Universität Rostock Traditio et Innovatio

A Classification of Partial Boolean Clones

Dietlinde Lau, Karsten Schölzel
Universität Rostock, Institut für Mathematik

Aim

Describe the structure of the lattice of partial boolean clones as far as possible using the known structure of total boolean clones, i.e. Post's lattice.

Partial functions

$E_2 := \{0, 1\}$
$\widetilde{E}_2 := \{0, 1, \infty\}$

Let $n \in \mathbb{N}$, $n \geq 1$ and $f^{(n)} : E_2^n \to \widetilde{E}_2$. Then $f := f^{(n)}$ is called an n-ary boolean function.

$\widetilde{P}_2$: the set of all partial boolean functions.

$C \subseteq \widetilde{P}_2$ is a partial clone if it is closed with respect to the superposition operations and contains the projections.

Intervals

Definition

Let $A \subseteq P_2$ be a (total) boolean clone. Then

$$\mathcal{I}_2(A) := \{C \subseteq \widetilde{P}_2 \mid C \text{ is a partial clone and } C \cap P_2 = A\}$$

is called the interval of A.

$\widetilde{P}_2$: the set of all partial boolean functions;
P_2: the set of all everywhere-defined boolean functions.

Example

The interval $\mathcal{I}_2(P_2) = \{P_2, P_2 \cup C_\infty, \widetilde{P}_2\}$ where C_∞ are all the functions with empty domain.

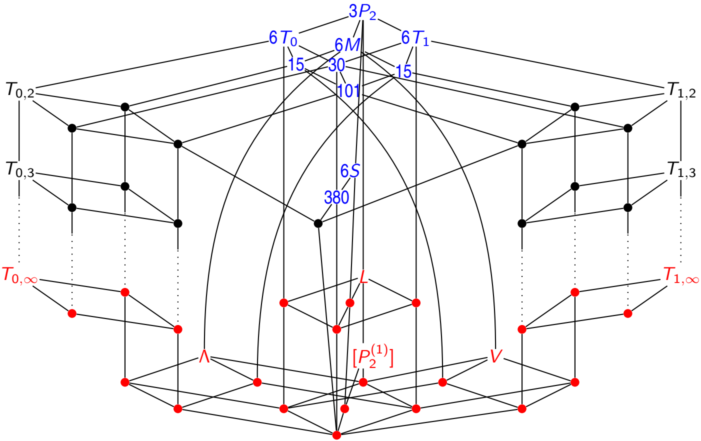
Introduction

Results

The last “interesting” interval: $\mathcal{I}_2\left(T_{0,2}\right)$?

Conclusions

Post's lattice



The finite intervals

Let $S \subseteq P_2$ be the set of all self-dual functions, $M \subseteq P_2$ the set of all monotone functions and $T_a := \{f \in P_2 \mid f(a, \ldots, a) = a\}$ for $a \in E_2$.

Already known (see e.g. Alekseev, Voronenko, Strauch, Lau):
Let $A \subseteq P_2$ be a clone. If $T_0 \cap T_1 \cap M \subseteq A$ or $T_0 \cap T_1 \cap S \subseteq A$ then $\mathcal{I}_2(A)$ is a finite set.

A	$\lvert\mathcal{I}_2(A)\rvert$	
P_2	3	
T_0, T_1, M, S	6	
$T_0 \cap M, T_1 \cap M$	15	
$T_0 \cap T_1$	30	
$T_0 \cap T_1 \cap M$	101	
$T_0 \cap T_1 \cap S$	380	corrected number

The interval $\mathcal{I}_2\,(T_0 \cap T_1 \cap S)$

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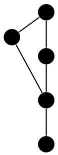

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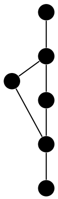

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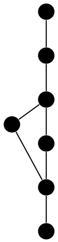

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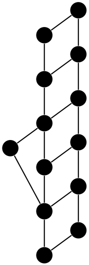

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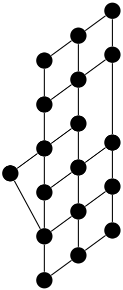

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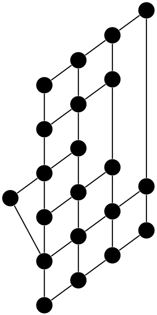

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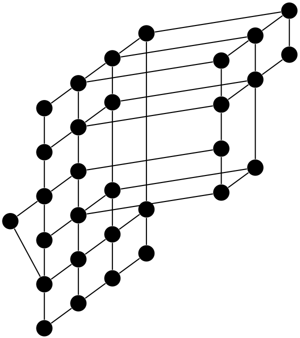

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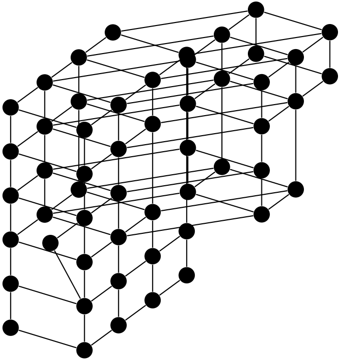

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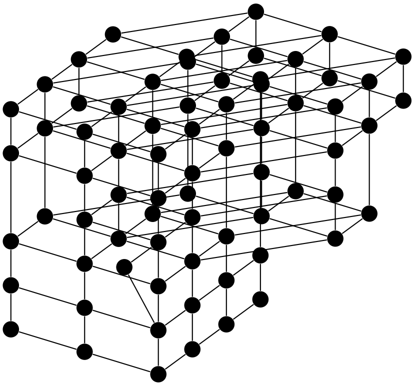

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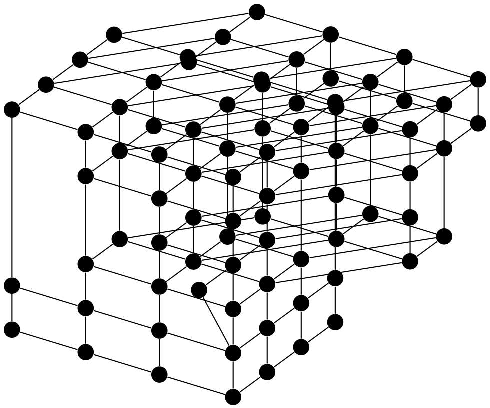

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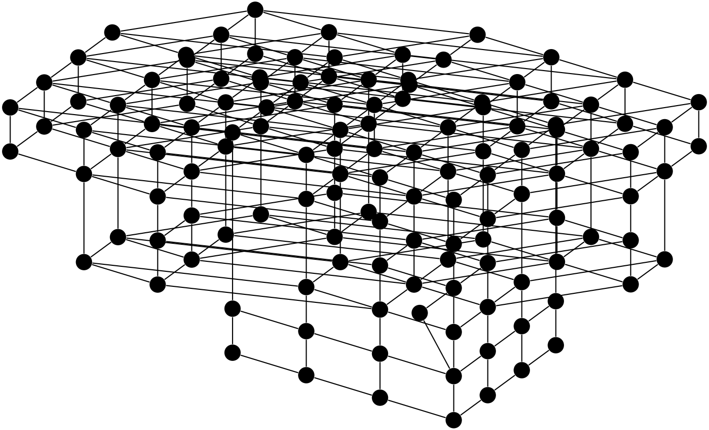

The other intervals are infinite

- construct an chain infinite chain in $\mathcal{I}_2\left(T_{0,2}\right)$,
- use Theorem 8, and
- combine with older results

and we get

Theorem

Let $A \subseteq P_2$ be a clone. If $T_0 \cap T_1 \cap M \not\subseteq A$ and $T_0 \cap T_1 \cap S \not\subseteq A$ then $\mathcal{I}_2\left(A\right)$ is an infinite set.

An infinite chain in $\mathcal{I}_2\left(T_{0,2}\right)$

For $n \in \mathbb{N}$ let $q^{(n)}(\mathbf{x}) = \begin{cases} \infty & \text{if } \mathbf{x} = \mathbf{0}, \\ 0 & \text{otherwise.} \end{cases}$

Let $Q'_n := [\{q^{(n)}\} \cup T_{0,2}]$.

Lemma

Let $i, j \in \mathbb{N}$ be arbitrary with $i < j$. Then $Q'_i \subsetneq Q'_j$.

Then $Q'_1 \subset Q'_2 \subset \dots$ is an infinite chain in $\mathcal{I}_2\left(T_{0,2}\right)$ and the sets $Q_i := \{f \in Q'_i \mid f(\mathbf{0}) = \infty\}$ can be used with Theorem 8.

Idea

If we find a partial clone $W \in \mathcal{I}_2\left(T_{0,2}\right)$ with an infinite basis then the intervals $\mathcal{I}_2\left(B\right)$ with $B \subseteq T_{0,2}$ have the cardinality of the continuum.

Now let $\mathfrak{B}$ let be an infinite basis of W.

Let $V_0 := \{f \in \mathrm{Str}(T_{0,2}) \mid f(0,0,\ldots,0) = 0\}$.

Only finitely many elements of $\mathfrak{B}$ belong to V_0 because

- $T_{0,2}$ is finitely generated, e.g. $T_{0,2} = \left[\{h^{(3)}, t^{(2)}\}\right]$ with $h^{(3)}(x,y,z) := (x \wedge y) \vee (x \wedge z) \vee (y \vee z)$ and $t^{(2)}(x,y) := x \wedge \overline{y}$.
- for all $f \in V_0 \setminus T_{0,2}$ we have $[T_{0,2} \cup \{f\}] = V_0$.

Example for $[T_{0,2} \cup \{f\}] = V_0$

Let $f \in V_0 \setminus T_{0,2}$, e.g. $f\left(\begin{smallmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ \infty \end{smallmatrix}\right)$. Then define $q(x) := f(x, c_0, c_0, x, x) \in [T_{0,2} \cup \{f\}]$ with $q(0) = 0$ and $q(1) = \infty$.

Now let $g \in V_0$ be arbitrary, e.g. the ternary function g as below, and $g_1, h_1, h_2, h_3 \in T_{0,2}$. Then $g = e_1^{(4)}(g_1, q(h_1), q(h_2), q(h_3))$.

x	y	z	g	g_1	h_1	h_2	h_3	$q(h_1)$	$q(h_2)$	$q(h_3)$
0	0	0	0	0	0	0	0	0	0	0
0	0	1	∞	0	1	0	0	∞	0	0
0	1	0	0	0	0	0	0	0	0	0
0	1	1	∞	0	0	1	0	0	∞	0
1	0	0	1	1	0	0	0	0	0	0
1	0	1	1	1	0	0	0	0	0	0
1	1	0	∞	0	0	0	1	0	0	∞
1	1	1	1	1	0	0	0	0	0	0

Some definitions

Let $f^{(n)}, g^{(m)} \in \widetilde{P}_2$.

Then

$$(f \star g)(x_1, \ldots, x_{m+n-1}) := \begin{cases} \infty & \text{if } g(x_1, \ldots, x_m) = \infty, \\ f\left(g\left(x_1, \ldots, x_m\right), x_{m+1}, \ldots, x_{m+n-1}\right) & \text{otherwise.} \end{cases}$$

Let $A, B \subseteq \widetilde{P}_2$. Then

$$A \star B := \{f \star g \mid f \in A, g \in B\}.$$

Using Theorem 8 – part I

Let $U_0 := \{f \in T_{0,2} \mid f(0,0,\ldots,0) = \infty\}$.

Let $Q := \{q_n \mid n \in \mathbb{N}\} \subseteq U_0$. W.l.o.g. the basis $\mathfrak{B} = \{h,t\} \cup Q$ with $q_n \notin [T_{0,2} \cup Q \setminus \{q_n\}]$ for all $n \in \mathbb{N}$.

We set $I := 2^{\mathbb{N}}$ and $Q_i := [T_{0,2} \cup \{q_t \mid t \in i\}] \setminus P_2$ for each $i \in I$.

Then holds the following and Theorem 8 of the paper will be applicable

- $Q_i \cap P_2 = \emptyset$ (by construction),
- $[Q_i] = Q_i$ (since $U_0 \star T_{0,2} \subseteq U_0$, $T_{0,2} \star U_0 \subseteq U_0$ we have $Q_i = [T_{0,2} \cup \{q_t \mid t \in i\}] \cap U_0$),
- $Q_i \star T_{0,2} \subseteq Q_i$ and $T_{0,2} \star Q_i \subseteq Q_i$.

Using Theorem 8 – part II

We know for all $i, j \in I$ that $Q_i \neq Q_j$ if and only if $i \neq j$.

Let $B \subseteq T_{0,2}$ a clone. Then Theorem 8 yields

- $[Q_i \cup B] = Q_i \cup B$ for all $i \in I$,
- $|\mathcal{I}_2(B)| \geq |2^{\mathbb{N}}| = |\mathbb{R}|$ and since $|\mathcal{I}_2(B)| \leq |\mathbb{R}|$ we get $|\mathcal{I}_2(B)| = |\mathbb{R}|$.

Conclusions

- Every finite interval has been determined now.
- If the interval of $T_{0,2}$ has continuum cardinality then the size of every interval would be determined.