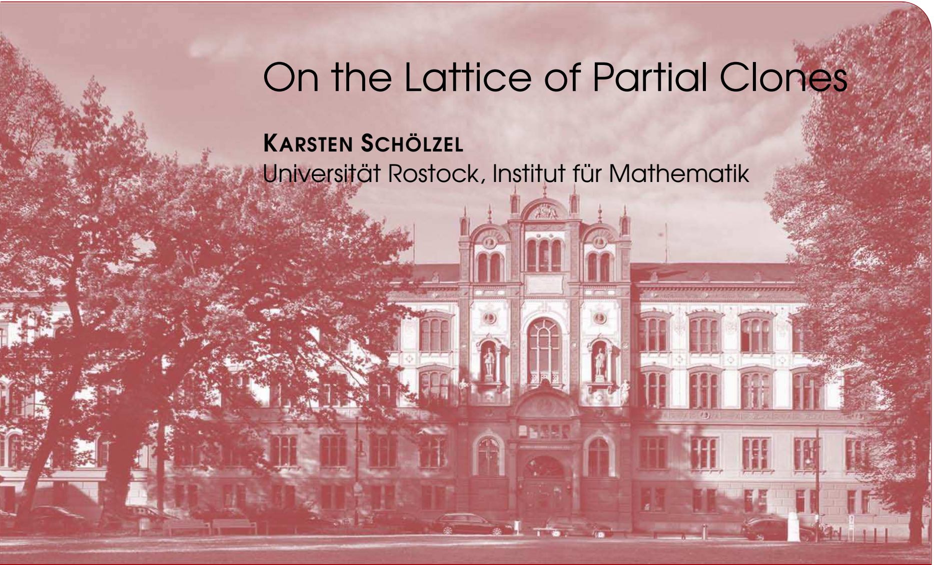
On the Lattice of Partial Clones

Karsten Schölzel

Universität Rostock, Institut für Mathematik



Content

Content

Content

Content

Definitions

$E_k := \{0, 1, \ldots, k-1\}$, $k \in \mathbb{N}$,
$\widetilde{E}_k := E_k \cup \{\infty\}$
$f^{(n)} : E_k^n \to \widetilde{E}_k$ is called a partial n-ary function
$\widetilde{P}_k$ is the set of all finitary partial functions on E_k
P_k is the set of all total functions, i.e., functions $f^{(n)}$ such that $f(\mathbf{x}) \in E_k$ for all $\mathbf{x}$.

Superposition

Let $f^{(n)}, g_1^{(m)}, \ldots, g_n^{(m)} \in \widetilde{P}_k$. Then $f(g_1, \ldots, g_n)$ is an m-ary partial function defined by

$$f(g_1, \ldots, g_m)(\mathbf{x}) := \begin{cases} \infty & \text{if } g_i(\mathbf{x}) = \infty \text{ for some } i \in \{1, \ldots, n\}; \\ f(g_1(\mathbf{x}), \ldots, g_n(\mathbf{x})) & \text{otherwise.} \end{cases}$$

Clone

$e_i^{(n)}$ is the n-ary projection on the i-th coordinate, i.e., $e_i^{(n)}(x_1, \ldots, x_n) = x_i$.
J_k is the set of all projections, i.e., $J_k := \{e_i^{(n)} \mid 1 \leq i \leq n\}$.

Definition

Let $C \subseteq \widetilde{P}_k$. Then C is a partial clone if C is closed with respect to the superposition contains the projections.

If $C \subseteq P_k$, i.e., C consists of total functions only, then C is a total clone.

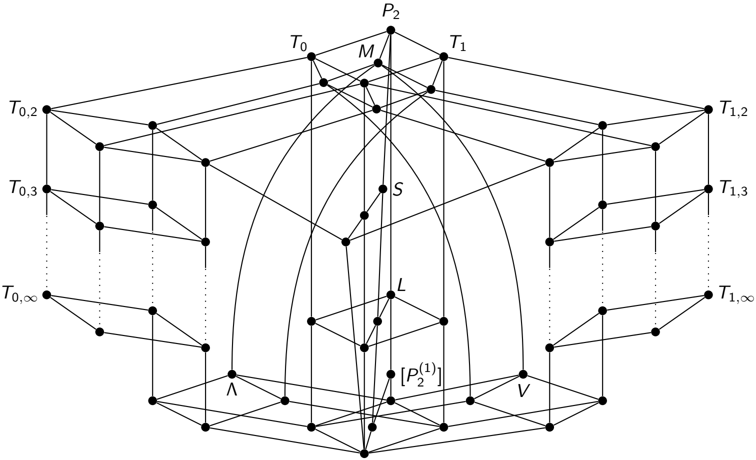
Lattice

The partial clones ordered by inclusion form a dually atomic lattice $\mathbb{L}(\widetilde{P}_k)$.
$\widetilde{P}_k$ is the top element of $\mathbb{L}(\widetilde{P}_k)$;
J_k (set of projections) is a clone and the bottom element of $\mathbb{L}(\widetilde{P}_k)$;

The total clones ordered by inclusion also form a dually atomic lattice $\mathbb{L}(P_k)$ which is a sublattice of $\mathbb{L}(\widetilde{P}_k)$.
$\mathbb{L}(P_2)$ is better known as Post's lattice.

The lattice $\mathbb{L}(\widetilde{P}_k)$ has the cardinality of the continuum for each $k \geq 2$.

Post's Lattice — total clones for $k = 2$



Aim

- Describe the partial clones via a Galois connection with relations on the other side.
- Characterize the Galois closed sets of relations (co-clones) yielding a lattice on the relation side.

Relations

We consider relations on $\widetilde{E}_k$. A subset $\varrho^{(h)} \subseteq \widetilde{E}_k^h$ is called h-ary relation. The set of all finitary relations on $\widetilde{E}_k$ is called $\widehat{\mathcal{R}}_k$. The set $\mathcal{R}_k$ contains all relations on E_k, i.e., all relations without ∞.

We write relations as matrices where the tuples are the columns of the matrix; e.g.

$$\begin{pmatrix} 0 & 2 \\ 3 & 1 \\ 1 & 2 \end{pmatrix} \in \mathcal{R}_4^{(3)} \subseteq \widehat{\mathcal{R}}_4^{(3)} \qquad \begin{pmatrix} 0 & 2 & \infty \\ 1 & \infty & \infty \\ 1 & 3 & 0 \end{pmatrix} \in \widehat{\mathcal{R}}_4^{(3)}$$

Preserving of relations I

Let $f^{(n)} \in \widetilde{P}_k$ and $\varrho^{(h)} \in \widehat{\mathcal{R}}_k$. Then f preserves ϱ if

$$f\begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{h1} & x_{h2} & \dots & x_{hn} \end{pmatrix} := \begin{pmatrix} f(x_{11}, x_{12}, \dots, x_{1n}) \\ f(x_{21}, x_{22}, \dots, x_{2n}) \\ \vdots \\ f(x_{h1}, x_{h2}, \dots, x_{hn}) \end{pmatrix} \in \varrho$$

holds for all $(x_{1i}, \dots, x_{hi}) \in \varrho$ and $i \in \{1, \dots, n\}$.

Preserving of relations II

$\mathrm{pPol}_k\,\varrho := \{f \in \widetilde{P}_k \mid f \text{ preserves } \varrho\}$
$\mathrm{pInv}_k\,f := \{\varrho \in \widehat{\mathcal{R}}_k \mid f \text{ preserves } \varrho\}$

$(\mathrm{pPol}_k, \mathrm{pInv}_k)$ form an antitone Galois-connection between

- subsets of $\widetilde{P}_k$ and
- subsets of $\widehat{\mathcal{R}}_k$.

Not so difficult:

Theorem

$C \subseteq \widetilde{P}_k$ is a clone if and only if $C = \mathrm{pPol}_k\,\mathrm{pInv}_k\,C$.

Characterize Galois

For the Galois connection in the total case between subsets of P_k and $\mathcal{R}_k$ there exists a characterization of the closed relation sets $R = \mathrm{Inv}_k \, \mathrm{Pol}_k \, R$ by elementary relations.

Question

Is there a characterization of the subsets of $\widehat{\mathcal{R}}_k$ *with* $R = \mathrm{pInv}_k \, \mathrm{pPol}_k \, R$*?*

The set $\widetilde{\mathcal{R}}_k$

The set $\mathrm{pInv}_k\, J_k$ does not contain all relations from $\widehat{\mathcal{R}}_k$ due to the special treatment of ∞.
For example, $e_1^{(2)} \in J_k$ does not preserve $\varrho^{(3)} \in \widehat{\mathcal{R}}_k$ defined by

$$\varrho := \begin{pmatrix} 0 & 1 & 0 & 1 \\ 2 & 3 & \infty & 3 \\ 1 & 0 & 2 & \infty \end{pmatrix} \quad \text{since} \quad e_1^{(2)} \begin{pmatrix} 1 & 0 \\ 3 & \infty \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 \\ \infty \\ 0 \end{pmatrix} \notin \varrho.$$

Let $\widetilde{\mathcal{R}}_k := \mathrm{pInv}_k\, J_k$. Since we ant to describe clones and $J_k \subseteq C$ for every clone C the Galois connection $(\mathrm{pPol}, \mathrm{pInv})$ is restricted to $\widetilde{\mathcal{R}}_k$.

Elementary operations on $\widetilde{\mathcal{R}}_k$

Let $\varrho^{(h)}, \sigma^{(\mu)} \in \widetilde{\mathcal{R}}_k$. Then

- $\zeta\varrho$ — cyclical exchange of the rows of ϱ
- $\tau\varrho$ — exchange of the first two rows of ϱ
- $\mathrm{pr}\ \varrho$ — projection onto the rows $2, \ldots, h$
- $\varrho \times \sigma$ — cartesian product of ϱ and σ
- $\varrho \wedge \sigma$ — intersection of ϱ and σ
- $\widetilde{\delta}^{(3)}_{k;\{1,2\}} := \{(x, y, z) \in \widetilde{E}^3_k \mid x = y\}$

are the elementary operations and relation known from the total case.

Elementary operations on $\widetilde{\mathcal{R}}_k$; the new one $\hat{\kappa}$

Let $\varrho^{(h)} \in \widetilde{\mathcal{R}}_k$.
If $\varrho = \emptyset$ or unary then let $\hat{\kappa}\varrho := \varrho \cup \{\infty\}$.
If $h \geq 2$ then define it by

$$\hat{\kappa}\varrho = \varrho \cup \{(a_1, a_2, \ldots, a_h) \in \widetilde{E}_k^h \mid a_1 = a_2 = \infty\}.$$

For example for $k = 3$

$$\hat{\kappa}\begin{pmatrix} 0 & 1 \\ 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & \infty & \infty & \infty & \infty \\ 2 & 0 & \infty & \infty & \infty & \infty \\ 1 & 1 & 0 & 1 & 2 & \infty \end{pmatrix}$$

Elementary operations on $\widetilde{\mathcal{R}}_k$ and pPol_k

Theorem

Let $\varrho^{(h)}, \sigma^{(\mu)} \in \widetilde{\mathcal{R}}_k$ *and* $f \in (\mathrm{pPol}_k\, \varrho) \cap (\mathrm{pPol}_k\, \sigma)$*. Then*

- $f \in \mathrm{pPol}_k(\alpha\varrho)$ *for* $\alpha \in \{\zeta, \tau, \mathrm{pr}, \hat{\kappa}\}$;
- $f \in \mathrm{pPol}_k(\varrho \times \sigma)$;
- $f \in \mathrm{pPol}_k(\varrho \wedge \sigma)$; *and*
- $f \in \mathrm{pPol}_k\, \widetilde{\delta}^{(3)}_{k;\{1,2\}}$.

In other words $\mathrm{pInv}_k\, f$ *is closed with respect to* $\{\zeta, \tau, \mathrm{pr}, \times, \wedge, \hat{\kappa}, \widetilde{\delta}^{(3)}_{k;\{1,2\}}\}$.

Co-Clones

Definition

Let $R \subseteq \widetilde{\mathcal{R}}_k$. Then R is called a partial co-clone if it is closed with respect to the elementary relations $\{\zeta, \tau, \mathrm{pr}, \times, \wedge, \hat{\kappa}\}$ and contains $\widetilde{\delta}^{(3)}_{k;\{1,2\}}$.

Theorem

$R \subseteq \widetilde{\mathcal{R}}_k$ is a partial co-clone if and only if $R = \mathrm{pInv}_k\,\mathrm{pPol}_k\,R$.

Let $\mathbb{L}(\widetilde{\mathcal{R}}_k)$ be the set of all partial co-clones of $\widetilde{\mathcal{R}}_k$.

Galois

Theorem

The mappings

$$\mathrm{pInv} : \mathbb{L}(\widetilde{P}_k) \longrightarrow \mathbb{L}(\widetilde{\mathcal{R}}_k), A \mapsto \mathrm{pInv}\, A$$
$$\mathrm{pPol} : \mathbb{L}(\widetilde{\mathcal{R}}_k) \longrightarrow \mathbb{L}(\widetilde{P}_k), Q \mapsto \mathrm{pPol}\, Q$$

are bijective mappings, which reverse the partial order $\subseteq$*, i.e., it holds*

$$\forall A, B \in \mathbb{L}(\widetilde{P}_k) : A \subseteq B \Longrightarrow \mathrm{pInv}\, B \subseteq \mathrm{pInv}\, A$$
$$\forall S, T \in \mathbb{L}(\widetilde{\mathcal{R}}_k) : S \subseteq T \Longrightarrow \mathrm{pPol}\, T \subseteq \mathrm{pPol}\, S.$$

In other words: The lattices $(\mathbb{L}(\widetilde{P}_k), \subseteq)$ *and* $(\mathbb{L}(\widetilde{\mathcal{R}}_k), \subseteq)$ *are antiisomorphic.*

What can we do with it now?

- characterize partial clones with irredundant relations
- characterize partial clones with relations with certain structure of the tuples which contain ∞
- a new proof of the Lemma of Romov, an important step to the Completeness Theorem of Haddad and Rosenberg (determination of the maximal partial clones)

Universität Rostock Traditio et Innovatio

pPOL_k

Let $\varrho^{(h)} \in \widetilde{\mathcal{R}}_k$. Then κ defined by

$$\kappa\varrho := \varrho \cup (\widetilde{E}_k^h \setminus E_k^h)$$

is derivable from the elementary operations on $\widetilde{\mathcal{R}}_k$.

Definition

For $\varrho \in \mathcal{R}_k$ define pPOL_k by

$$\mathrm{pPOL}_k\, \varrho := \mathrm{pPol}_k(\kappa\varrho).$$

Theorem of Haddad and Rosenberg

Let $\widetilde{\mathcal{R}}_k^{\max} \subseteq \mathcal{R}_k$ be the set of coherent relations. Let

$$p\mathscr{M}_k := \{P_k \cup C_\infty\} \cup \{\mathrm{pPOL}_k\, \varrho\}.$$

Theorem (Haddad and Rosenberg (1989,1992))

Let $k \geq 2$. For each clone $C \subset \widetilde{P}_k$ there is a maximal partial clone M_C with $C \subseteq M_C$. A clone M is a maximal partial clone of $\widetilde{P}_k$ if and only if $M \in p\mathscr{M}_k$, i.e., $p\mathscr{M}_k$ is the set of all maximal partial clones of $\widetilde{P}_k$.

List of all maximal partial clones for some k

equivalently: a list of all coherent relations $\widetilde{\mathcal{R}}_k^{\max}$

What do we need such a list for?

- some applications need to check all maximal partial clones
- get examples to check some hypotheses
- ...

Can we do it by hand? Maybe up to $k = 4$ (reasonably well only for $k = 2$ and $k = 3$)

A computer might do it!?

Pro:

- faster
- makes no mistakes when doing tedious work

Difficulties:

- Is there a scalable algorithm? Probably not!
- Is the implementation of the algorithm correct? Needs a clear implementation and documentation to make it possible for a human to check!
- Size (memory) and Time (how long do we want to wait?) constraints

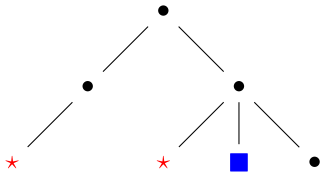
First try

- list all coherent relations
- then check if some of them describe the same maximal partial clone (duplicates)

Problems:

- for a maximal partial clone described by an h-ary coherent relation there are up to $h!$ different coherent relations
- the check for duplicates is expensive
- needs too much space since all coherent relations have to be in memory

Second try: let's look at the backtracking tree



The red stars should mark two coherent relations describing the same maximal partial clone.
Can we select only one star by just investigating that star, i.e., some property of the coherent relation which can be computed just from the relation itself?

Second try: Select the star

Let $\prec$ be the lexicographical order on tuples and inherently on relations.

Let $\varrho^{(h)} \in \mathcal{R}_k$ be a coherent relation. Then we can define $P(\varrho)$ be the set of all relations derivable from ϱ by permutations of the rows of ϱ.

Select that relation ϱ where $\varrho = \min_{\prec} T(\varrho)$.

Second try: it's not perfect!

- it's fairly simple to implement
- every maximal partial clone is hit once
- **but** there are isomorphic clones which could be listed together
  there are 720 different clones in nearly every isomorphism class for $k = 6$

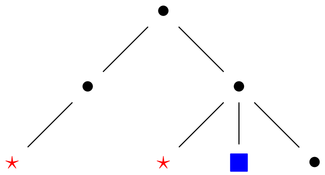
Definition (Isomorphic clones)

Let C, D be partial clones. Then C and D are isomorphic if there is some permutation π on the set E_k such that

- for every $f^{(n)} \in C$ we have $f^{\pi} \in D$; and
- for every $g^{(n)} \in D$ we have $g^{\pi^{-1}} \in C$.

$$\forall x_1, \ldots, x_n \in E_k : f^{\pi}(x_1, \ldots, x_n) := \pi(f(\pi^{-1}(x_1), \ldots, \pi^{-1}(x_n)))$$

Third try; the backtracking tree again



The blue box marks a clone isomorphic to the red stars.
Can we select only one of isomorphic clones?

Relation class

Let $\varrho^{(h)} \in \mathcal{R}_k$ be a coherent relation. Then we can define the relation class $\mathrm{class}(\varrho)$ to be the set of all relations derivable from ϱ by permutations of the rows of ϱ together with applying a bijection π of E_k onto the elements.

Now we take just relations ϱ with

$$\varrho = \min_{\prec} \mathrm{class}(\varrho).$$

Bonus point: with a variant of the relation class we can eliminate many subtrees of the backtracking tree, i.e., in most cases we would touch just one relation for each isomorphism class

Third try: it works well enough

- less space needed for the list
- fast enough to get $k = 6$ (still two days worth of computer time)
- the algorithm is more complex

Number of maximal partial clones

k	$\lvert\mathscr{M}_k\rvert$	$\lvert p\mathscr{M}_k\rvert$	$\lvert p\mathscr{M}_k^{\mathrm{C}}\rvert$	$\frac{\lvert p\mathscr{M}_k\rvert}{\lvert p\mathscr{M}_k^{\mathrm{C}}\rvert\cdot k!}$
2	5	8	7	0.57
3	18	58	26	0.37
4	82	1 102	138	0.33
5	643	325 722	3 287	0.82
6	15 182	5 242 621 816	7 322 017	0.99
7	7 848 984	?	?	$>$ 0.99?
8	549 761 933 169	?	?	$>$ 0.99?

$\mathscr{M}_k$	maximal total clones
$p\mathscr{M}_k$	maximal partial clones
$\lvert p\mathscr{M}_k^{\mathrm{C}}\rvert$	number of isomophism classes

Aim

Describe the structure of the lattice of partial Boolean clones as far as possible using the known structure of total Boolean clones, i.e. Post's lattice.

Intervals

Definition

Let $A \subseteq P_2$ be a (total) Boolean clone. Then

$$\mathcal{I}_2(A) := \{C \subseteq \widetilde{P}_2 \mid C \text{ is a partial clone and } C \cap P_2 = A\}$$

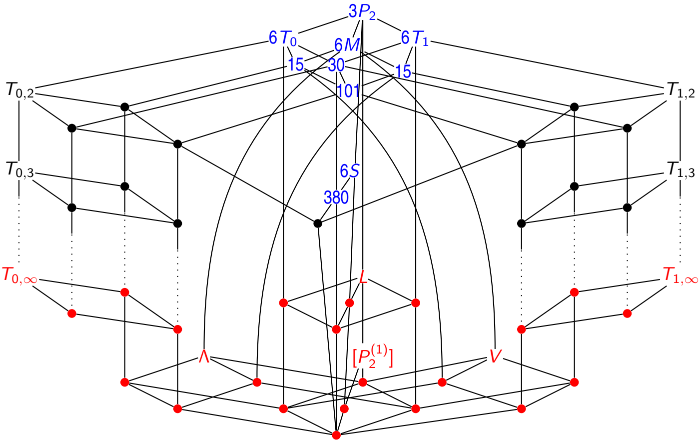
is called the interval of A.

$\widetilde{P}_2$: the set of all partial Boolean functions;
P_2: the set of all everywhere-defined Boolean functions.

Example

The interval $\mathcal{I}_2\left(P_2\right) = \{P_2, P_2 \cup C_\infty, \widetilde{P}_2\}$ where C_∞ are all the functions with empty domain.

Post's lattice



The finite intervals

Let $S \subseteq P_2$ be the set of all self-dual functions, $M \subseteq P_2$ the set of all monotone functions and $T_a := \{f \in P_2 \mid f(a, \ldots, a) = a\}$ for $a \in E_2$.

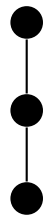
Already known (see e.g. Alekseev, Voronenko, Strauch, Lau):
Let $A \subseteq P_2$ be a clone. If $T_0 \cap T_1 \cap M \subseteq A$ or $T_0 \cap T_1 \cap S \subseteq A$ then $\mathcal{I}_2(A)$ is a finite set.

A	$\lvert\mathcal{I}_2(A)\rvert$
P_2	3
T_0, T_1, M, S	6
$T_0 \cap M, T_1 \cap M$	15
$T_0 \cap T_1$	30
$T_0 \cap T_1 \cap M$	101
$T_0 \cap T_1 \cap S$	380

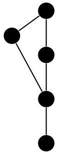
The interval $\mathcal{I}_2(T_0 \cap T_1 \cap S)$

The interval $\mathcal{I}_2\,(T_0 \cap T_1 \cap S)$

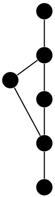
The interval $\mathcal{I}_2(T_0 \cap T_1 \cap S)$



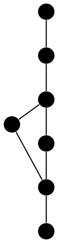
The interval $\mathcal{I}_2(T_0 \cap T_1 \cap S)$



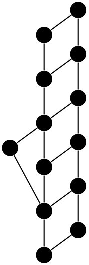
The interval $\mathcal{I}_2(T_0 \cap T_1 \cap S)$



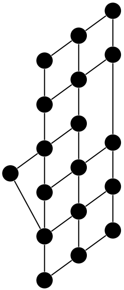
The interval $\mathcal{I}_2(T_0 \cap T_1 \cap S)$



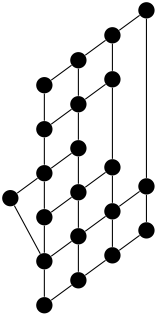
The interval $\mathcal{I}_2(T_0 \cap T_1 \cap S)$



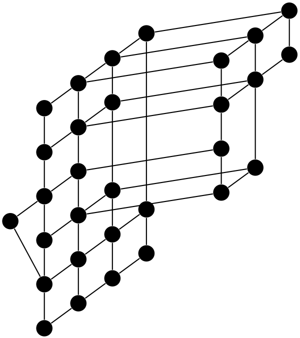
The interval $\mathcal{I}_2\,(T_0 \cap T_1 \cap S)$



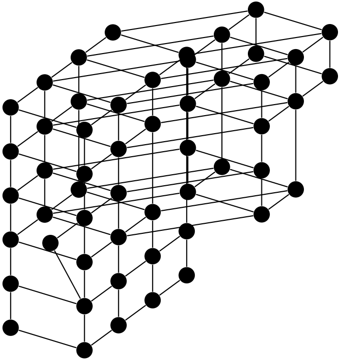
The interval $\mathcal{I}_2(T_0 \cap T_1 \cap S)$



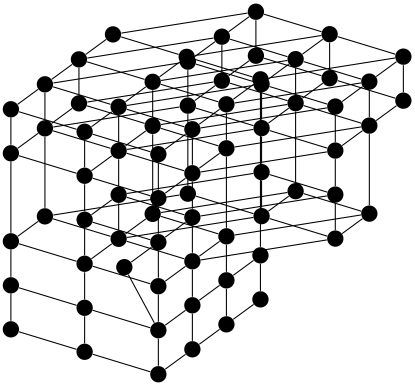
The interval $\mathcal{I}_2(T_0 \cap T_1 \cap S)$



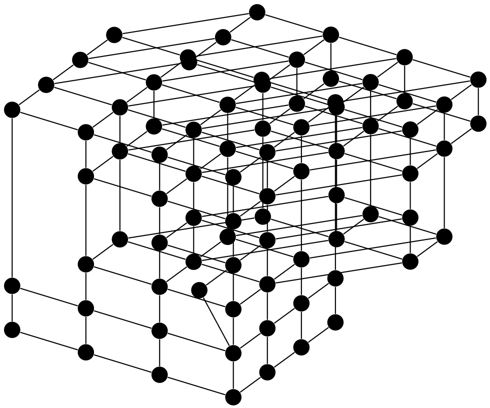
The interval $\mathcal{I}_2(T_0 \cap T_1 \cap S)$



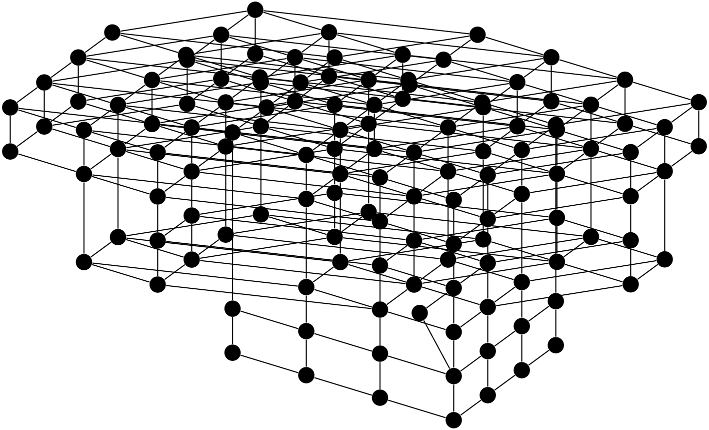
The interval $\mathcal{I}_2(T_0 \cap T_1 \cap S)$



The interval $\mathcal{I}_2(T_0 \cap T_1 \cap S)$



The interval $\mathcal{I}_2\,(T_0 \cap T_1 \cap S)$



The other intervals are infinite

- construct an chain infinite chain in $\mathcal{I}_2\,(T_{0,2})$,
- use some Theorem, and
- combine with older results

and we get

Theorem

Let $A \subseteq P_2$ be a clone. If $T_0 \cap T_1 \cap M \not\subseteq A$ and $T_0 \cap T_1 \cap S \not\subseteq A$ then $\mathcal{I}_2\,(A)$ is an infinite set.

An infinite chain in $\mathcal{I}_2\left(T_{0,2}\right)$

For $n \in \mathbb{N}$ let $q^{(n)}(\mathbf{x}) = \begin{cases} \infty & \text{if } \mathbf{x} = \mathbf{0}, \\ 0 & \text{otherwise.} \end{cases}$

Let $Q'_n := [\{q^{(n)}\} \cup T_{0,2}]$.

Lemma

Let $i, j \in \mathbb{N}$ be arbitrary with $i < j$. Then $Q'_i \subsetneq Q'_j$.

Then $Q'_1 \subset Q'_2 \subset \ldots$ is an infinite chain in $\mathcal{I}_2\left(T_{0,2}\right)$ and the sets $Q_i := \{f \in Q'_i \mid f(\mathbf{0}) = \infty\}$ can be used with aforementioned Theorem.

The last “interesting” interval: $\mathcal{I}_2(T_{0,2})$?

Idea

If we find a partial clone $W \in \mathcal{I}_2(T_{0,2})$ with an infinite basis then the intervals $\mathcal{I}_2(B)$ with $B \subseteq T_{0,2}$ have the cardinality of the continuum.

Now let $\mathfrak{B}$ be an infinite basis of W.

Let $V_0 := \{f \in \mathrm{Str}(T_{0,2}) \mid f(0,0,\ldots,0) = 0\}$.

Only finitely many elements of $\mathfrak{B}$ belong to V_0 because

- $T_{0,2}$ is finitely generated, e.g. $T_{0,2} = \left[\left\{h^{(3)}, t^{(2)}\right\}\right]$ with $h^{(3)}(x,y,z) := (x \wedge y) \vee (x \wedge z) \vee (y \vee z)$ and $t^{(2)}(x,y) := x \wedge \overline{y}$.
- for all $f \in V_0 \setminus T_{0,2}$ we have $[T_{0,2} \cup \{f\}] = V_0$.

Example for $[T_{0,2} \cup \{f\}] = V_0$

Let $f \in V_0 \setminus T_{0,2}$, e.g. $f\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \infty \end{pmatrix}$. Then define $q(x) := f(x, c_0, c_0, x, x) \in [T_{0,2} \cup \{f\}]$ with $q(0) = 0$ and $q(1) = \infty$.

Now let $g \in V_0$ be arbitrary, e.g. the ternary function g as below, and $g_1, h_1, h_2, h_3 \in T_{0,2}$. Then $g = e_1^{(4)}(g_1, q(h_1), q(h_2), q(h_3))$.

x	y	z	g	g_1	h_1	h_2	h_3	$q(h_1)$	$q(h_2)$	$q(h_3)$
0	0	0	0	0	0	0	0	0	0	0
0	0	1	∞	0	1	0	0	∞	0	0
0	1	0	0	0	0	0	0	0	0	0
0	1	1	∞	0	0	1	0	0	∞	0
1	0	0	1	1	0	0	0	0	0	0
1	0	1	1	1	0	0	0	0	0	0
1	1	0	∞	0	0	0	1	0	0	∞
1	1	1	1	1	0	0	0	0	0	0

Some definitions

Let $f^{(n)}, g^{(m)} \in \widetilde{P}_2$.

Then

$$(f \star g)(x_1, \ldots, x_{m+n-1}) := \begin{cases} \infty & \text{if } g(x_1, \ldots, x_m) = \infty, \\ f\left(g\left(x_1, \ldots, x_m\right), x_{m+1}, \ldots, x_{m+n-1}\right) & \text{otherwise.} \end{cases}$$

Let $A, B \subseteq \widetilde{P}_2$. Then

$$A \star B := \{f \star g \mid f \in A, g \in B\}.$$

The intervals below are also infinite – part I

Let $U_0 := \{f \in T_{0,2} \mid f(0,0,\dots,0) = \infty\}$.

Let $Q := \{q_n \mid n \in \mathbb{N}\} \subseteq U_0$. W.l.o.g. the basis $\mathfrak{B} = \{h, t\} \cup Q$ with $q_n \notin [T_{0,2} \cup Q \setminus \{q_n\}]$ for all $n \in \mathbb{N}$.

We set $I := 2^{\mathbb{N}}$ and $Q_i := [T_{0,2} \cup \{q_t \mid t \in i\}] \setminus P_2$ for each $i \in I$.

Then the following holds

- $Q_i \cap P_2 = \emptyset$ (by construction),
- $[Q_i] = Q_i$ (since $U_0 \star T_{0,2} \subseteq U_0$, $T_{0,2} \star U_0 \subseteq U_0$ we have $Q_i = [T_{0,2} \cup \{q_t \mid t \in i\}] \cap U_0$),
- $Q_i \star T_{0,2} \subseteq Q_i$ and $T_{0,2} \star Q_i \subseteq Q_i$.

The intervals below are also infinite – part II

We know for all $i, j \in I$ that $Q_i \neq Q_j$ if and only if $i \neq j$.

Let $B \subseteq T_{0,2}$ a clone. From the statements before we conclude

- $[Q_i \cup B] = Q_i \cup B$ for all $i \in I$,
- $|\mathcal{I}_2(B)| \geq |2^{\mathbb{N}}| = |\mathbb{R}|$ and since $|\mathcal{I}_2(B)| \leq |\mathbb{R}|$ we get $|\mathcal{I}_2(B)| = |\mathbb{R}|$.

Conclusions

- Every finite interval has been determined now.
- If the interval of $T_{0,2}$ has continuum cardinality then the size of every interval would be determined.

The end

Thank you for your attention!