a \in E_4$.

The next step f_1

Proof.

Now pick a $X \in p\mathscr{M}_4$ with $f_0 \in X$ and $X \neq C$, e.g.

$X = pPOL_4\psi$ with $\psi = \begin{pmatrix} 0 & 2 & 0 & 1 & 2 & 3 \\ 1 & 3 & 0 & 1 & 2 & 3 \end{pmatrix}$

Define f_1 with $(\mathrm{dom}\, f_1)' = \begin{pmatrix} 0 & 2 & 0 & 1 & 2 & 3 \\ 1 & 3 & 0 & 1 & 2 & 3 \end{pmatrix}$ and

$f_1 \begin{pmatrix} 0 & 2 & 0 & 1 & 2 & 3 \\ 1 & 3 & 0 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

Then $f_1 \in C$ and $f_1 \notin X$.

Continue for all other $Y \in p\mathscr{M}_4$ and find $f_2, \ldots, f_l$.

Let $f := f_0 \otimes f_1 \otimes \cdots \otimes f_l$. Then $f \in C$ by construction. □

Not a member of the minimal covering $\hat{\mathcal{X}}_4$ of $p\mathscr{M}_4$

Lemma

Let $\varrho := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 3 & 3 & 3 \end{pmatrix} \cup \left\{ \begin{pmatrix} a \\ a \\ b \end{pmatrix} \,\middle|\, a, b \in E_4 \right\} \in \mathcal{Q}_1.$

Then $C := pPOL_4\varrho \notin \hat{\mathcal{X}}_4.$

Proof.

Let $\psi = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 2 & 2 \end{pmatrix} \cup \left\{ \begin{pmatrix} a \\ a \end{pmatrix} \,\middle|\, a \in E_4 \right\} \in \mathcal{Q}_2.$

Let $f^{(n)} \in C$ and assume $f \notin pPOL_4\{3\}$ and $f \notin pPOL_4\psi$.

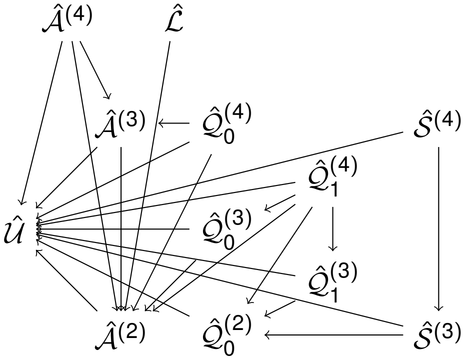
Then exist $x_1, \dots, x_n \in \{3\}$ with $x := f(x_1, \dots, x_n) \in E_4 \setminus \{3\}$ and $y_1, \dots, y_n \in \psi$ with $y := f(y_1, \dots, y_n) \in E_4^2 \setminus \psi$.

Then $\begin{pmatrix} y_i \\ x_i \end{pmatrix} \in \varrho$ for all i but $\begin{pmatrix} y \\ x \end{pmatrix} \notin \varrho$, i.e. $f \notin C$.

Contradiction. Thus $C \subseteq pPOL_4\{3\} \cup pPOL_4\psi$. □

Theorem

The minimal covering of $p\mathscr{M}_4$ has 449 elements and all the elements of the covering have been determined.



The End

Thank you for your attention.