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Supermodular subgroups in finite groups

The concept of a modular subgroup originally comes from the theory of lattices. A subgroup M of a group G is called modular in G if the following equalities are satisfied:

$$(W \cup M) \cap V = W \cup (M \cap V) \text{ for all } W, V \text{ in } G \text{ with } W \leq V \text{ and}$$
$$(W \cup M) \cap V = M \cup (W \cap V) \text{ for all } W, V \text{ in } G \text{ with } M \leq V$$

where $X \cup Y = \langle X, Y \rangle$ denotes the subgroup generated by X and Y. Thus M is a modular element of the subgroup lattice in G. Detailed analysis of modular subgroups has been carried out by R. Schmidt (cf. [2], [3]).

In this paper we define what are called supermodular subgroups and study them, where the author in [4] and [1] has studied lattices satisfying supermodular identity. It has been proved in [4] and [1] that supermodular lattices form an equational class of lattices lying properly between the equational class of distributive lattices and the equational class of modular lattices.

Definition 1 *Let G be a finite group. A subgroup M of the group G is called supermodular if for all subgroups B, C, D in G we have the following identity to be true:*

$$\begin{aligned}&(M \cup B) \cap (M \cup C) \cap (M \cup D) \\ &\quad = M \cup \{[(B \cap C) \cap (M \cup D)] \cup [(C \cap D) \cap (M \cup B)] \cup [(B \cap D) \cap (M \cup C)]\} \qquad (1)\end{aligned}$$

This identity is known as the supermodular identity.

Proposition 2 *Every supermodular subgroup M of a group G is a modular subgroup of G; but a modular subgroup M of G need not necessarily be a supermodular subgroup of G.*

Proof. Putting $C = D$ in the supermodular identity we get

$$\begin{aligned}(M \cup B) \cap (M \cup C) &= M \cup \{(B \cap C) \cup [C \cap (M \cup B)] \cup (B \cap C)\} \\ &= M \cup [C \cap (M \cup B)]\end{aligned}$$

to be true for all subgroups B, C in G, which is easily recognized as the condition for a subgroup M to be modular.

Now we prove that every modular subgroup M of G need not necessarily be a supermodular subgroup of G by an example. Let us consider the commutative group $G = \langle a, b, c, d \,/\, a^2 = b^2 = c^2 = d^2 = 1\rangle$. Take $M = \langle a, b, c\rangle$, $B = \langle b, c, d\rangle$, $C = \langle abd\rangle$ and $D = \langle acd\rangle$. The supermodular identity of M would claim

$$(M \cup B) \cap (M \cup C) \cap (M \cup D)$$
$$= M \cup \{[(B \cap C) \cap (M \cup D)] \cup [(C \cap D) \cap (M \cup B)] \cup [(B \cap D) \cap (M \cup C)]\} \; .$$

For the L.H.S. we obtain $(M \cup B) \cap (M \cup C) \cap (M \cup D) = G \cap G \cap G = G$, and for the R.H.S. we get $M \cup \{[\{1\} \cap (M \cup D)] \cup [\{1\} \cap (M \cup B)] \cup [\{1\} \cap (M \cup C)]\} = M$. Because obviously L.H.S. $\neq$ R.H.S. we are done.

Theorem 3 *Let G be a finite group in which any two subgroups are comparable. Then every subgroup of G is supermodular.*

Proof. It is known that a finite group in which any two subgroups are comparable necessarily is cyclic of prime power order. Then the lattice of subgroups is a chain, hence distributive and so is supermodular.

Theorem 4 *Let G be a group which has no supermodular subgroups then G has a set of four subgroups M, B_1, C_1, D_1 such that*

$$(M \cup B_1) \cap (M \cup C_1) \cap (M \cup D_1) \gneqq M$$

while $M > (B_1 \cap C_1) \cap (M \cup D_1)$, $(C_1 \cap D_1) \cap (M \cup B_1)$, $(B_1 \cap D_1) \cap (M \cup C_1)$ holds.

Remark. B_1, C_1, D_1 being distinct subgroups of $G, B_1 \neq C_1$, otherwise $B_1 = B_1 \cap C_1$ and $M \cup B_1 = M \cup (B_1 \cap C_1)$ a contradiction as it will imply equality of $(M \cup B_1) \cap (M \cup C_1) \cap (M \cup D_1) = M$.

Proof. As G has no supermodular subgroups there exists subgroups P, X, Y, Z such that

$$(P \cup X) \cap (P \cup Y) \cap (P \cup Z) \gneqq P \cup \{[(X \cap Y) \cap (P \cup Z)]$$
$$\cup [(Y \cap Z) \cap (P \cup X)] \cup [(Z \cap X) \cap (P \cup Y)]\} \; .$$

Put $M = P \cup \{[(X \cap Y) \cap (P \cup Z)] \cup [(Y \cap Z) \cap (P \cup X)] \cup [(Z \cap X) \cap (P \cup Y)]\}$. Let X, Y and Z denote the subgroups B_1, C_1 and D_1, respectively, and the subgroup M given by the above relation: $B_1 = X$, $C_1 = Y$, $D_1 = Z$.

$$M \cup B_1 = P \cup \{[(X \cap Y) \cap (P \cup Z)] \cup [(Y \cap Z) \cap (P \cup X)] \cup [(Z \cap X)$$
$$\cap (P \cup Y)]\} \cup X = (P \cup X) \cup (Y \cap Z) \cap (P \cup X) = P \cup X \; .$$

Similary $M \cup C_1 = P \cup Y$ and $M \cup D_1 = P \cup Z$.

So $(M \cup B_1) \cap (M \cup C_1) \cap (M \cup D_1) \gneq M$. Hence the result.

Theorem 5 *Let G be a group in which every subgroup is modular but having no subgroup which is supermodular. Then G has 4 distinct subgroups M, B, C, D such that*

$$M \cup B = M \cup C = M \cup D \gneq M \ ,$$

$$M > B \cap C \ , \quad C \cap D \ , \quad D \cap B$$

holds.

Proof. By our Theorem 4 we can assert the existence of a set of four elements M, B_1, C_1, D_1 in G such that

$$(M \cup B_1) \cap (M \cup C_1) \cap (M \cup D_1) \gneq M$$

and

$$M > (B_1 \cap C_1) \cap (M \cup D_1), \ (C_1 \cap D_1) \cap (M \cup B_1), (B_1 \cap D_1) \cap (M \cup C_1) \ .$$

Put

$$\begin{aligned} B &= B_1 \cap (M \cup C_1) \cap (M \cup D_1) \\ C &= C_1 \cap (M \cup B_1) \cap (M \cup D_1) \\ D &= D_1 \cap (M \cup B_1) \cap (M \cup C_1) \ . \end{aligned}$$

Now consider the four subgroups M, B, C, D; we have, using the value of B:

$$M \cup B = M \cup B_1 \cap (M \cup C_1) \cap (M \cup D_1) = (M \cup B_1) \cap (M \cup C_1) \cap (M \cup D_1)$$

as $(M \cup C_1) \cap (M \cup D_1) > M$ and as every subgroup is modular. Similary

$$M \cup C = (M \cup B_1) \cap (M \cup C_1) \cap (M \cup D_1)$$

and

$$M \cup D = (M \cup B_1) \cap (M \cup C_1) \cap (M \cup D_1) \ .$$

So $M \cup B = M \cup C = M \cup D > M$. Hence the result. Now consider $B \cap C$ where B and C are given by the above equations.

$$\begin{aligned} B \cap C &= [B_1 \cap (M \cup C_1) \cap (M \cup D_1)] \cap [C_1 \cap (M \cup B_1) \cap (M \cup D_1)] \\ &= (B_1 \cap C_1) \cap (M \cup D_1) \end{aligned}$$

We get $M > B \cap C$. Similary $M > C \cap D$ and $M > D \cap B$. Hence the theorem.

References

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