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Meromorphic function fields of Albanese bundles[1]

Introduction

Let X be a (connected) compact complex manifold and Aut X the complex Lie group of biholomorphic transformations of X. The Albanese torus $A(X)$ of X is a bimeromorphic invariant of X, and the holomorphic Albanese map $\alpha : X \to A(X)$ often yields important information about the holomorphic or meromorphic structure of X. For example, if X is kählerian and if $c_1(X) \geq 0$ or Aut X has an open orbit on X, then $\alpha : X \to A(X)$ is a holomorphic fiber bundle and much is known about the topological and holomorphic structure of X (cf. [18], [19], [20], [13], [1], [14]). In particular, if Aut X operates transitively on X, then X splits biholomorphically into a product $A(X) \times F$ with a projective rational manifold F ([3]).

In general not much can be said about the structure of the general fiber F of the Albanese map, even if $\alpha : X \to A(X)$ is holomorphically locally trivial. Therefore we assume that Aut X acts transitively on $A(X)$ via α and call $\alpha : X \to A(X)$ a *homogeneous* bundle. We want to find sufficient criteria for a bimeromorphic splitting of X into a product $A(X) \times F$. Simple examples show that such a splitting is in general impossible if X is not projective algebraic.

Lieberman [14] and Fujiki [6] analyzed the structure of $\mathrm{Aut}^0 X$ for a compact Kähler manifold X. Using results of Barlet and Douady about cycle spaces they showed in particular that every $\mathrm{Aut}^0 X$–orbit on X is Zariski–open in its closure and that in case of a homogeneous Albanese bundle $\mathrm{Aut}^0 X$ is an extension of the kernel of $\alpha_* : \mathrm{Aut}^0 X \to \mathrm{Aut}^0 A(X)$ by the center of $\mathrm{Aut}^0 X$. If X is projective algebraic it follows immediately from their work together with results of Rosenlicht [22] and Matsumura [15] that a finite unramified covering $\tilde{X}$ of X splits birationally into a product $A(\tilde{X}) \times F$.

Therefore the problem of birational splitting of the (homogeneous) Albanese bundle of a projective algebraic manifold X can be reduced to the study of bundles with finite abelian structure group.

[1]Ausgearbeitete und erweiterte Fassung eines Vortrags, gehalten auf dem 3. Bremen-Rostocker Kolloquium, Rostock, 22. und 23. Mai 1992

Let K be a rational function field over $\mathbb{C}$ and $\Gamma \subset \mathrm{Aut}_{\mathbb{C}} K$ a subgroup. If $(z_1, \ldots, z_n)$ is a transcendence basis of $K = \mathbb{C}(z_1, \ldots, z_n)$ over $\mathbb{C}$ with $\Gamma(\mathbb{C}(z_1, \ldots, z_i)) = \mathbb{C}(z_1, \ldots, z_i)$, $1 \leq i \leq n$, then we say that Γ *fixes a flag* in K. In [24] it is shown that the invariant fields of finite groups Γ of odd order with the flag–fixing property fullfill strong rationality conditions. We apply these rationality properties to prove our main result (Theorem 15):

Let X be a projective algebraic manifold with homogeneous Albanese bundle with projective rational fiber F. There is an abelian variety T, a projective unirational manifold $\tilde{F}$ and a finite subgroup $\Gamma \subset T$, embedded into Aut $\tilde{F}$, such that X is birationally equivalent to $(T \times_\Gamma \tilde{F}) \times \mathbb{P}_k$. If Γ is of odd order and fixes a flag in the meromorphic function field $\mathfrak{M}(\tilde{F})$ of $\tilde{F}$, then X splits birationally into a product $A(X) \times \mathbb{P}_r$.

The splitting property of the Albanese bundle depends strongly on the fact that there are enough meromorphic functions on $A(X)$. If $A(X)$ is not an abelian variety then in general it is impossible to get a bimeromorphic splitting by going to a finite unramified covering, even if the fiber of the bundle is a rational curve:

Proposition 14 *Let X be a compact Kähler manifold and $G \subset$ Aut X a connected complex subgroup with an open orbit Ω on X. Assume that Ω cannot be holomorphically fibered by compact complex tori. The following statements are equivalent:*

i) There is a finite unramified covering $\tilde{X}$ of X which splits bimeromorphically into a product $A(\tilde{X}) \times F$.

ii) $A(X)$ is an abelian variety.

iii) X is projective algebraic.

Using cycle space methods, Campana has proved uniqueness properties for the algebraic reduction of a compact complex manifold X. Using his results (see [4]) we can prove:

Proposition 19 *Let X be an almost homogeneous compact complex manifold. There is a projective algebraic almost homogeneous manifold $\hat{X}$ with $\mathfrak{M}(\hat{X}) = \mathfrak{M}(X)$. The Albanese torus $A(\hat{X})$ is the algebraic reduction of $A(X)$.*

This note is organized as follows: In section 1 we discuss the main structural properties of homogeneous kählerian Albanese bundles. In section 2 we give a straightforward and elementary proof of the complex version of a splitting theorem of Rosenlicht [22] and Matsumura [15]. This direct approach yields the detailed information about the orbits of an algebraic action of a connected solvable linear algebraic group which we need for the proof of theorem 15. In particular it enables us in our special situation to transform rational group actions of algebraic groups into regular ones via a birational mapping without using the structure theorems of A. Weil [26] in their general form.

1 G-induced Albanese bundles

For a (connected) compact complex manifold X the Albanese torus $A(X)$ and the holomorphic Albanese map $\alpha : X \to A(X)$ are characterized by the universality property that every holomorphic map $\pi : X \to T$ into a compact complex torus T induces a unique holomorphic map $\pi_A : A(X) \to T$ with $\pi_A \circ \alpha = \pi$. Consequently α is $\mathrm{Aut}X$–equivariant and defines a holomorphic homomorphism

$$\alpha_* : \ \mathrm{Aut}^0 X \to \mathrm{Aut}^0 A(X) = A(X), \quad \alpha \circ g = \alpha_*(g) \circ \alpha,$$

where $\mathrm{Aut}^0 A(X)$ is identified with $A(X)$ by fixing a basepoint $0 \in A(X)$. The Albanese torus $A(X)$ is a bimeromorphic invariant of X with $\dim_{\mathbb{C}} A(X) \leq \frac{1}{2} b_1(X)$ and $\dim_{\mathbb{C}} A(X) = \frac{1}{2} b_1(X)$ if X is kählerian. The Albanese torus of a projective algebraic manifold is an abelian variety (for details see [2]).

In general not much can be said about the structure of the general fiber F of the Albanese map. Even if X is projective algebraic and α is a locally trivial holomorphic fibration it may happen that F is a complex torus with $b_1(F) = b_1(X)$, e.g. if X is a hyperelliptic surface. If $X \neq A(X)$ is a compact complex solvmanifold, then X is a holomorphic fiber bundle over $A(X)$ and F is again a compact complex solvmanifold; in particular $b_1(F) \neq 0$.

If X is projective algebraic, the group $I := \ker\, \alpha_*$ is a linear algebraic group: By [2], Th. I.3.2 the canonical operation of I on $H^1(X, \mathcal{O}^*)$ is trivial. Take a very ample line bundle ζ on X and an embedding $\varphi : X \to \mathbb{P}_m$ defined by a basis of $\Gamma(X, \mathcal{O}(\zeta))$. Then the I-action on $\varphi(X)$ is induced by a linear action I on $\mathbb{P}_m$. Therefore the algebraic closure $\overline{I}$ of I in $\mathrm{PGL}(m, \mathbb{C})$ stabilizes $\varphi(X)$ and I is a linear algebraic group.

Let G be a connected complex subgroup of Aut X. We call $\alpha : X \to A(X)$ *G-induced,* if $\alpha_*(G) = A(X)$. In this case $F := \alpha^{-1}(0)$ is connected and α is a locally trivial holomorphic fibration, namely the quotient of $G \times F$ by the following action ρ of $G \cap \ker\, \alpha_*$ on $G \times F$:

$$\rho(h)(g, z) := (gh^{-1}, h(z)), \ \ h \in G \cap \ker \alpha_*, g \in G, z \in F.$$

Remarks 1) If G has an open orbit on X, then $\alpha : X \to A(X)$ is G-induced. Nevertheless it may happen that $b_1(F) \neq 0$, e.g. if X is a non-kählerian compact complex solvmanifold. But if X is kählerian, then $H^1(F, \mathcal{O})) = H^2(F, \mathcal{O}) = 0$ ([21]). In particular F is projective algebraic and $b_1(F) = 0$.

2) In [1] the structure of $\mathrm{Aut}^0 X$ was analyzed for a compact Kähler manifold X with G-induced Albanese bundle $\alpha : X \to A(X)$ and projective algebraic fiber F with $b_1(F) = 0$. It was shown that ker α_* is a linear algebraic group and that the canonical homomorphism ker $\alpha_* \to$ Aut F is injective.

3) D. Liebermann [14] and A. Fujiki [6] independently studied the structure of $\mathrm{Aut}^0 X$ for arbitrary Kähler manifolds X. They used cycle space methods and got very general results. In particular they showed that every $\mathrm{Aut}^0 X$-orbit is Zariski-open in its Zariski-closure. If $\alpha : X \to A(X)$ is G-induced, then ker α_* is a linear algebraic group. ■

For G-induced Albanese bundles the situation $b_1(F) \neq 0$ can only occur when ker α_* has infinitely many components.

Lemma 1 *Let X be a compact complex manifold with G-induced Albanese bundle $\alpha : X \to A(X)$ with typical fiber F. If $I := \ker \alpha_*$ has only finitely many components, then $b_1(F) = 0$.*

Proof: We set $G := \mathrm{Aut}^0 X$. The Albanese map $\beta : F \to A(F)$ yields a representation $\beta_* : I \to \mathrm{Aut}\ A(F)$. Since $\beta_*(I^0) \subset \mathrm{Aut}^0 A(F) = A(F)$, the group $G \times A(F)$ operates holomorphically and transitively on the compact manifold $\tilde{Y} := G \times_{I^0} A(F)$, which is a holomorphic $A(F)$-bundle over the complex torus G/I^0. It follows that $\tilde{Y}$ is complex parallelisable and a complex Lie group, since the isotropy group L of $[(e,0)] \in \tilde{Y}$ is connected:

$$L = \{(h, \beta_*(h^{-1})) \in I^0 \times A(F)\} = L^0.$$

Therefore $\tilde{Y}$ and its finite quotient $Y := G \times_I A(F)$ are complex tori. The canonical G-equivariant holomorphic map $X = G \times_I F \to Y$ together with the universality of $\alpha : X \to A(X)$ yield $Y = A(X)$, i.e. $A(F) = \{0\}$. ∎

From the fact that ker α_* is a linear algebraic group it follows that $\mathrm{Aut}^0 X$ is an extension of ker α_* by the center $C(\mathrm{Aut}^0 X)$ of $\mathrm{Aut}^0 X$. More precisely:

Lemma 2 *Let X be a compact complex manifold and G a connected complex subgroup of Aut X. If the Albanese map $\alpha : X \to A(X)$ is G-induced and if $G \cap$ker α_* is a linear algebraic group, then G is an extension of $G \cap$ker α_* by the center $C(G)$ of G; in other words $G = (G \cap \text{ker } \alpha_*) \cdot C(G)$. In particular α is $C(G)$-induced.*

Proof: Let i denote the Lie algebra of $I := G \cap \ker \alpha_*$ and define $\mathrm{Ad}_I G$ to be the image of Ad G under the restriction map Ad $G \to \mathrm{Aut}\ i$, Ad $G \mapsto$ Ad $G_{|i}$. Let $C_G(I^0)$ and $C_I(I^0)$ denote the centralizer of I^0 in G and I, respectively.

The adjoint representation of the linear algebraic group I is regular, hence $\mathrm{Ad}I$ is a linear algebraic normal subgroup of $\mathrm{Ad}_I G$ and of the algebraic closure of $\mathrm{Ad}_I G$ in $\mathrm{GL}(i)$. Therefore $\mathrm{Ad}_I G/\mathrm{Ad}I$ is holomorphically separable and $A(X) = G/I \to \mathrm{Ad}_I G/\mathrm{Ad}I$ has to be constant. Then $\mathrm{Ad}_I G = \mathrm{Ad}I$, $G/I = C_G(I^0)/C_I(I^0) = C_G(I^0)/I \cap C_G(I^0)$, and $G = I \cdot C_G(I^0) = I^0 \cdot C_G^0(I^0)$.

Since $G/I = C_G^0(I^0)/C_I(I^0) \cap C_G^0(I^0)$ is compact and since $C_I(I^0)$ is linear algebraic, the group $C_G^0(I^0)/C_I^0(I^0)$ is a compact complex torus. The relation $C_I^0(I^0) \subset C(C_G^0(I^0))$ implies that $C_G^0(I^0)$ is abelian and $C_G^0(I^0) \subset C^0(G)$. ∎

If the Albanese map $\alpha : X \to A(X)$ is G-induced with typical fiber $F = \mathbb{P}_1$, then X is kählerian and again we can apply Remark 2) together with Lemma 2. This leads to

Proposition 3 *Let G be a connected complex Lie group and $H \subset I \subset G$ closed complex subgroups with $T := G/I$ a compact complex torus and $I/H \in \{\mathbb{C}^*, \mathbb{C}, \mathbb{P}_1\}$. If G is non-abelian, then the bundle $G/H \to G/I$ is holomorphically trivial.*

Proof: There is a natural G-equivariant kählerian compactification X of G/H, namely a holomorphic $\mathbb{P}_1$-bundle $\alpha : X \to T = A(X)$, and we may assume that $G = \mathrm{Aut}^0 X$ and $I = \ker \alpha_*$. By Lemma 2 we have $G = I \cdot C^0(G)$ and since G is non-abelian it follows that dim $I \geq 2$. Therefore $I = \mathrm{PGL}(2, \mathbb{C})$ or I is a parabolic subgroup of $\mathrm{PGL}(2, \mathbb{C})$. In particular $C(G) \cap I \subset C(I) = \{e\}$. This implies $G = I \times C(G) = I \times C^0(G) = I \times A(X)$, $G/H = I/H \times A(X)$ and $X = \mathbb{P}_1 \times A(X)$. ■

Corollary *If $\alpha : X \to A(X)$ is a G-induced $\mathbb{P}_1$-bundle and if $Aut^0 X$ is non-abelian, then X splits biholomorphically into a product $\mathbb{P}_1 \times A(X)$.* ■

The condition for $\mathrm{Aut}^0 X$ to be non-abelian is really essential. Otherwise we meet counterexamples already for projective algebraic surfaces:

Example 1: The embedding $\mathbb{C} \to \mathbb{C}^* \times \mathbb{C}^*$, $z \to (e^z, e^{\tau z})$, $\tau \in \mathbb{C}$, Im $\tau > 0$, realizes $\mathbb{C}$ as a closed complex subgroup of $\mathbb{C}^* \times \mathbb{C}^*$, the quotient $T := \mathbb{C}^* \times \mathbb{C}^* / \mathbb{C}$ being an elliptic curve. The canonical compactification of the $\mathbb{C}$-principal bundle $\mathbb{C}^* \times \mathbb{C}^* \to T$ is a projective algebraic $\mathbb{P}_1$-bundle X over $T = A(X)$. No finite unramified covering $\tilde{X}$ of X splits biholomorphically into a product $\mathbb{P}_1 \times A(\tilde{X})$, since $\mathbb{C}^* \times \mathbb{C}^* \not\subset A(\tilde{X}) \times \mathrm{PGL}(2, \mathbb{C}) = \mathrm{Aut}^0(A(\tilde{X}) \times \mathbb{P}_1)$. Note that $\mathrm{Aut}^0 X = \mathbb{C}^* \times \mathbb{C}^*$ has an open orbit on $\tilde{X}$. ■

This example is also interesting under the following aspect: For any projective algebraic manifold X the group $\mathrm{Aut}^0 X$ carries the structure of an algebraic group (see [17], [16]), and for a G-induced Albanese bundle $\alpha : X \to A(X)$ the group ker α_* is linear algebraic. In the above example $\mathrm{Aut}^0 X = \mathbb{C}^* \times \mathbb{C}^*$ is a linear algebraic group, but ker α_* is not a linear algebraic subgroup of $\mathrm{Aut}^0 X$. In particular $\mathrm{Aut}^0 X$ is not a linear algebraic transformation group of X.

Remember that a (connected) compact complex manifold X is called *homogeneous,* if Aut X operates transitively on X. We call X *almost homogeneous,* if Aut X has an open orbit on X. Meanwhile it is a classical theorem that the Albanese bundle $\alpha : X \to A(X)$ of a compact homogeneous Kähler manifold splits biholomorphically into a product $A(X) \times F$, where F is projective rational. It was proven by Matsushima ([18]) for the case of reductive groups and in general by Borel-Remmert in [3]. This splitting is no longer true for almost homogeneous manifolds, as the above example shows. For almost homogeneous kählerian manifolds X the fiber F of the Albanese bundle is unirational, but in general not rational. A finite unramified covering $\tilde{X}$ of X splits topologically into a product $A(\tilde{X}) \times F$, but $\alpha : X \to A(X)$ is in general not topologically trivial ([1]).

We are interested in the birational structure of a G-induced Albanese bundle and in the following we will study this problem. It seems to be still an open problem whether the Albanese bundle of an almost homogeneous projective algebraic manifold X splits birationally into a product $A(X) \times F$.

As is known and will be seen (Corollary to Proposition 11) a suitable finite unramified covering $\tilde{X}$ of a projective algebraic G-induced Albanese bundle $\alpha : X \to A(X)$ splits birationally into a product $A(\tilde{X}) \times F$. This is not true for compact Kähler manifolds, even if they are almost homogeneous:

Example 2: Let Γ_{2n-1} be a discrete subgroup of $\mathbb{C}^n$ of rang $2n-1$. The group $\mathbb{C}^n/\Gamma_{2n-1}$ can be fibered as a holomorphic $\mathbb{C}^*$-principal bundle over a compact complex torus T. The canonical compactification X of this bundle is an almost homogeneous $\mathbb{P}_1$-bundle over $T = A(X)$. For $n \geq 3$ the group Γ_{2n-1} can be chosen such that $\mathbb{C}^n/\Gamma_{2n-1}$ has no non-constant meromorphic functions. Then no finite unramified covering $\tilde{X}$ of X splits birationally into a product $A(\tilde{X}) \times \mathbb{P}_1$. ∎

For a compact complex manifold Y we define $a(Y)$ to be the transcendence degree over $\mathbb{C}$ of the field of meromorphic functions on Y and $d(Y) := \dim_{\mathbb{C}} Y - a(Y)$. A compact almost homogeneous Kähler manifold X is positive algebraic if and only if $d(A(X)) = 0$ (Proposition 5).

Lemma 4 *Let X be a compact Kähler manifold and $G \subset Aut\ X$ a connected complex subgroup with an open orbit $\Omega = G/H$ in X. Assume that a finite unramified covering $\tilde{X}$ of X splits bimeromorphically into a product $F \times A(\tilde{X})$. There is a closed complex subgroup I of G containing H and a holomorphic fibration $\Omega = G/H \to G/I$ with $T := I/H$ a $d(A(X))$-dimensional compact complex torus.*

Proof: For $x \in X$ define A_x to be the intersection of all closed complex hypersurfaces in X containing x. Then $g(A_x) = A_{g(x)}$ for all $g \in \text{Aut } X$ and $A_x \cap \Omega = A_y \cap \Omega$ or $A_x \cap A_y \cap \Omega = \emptyset$ for all $x, y \in \Omega$. We fix $x \in \Omega$. The group $I := \{g \in G | g(A_x) = A_x\}$ operates transitively on $A_x \cap \Omega$. Let A be the irreducible component of A_x containing $A_x \cap \Omega$.
Then $a(X) = \text{codim } A$ since G operates transitively on Ω (see [5]).

Now $\tilde{X}$ splits and the fiber F of the Albanese bundle $\alpha : X \to A(X)$ is projective algebraic, therefore

$$a(X) = \dim F + a(A(X)), \quad \dim(A \cap F) = 0,$$

and

$$\dim \alpha(A) = \dim A = \dim X - a(X) = \dim A(X) - a(A(X)) = d(A(X)).$$

Since I operates with an open orbit on A and on $\alpha(A)$ and since all I-orbits in $A(X)$ have the same dimension, the group I operates already transitively on $\alpha(A)$, and $\alpha(A)$ is a compact complex torus. The assumption $A \not\subset \Omega$ leads to the contradiction

$$d(A(X)) = \dim \alpha(A) \leq \dim(A \cap (X \backslash \Omega)) \leq \dim A - 1 \leq d(A(X)) - 1.$$

This completes the proof. ∎

Example 3: Embed $\mathbb{C}^2$ as a closed complex subgroup into $G := (\mathbb{C}^*)^4$ such that the compact complex torus $T := G/\mathbb{C}^2$ is not projective algebraic. The canonical G-equivariant compactification X of G as a holomorphic $\mathbb{P}_2$-bundle over $T = A(X)$ is an almost homogeneous Kähler manifold. Since $\Omega := G$ is Stein, no finite unramified covering $\tilde{X}$ of X splits meromorphically into a product $\mathbb{P}_2 \times A(\tilde{X})$.

A holomorphic $\mathbb{P}_n$-bundle X over a projective algebraic base Y splits birationally into a product if it is induced by a rang $(n+1)$-vector bundle over Y, because every such vector bundle is an algebraic bundle ([23]). In particular X splits birationally into a product $\mathbb{P}_n \times Y$, if $H^2(Y, \mathcal{O}^*) = 0$, e.g. if Y is a compact Riemann surface.

Every G-induced kählerian Albanese bundle $\alpha : X \to A(X)$ over an abelian variety $A(X)$ with projective algebraic fiber F is a subbundle of a holomorphic $\mathbb{P}_n$-bundle $\tilde{X} \to A(X)$.

Proposition 5 *Let X be a compact Kähler manifold with G-induced Albanese bundle $\alpha : X \to A(X)$ where the fiber F is projective algebraic. The following statements are equivalent:*

i) $A(X)$ is an abelian variety.

ii) The Albanese bundle $X \to A(X)$ is a subbundle of a holomorphic $\mathbb{P}_n$-bundle $\tilde{X} \to A(X)$.

iii) X is projective algebraic.

Proof: It remains to verify the implication i) $\Rightarrow$ ii). Since $I = \ker \alpha_*$ has only finitely many components and since $b_1(F) = 0$ there is a proper embedding of F into $\mathbb{P}_n$ such that the I-action on F is induced by a linear I-action on $\mathbb{P}_n$. Choosing n minimal we see that $X = G \times_I F$ is a closed submanifold of $G \times_I \mathbb{P}_n$, which is a holomorphic $\mathbb{P}_n$-bundle over $A(X)$ and therefore projective algebraic ([2]). ■

2 Reduction to finite structure groups

Studying the birational structure of the Albanese bundle we are lead to meromorphic G-actions. Let G be a complex Lie group and Y an irreducible compact complex space. A meromorphic map $\psi : G \times Y \rightharpoonup Y$ is called a *meromorphic* G-action on Y if $\psi(g, -) : Y \to Y$ defines a bimeromorphic map of Y for every $g \in G$ with $\psi(e, -) = \mathrm{id}_Y$ and $\psi(gh, y) = \psi(g, \psi(h, y))$ whenever $\psi(gh, y)$, $\psi(h, y)$ and $\psi(g, \psi(h, y)$ are defined. If G is an algebraic group, Y a projective algebraic variety and ψ a rational (resp. regular) map, we call ψ a *rational* (resp. *regular*) G-action on Y. In the above situation G operates as a group of $\mathbb{C}$-automorphisms on the meromorphic function field $\mathfrak{M}(Y)$ of Y with invariant field $\mathfrak{M}(Y)^G$.

If B is a connected solvable linear algebraic group with a rational action on a projective algebraic variety Y, then, by the cross-section-theorem of Rosenlicht ([22]) generalized by Matsumura ([15]), the field $\mathfrak{M}(Y)$ is a purely transcendental extension of $\mathfrak{M}(Y)^B$, say $\mathfrak{M}(Y) = \mathfrak{M}(Y)^B(z_1, \dots, z_k)$. Let Y_B be a projective variety with $(Y_B) = \mathfrak{M}(Y)^B$. The above result means that Y is birationally equivalent to $Y_B \times \mathbb{P}_k$.

Assume now that Y is projective rational. Then Y_B is stably rational but need not be rational. There is a rational manifold Z on which B acts regularly and effectively with an open orbit. Take a rational manifold Y of the form $Y = W \times Z$ where W is unirational but not rational. The B-action on Z leads to an example where $\mathfrak{M}(Y)^B = \mathfrak{M}(W)$ is not a rational function field. Note (see [25]) that $\mathfrak{M}(Y)^B$ is a rational function field if and only if B *fixes a flag in* $\mathfrak{M}(Y)$, i.e. there is a transcendence basis $(z_1, \dots, z_n)$ of $\mathfrak{M}(Y) = \mathbb{C}(z_1, \dots, z_n)$ with $B(\mathbb{C}(z_1, \dots, z_i)) = \mathbb{C}(z_1, \dots, z_i)$ for $i \le i \le n$.

The purpose of this section is a straightforward and rather elementary proof of a special version of the Rosenlicht-theorem. This direct approach yields detailed information about the orbits of an algebraic action of a connected solvable linear algebraic group which we need later. In particular it allows us in our situation to transform rational group actions into regular ones via a birational mapping without using the general structure theorems of A. Weil ([26]).

For the sake of completeness we start with the regular-section-theorem for algebraic actions of solvable linear groups on affine varieties over $\mathbb{C}$.

Proposition 6 *Let Z be a normal irreducible affine algebraic variety and B a connected solvable linear algebraic group operating regularly and effectively on Z.*

1) *There is a B-invariant Zariski-open affine subvariety U of Z, a Zariski-closed subvariety V of U and a biregular map*

$$\sigma : \mathbb{C}^m \times (\mathbb{C}^*)^n \times W \to U, \quad m, n \geq 0,$$

with $\sigma(\mathbb{C}^m \times (\mathbb{C}^*)^n, w) = B(w)$ for all $w \in W$.

2) *Every B-orbit in Z is biregularly equivalent to a product of the form $\mathbb{C}^r \times (\mathbb{C}^*)^s$.*

The main step for the proof consists in analyzing regular $\mathbb{C}$- and $\mathbb{C}^*$-actions:

Lemma 7 *Let Z and B be as in Proposition 6. For every onedimensional connected normal algebraic subgroup N of B there is a B-invariant Zariski-open affine subvariety U_1 of Z and a Zariski-closed subvariety W_1 of U such that*

$$\sigma_1 : N \times W_1 \to U_1, \quad \sigma_1(n, w) := n(w),$$

is biregular. With $p : N \times W_1 \to W_1$ the canonical surjection, the map

$$\psi : B/N \times W_1 \to W_1, \quad \psi(gN, w) := p \circ \sigma^{-1}(g(w)),$$

defines a regular B/N-action on W_1 such that $N(\psi(B/N)(w)) = B(w)$ for all $w \in W_1$.

Proof of Proposition 6: By induction on $\dim_{\mathbb{C}} B$. Let $N, U_1, W_1, \sigma_1 : N \times W_1 \to U_1$ and $\psi : B/N \times W_1 \to W_1$ be as in Lemma 7 and let N_1/N be the kernel of ineffectivity of the ψ-action of B/N on W_1. By induction there is a biregular map

$$\sigma_2 : \mathbb{C}^m \times (\mathbb{C}^*)^n \times W \to U_2$$

with U_2 an affine algebraic Zariski-open subvariety of W_1 and W a Zariski-closed subvariety of U_2 and

$$B/N_1(w) = \sigma_2(\mathbb{C}^m \times (\mathbb{C}^*)^n, w) \quad \text{for all} \quad w \in W.$$

The composition

$$N \times \mathbb{C}^m \times (\mathbb{C}^*)^n \times W \overset{\mathrm{id} \times \sigma_2}{\longrightarrow} N \times U_2 \overset{\sigma_1}{\longrightarrow} \sigma_1(N \times U_2) =: U$$

defines a biregular map $\sigma := \sigma_1 \circ (\mathrm{id} \times \sigma_2)$ from $N \times \mathbb{C}^m \times (\mathbb{C}^*)^n \times W$ onto U. For $w \in W$ we have $\sigma(N \times \mathbb{C}^m \times (\mathbb{C}^*)^n, w) = N(\psi(B/N)(w)) = B(w)$. This completes the proof of 1). Statement 2) follows immediately from 1) by induction on $\dim Z$. ∎

Proof of Lemma 7: We embed Z as a closed affine subvariety into $\mathbb{C}^n$ such that the B-action on Z is induced by a linear action of B on $\mathbb{C}^n$ (see [12]) and choose n minimal. Identifying $\mathbb{C}^n$ with $\{(u_0 : \dots : u_n) \in \mathbb{P}_n \mid u_n \neq 0\}$ we study the connected solvable complex groups $\tilde{N} \subset \tilde{B} \subset \mathrm{SL}(n+1, \mathbb{C})$ with $\tilde{B}/\Gamma \cap \tilde{B} = B$, $\tilde{N}/\Gamma \cap \tilde{N} = N$, where $\Gamma \simeq \mathbb{Z}_{n+1}$ denotes the center of $\mathrm{SL}(n+1, \mathbb{C})$.

By the flag theorem of Lie we may assume that $\tilde{B}$ is contained in the subgroup of upper triangular matrices of $\mathrm{SL}(n+1, \mathbb{C})$ and handle the two possibilties $\tilde{N} = \mathbb{C}^*$ and $\tilde{N} = \mathbb{C}$ separately.

a) $\tilde{N} = \mathbb{C}^*$
Since $\tilde{N}$ is a connected normal subgroup of $\tilde{B}$ consisting only of semisimple elements, it is contained in the center of $\tilde{B}$ (see [10]). All elements of $\tilde{N}$ are diagonal matrices because the maximal tori in $\tilde{B}$ are conjugate to each other. Assume that

$$\tilde{N} = \{\mathrm{D}iag(\alpha^{k_0}, \dots, \alpha^{k_n}) \in \mathrm{S}L(n+1, \mathbb{C}) | \alpha \in \mathbb{C}^*, \, k_j \in \mathbb{Z}, \sum_{j=0}^{n} k_j = 0\}.$$

Define

$$\kappa(j) := \max\{i \in \{0, 1, \dots, n\} \| k_i = k_j\} \text{ for } j \in \{0, 1, \dots, n\}$$

and assume that there are exactly $r+1$ elements $\kappa_0, \dots, \kappa_r$ in the set $\{\kappa(j) | j \in \{0, 1, \dots, n\}\}$ for some $r \geq 1$. For $d := gcd(k_{\kappa_1} - k_{\kappa_0}, \dots, k_{\kappa_r} - k_{\kappa_0})$ there is a canonical isomorphism

$$\varphi : \mathbb{C}^*/\mathbb{Z}_d \to N, \, \varphi([\alpha]) := \mathrm{Diag}(\alpha^{k_0}, \dots, \alpha^{k_n}) \operatorname{mod} \Gamma.$$

Take $a_\nu \in \mathbb{Z}$ with $d = \sum_{\rho=1}^{r} a_\rho (k_{\kappa_\rho} - k_{\kappa_0})$ and choose $t_j, t'_j \in \mathbb{N}$ with

$$t_0 - t'_0 = \sum_{\rho=1}^{r} a_\rho, \quad t'_\rho - t_\rho = a_\rho, \quad \rho \in \mathbb{N}_r.$$

In particular

$$\sum_{j=0}^{r} t_j = \sum_{j=0}^{r} t'_j, \; d = \sum_{j=0}^{r} k_{\kappa_j}(t'_j - t_j).$$

Define

$$f(x_0, \dots, x_n) := \prod_{j=0}^{r} x_{\kappa_j}^{t_j} + \prod_{j=0}^{r} x_{\kappa_j}^{t'_j} \in C[x_0, \dots, x_n]$$

and

$$H := \{(u_0 : \dots : u_n) \in \mathbb{P}_n | f(u_0, \dots, u_n) = 0\} \cap Z.$$

Since $\tilde{N}$ centralizes $\tilde{B}$, the hyperplanes

$$A_{\kappa_j} := \{(u_0 : \dots : u_n) \in \mathbb{P}_n | u_{\kappa_j} = 0\}, \quad 0 \leq j \leq r,$$

are $\tilde{B}$-invariant by construction, and it follows that the Zariski-open affine subvariety

$$U_1 := \{(u_0 : \dots : u_n) \in \mathbb{P}_n \mid \prod_{j=0}^{r} u_{\kappa_j} \neq 0\} \cap Z$$

of Z is B-invariant. For $z = (u_0 : \ldots : u_n) \in U$ we get

$$\begin{aligned} f(\varphi([\alpha])(z)) &= \alpha^{\sum\limits_{j=0}^{r} k_{\kappa_j} t_j} (\prod_{j=0}^{r} u_{\kappa_j}^{t_j}) + \alpha^{\sum\limits_{j=0}^{r} k_{\kappa_j} t'_j} (\prod_{j=0}^{r} u_{\kappa_j}^{t'_j}) \\ &= \alpha^{\sum\limits_{j=0}^{r} k_{\kappa_j} t_j} (\prod_{j=0}^{r} u_{\kappa_j}^{t_j} + \alpha^d \cdot \prod_{j=0}^{r} u_{\kappa_j}^{t'_j}). \end{aligned}$$

This shows that there is exactly one $[\alpha] \in \mathbb{C}^* / \mathbb{Z}_d \cong N$ with $\varphi([\alpha])(z) \in H \cap U_1 =: W_1$. It follows that $\sigma_1 : N \times W_1 \to U_1$, $\sigma(g, w) := g(w)$, is a bijective regular open map. Since U_1 is normal, σ_1 is biregular.

b) $\tilde{N} = \mathbb{C}$
In this case $\tilde{N} \cap \Gamma = \{e\}$ and we may assume $n \geq 2$. There is a nilpotent upper triangular matrix $A \in \mathbb{C}^{(n+1,n+1)}$ of the form

$$\begin{pmatrix} 0 & a_{01} & \ldots & a_{0k} & \ldots & \ldots & a_{0n} \\ 0 & 0 & & & & & \vdots \\ \vdots & \vdots & \ddots & a_{k-1,k} & \ldots & \ldots & a_{k-1,n} \\ & & & 0 & r_{k+1} & \ldots & r_n \\ & & & & 0 & \ldots & 0 \\ & & & & & \ddots & \vdots \\ 0 & 0 & \ldots & & & & 0 \end{pmatrix},$$

$0 \leq k \leq n-1$, $(r_{k+1}, \ldots, r_n) \neq (0, \ldots, 0)$, with $\tilde{N} = \{e^{\lambda A} | \lambda \in \mathbb{C}\}$. The linear hyperplane

$$\tilde{L} := \{z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} | (Az)_k = \sum_{j=1}^{n-k} r_{k+j} z_{k+j} = 0\}$$

is $\tilde{B}$-invariant: Take $P = (p_{ij}) \in \tilde{B}$, $0 \leq i, j \leq n$. There exists $\alpha \in \mathbb{C}^*$ with

$$e^{PAP^{-1}} = Pe^A P^{-1} = e^{\alpha A}.$$

Since PAP^{-1} and αA are upper trigonal nilpotent matrices it follows that $PAP^{-1} = \alpha A$. For $PA = (c_{ij})$, $0 \leq i, j \leq n$, we get

$$c_{kj} = \begin{cases} 0 & 0 \leq j \leq k \\ p_{kk} r_j & k+1 \leq j \leq n \end{cases}$$

and

$$(PAz)_k = \sum_{j=k+1}^{n} p_{kk} r_j z_j = p_{kk} \sum_{j=1}^{n-k} r_{k+j} z_{k+j} = p_{kk} (Az)_k.$$

This means that

$$(A(Pz))_k = (\alpha^{-1} PAz)_k = \alpha^{-1} p_{kk} (Az)_k$$

and that $\tilde{L}$ is $\tilde{B}$-invariant. In particular, the hyperplane

$$L := \{(u_0 : \ldots : u_n) \in \mathbb{P}_n \mid \sum_{j=1}^{n-k} r_{k+j} u_{k+j} = 0\}$$

is B-invariant. With

$$\begin{aligned} U_1 &:= \{(u_0 : \ldots : u_n) \in \mathbb{P}_n \mid u_n \cdot \sum_{j=1}^{n-k} r_{k+j} u_{k+j} \neq 0\} \cap Z, \\ H &:= \{(u_0 : \ldots : u_n) \in \mathbb{P}_n \mid u_k = 0\} \cap Z, \end{aligned}$$

and $u := (u_0 \ldots : u_n) \in U_1$ we calculate $e^{\lambda A}(u) =: (w_0 : \ldots : w_n)$ and find

$$w_k = u_k + \sum_{j=1}^{n-k} \lambda r_{k+j} u_{k+j} = u_k + \lambda \cdot \sum_{j=1}^{n-k} r_{k+j} u_{k+j}.$$

Since $\sum_{j=1}^{n-k} r_{k+j} u_{k+j} \neq 0$, there is exactly one $\lambda \in \mathbb{C}$ with $e^{\lambda A}(u) \in H \cap U_1 =: W_1$. Just as above we conclude that $\sigma_1 : N \times W_1 \to U_1$ is biregular. The second statement of the Lemma is trivial. ∎

The splitting behaviour described in Proposition 6 leads immediately to a birational splitting of a normal projective algebraic variety X. Note that the kernel L of the canonical representation of Aut$X$ on $H^1(X, \mathcal{O}^*)$ is a linear algebraic group with $L^0 = (\ker \alpha)^0 \subset \ker \alpha \subset L$ if X is a projective algebraic manifold with Albanese map α.

Proposition 8 *Let X be a normal projective algebraic variety and B a connected solvable linear algebraic subgroup of L. There is a closed irreducible subvariety Y of X and a birational map*

$$\tau : \mathbb{P}_k \times Y \rightharpoonup X, \quad k = \max\{\dim B(x) | x \in X\}.$$

There is a Zariski-open subvariety V of Y with $\tau(\mathbb{P}_k \times \{v\}) = \overline{B(v)}$ for every $v \in V$. In particular $\mathfrak{M}(Y) = \mathfrak{M}(X)^B$.

Proof: We embed X into $\mathbb{P}_n$ such that the B-action on X is induced by a linear action of B on $\mathbb{P}_n$. By the flag theorem of Lie B stabilizes a hyperplane section D of X and the Zariski-open affine submanifold $Z := X \backslash D \neq \emptyset$. Using the notation of Proposition 6 we get a Zariski-open B-invariant affine subvariety U of Z, a closed subvariety W of U and a biregular map

$$\sigma : \mathbb{C}^m \times (\mathbb{C}^*)^n \times W \to U, \quad B(w) = \sigma(\mathbb{C}^m \times (\mathbb{C}^*)^n, w).$$

With $Y := \overline{W}$ (Zariski-closure of W in X) and $\hat{\sigma} : \mathbb{P}_m \times \mathbb{P}_n \times Y \to X$ the birational extension of σ we get the desired $\tau : \mathbb{P}_{m+n} \times Y \rightharpoonup \mathbb{P}_m \times \mathbb{P}_n \times Y \overset{\hat{\sigma}}{\rightharpoonup} X$. ∎

We want to study the subvariety Y more precisely. For a subgroup $B \subset \mathrm{Aut}X$ we define $N(B)$ to be the normalizer of B in Aut$X$.

Proposition 9 *Let X be a projective algebraic manifold with Albanese map $\alpha : X \to A(X)$ and B a connected solvable linear algebraic subgroup of ker α_* and G a connected complex subgroup of $(N(B))^0$ containing B. Let $\tau : \mathbb{P}_k \times Y \rightharpoonup X$ be a birational splitting as in Proposition 8; in particular $\alpha(Y) = \alpha(X)$.*

1) *The meromorphic map*
   $\psi : G/B \times Y \rightharpoonup Y$, $\psi(gB, y) := p \circ \tau^{-1} \circ g(y)$, *where $p : \mathbb{P}_k \times Y \to Y$ is the canonical surjection, defines a meromorphic G/B-action on Y and is compatible with the holomorphic G/B-action on $A(X)$:*

$$\alpha(\psi(gB, y)) = \alpha_*(g) \circ \alpha(y) \quad \textit{for all} \quad g \in G, y \in Y.$$

2) *There is a Zariski-open subset U of X such that $\alpha^{-1}(\alpha(u)) =: F_u$ is birationally equivalent to $\mathbb{P}_k \times (Y \cap F_u)$ for all $u \in U$.*

Proof: 1) Because of the special properties of Y (see Proposition 8) a short calculation shows that ψ defines in fact a meromorphic G/B-action on Y. Since α is G-equivariant and since $B \subset \ker \alpha_*$, this action is compatible with the G/B-action on $A(X)$.
2) With U, W, σ as in Proposition 8 the map

$$\sigma_{|F_u} : \mathbb{C}^m \times (\mathbb{C}^*)^n \times (W \cap F_u) \to U \cap F_u$$

is biregular for every $u \in U$ and induces a birational map

$$\mathbb{P}_k \times (Y \times F_u) \rightharpoonup \mathbb{P}_m \times \mathbb{P}_n \times (Y \times F_u) \rightharpoonup F_u.$$

∎

A meromorphic G-action on a complex space Y may be quite unpleasant for obvious reasons. We want to transform the meromorphic G/B-action on Y into a holomorphic G/B-action on a birationally equivalent projective algebraic variety $\tilde{Y}$. In our special situation this is possible.

Lemma 10 *Let T be a compact complex torus and $\psi : T \times Y \to Y$ a meromorphic T-action on an irreducible compact complex variety Y. Let Γ be a finite subgroup of T and $p : T \to T/\Gamma$ the canonical surjection. Assume that $\alpha : Y \to T/\Gamma$ is a holomorphic map with*

$$p(t)(\alpha(y)) = \alpha \circ \psi(t, y), \quad t \in T,\ y \in Y,$$

whenever $\psi(t, y)$ is defined.
There exists a normal compact complex variety $\tilde{F}$ and a holomorphic Γ-action on $\tilde{F}$ with the following properties:

1) *$\tilde{Y} := T \times_\Gamma \tilde{F}$ is bimeromorphically equivalent to Y.*

2) *$\tilde{F}$ is bimeromorphically equivalent to the general α-fiber.*

3) *If Y is projective algebraic, then $\tilde{F}$ can be chosen to be projective algebraic, too.*

Proof: Left translation of $\Gamma = \{\gamma_1 := e, \gamma_2, \ldots, \gamma_n\}$ on itself induces an embedding ϕ of Γ into the permutation group of n elements. Let $Y := Y \times \ldots \times Y$ (k factors) and define $\hat{Y} \subset Y_n$ to be the graph of the meromorphic map

$$(\gamma_2, \ldots, \gamma_n) : Y \to Y_{n-1},\ y \rightharpoonup (\gamma_2(y), \ldots, \gamma_n(y)),$$

whenever $\gamma_2(y), \ldots, \gamma_n(y)$ are defined. Then Γ operates holomorphically on $\hat{Y}$ by permutation:

$$\gamma(y_1, \ldots, y_n) := (y_{\phi(\gamma)(1)}, \ldots, y_{\phi(\gamma)(n)}), \quad \gamma \in \Gamma.$$

Since the group T is abelian, the Γ-action can be extended to a meromorphic T-action $T \times \hat{Y} \rightharpoonup \hat{T}$ with $t(y_1, \ldots, y_n) := (t(y_1), \ldots, t(y_n))$, $t \in T$, whenever $t(y_1), \ldots, t(y_n)$ are defined.

The projection of Y_n onto the first factor induces a holomorphic and bimeromorphic map of $\hat{Y}$ onto Y. Replacing Y by $\hat{Y}$ we may assume that Γ operates holomorphically on Y.

Let $\tilde{F}$ be the normalization of a general fiber of $\alpha : Y \to T/\Gamma$. The meromorphic T-action ψ induces a generically surjective meromorphic map $\tilde{\psi} : T \times \tilde{F} \rightharpoonup Y$ and T operates holomorphically on the quotient $\tilde{Y} := T \times_\Gamma \tilde{F}$, $\gamma(t, z) := (t - \gamma, \gamma(z))$, $\gamma \in \Gamma$, $t \in T$, $t \in \tilde{F}$. The meromorphic map

$$\tilde{Y} \rightharpoonup Y,\ [t, z] \rightharpoonup \tilde{\psi}(t, z) = t(z),$$

is bimeromorphic.

Remark: A general theorem of A. Weil ([26]) says, that every rational G-action on a projective algebraic variety Y can be transformed into a regular one: There is a birational map $\varphi : Y \rightharpoonup \hat{Y}$ onto a Moishezon variety $\hat{Y}$ such that the induced G-action on $\hat{Y}$ is regular. ∎

Proposition 11 *Let X be a projective algebraic manifold with G-induced Albanese bundle $\alpha : X \to A(X)$ with typical fiber F such that $I := G \cap \ker \alpha_*$ is a linear algebraic group (e.g. $G = Aut^0 X$). Let C be the identity component of the center of G, $T := C/C \cap I^0$ and $\Gamma := C \cap I / C \cap I^0$. Let B be a Borel subgroup of I and $k := \max\{\dim Bx | x \in X\}$. There is a normal projective algebraic variety $\tilde{F}$ and a holomorphic Γ-action on $\tilde{F}$ with the following properties:*

1) *There is a birational map*
$$\tau : \mathbb{P}_k \times (T \times_\Gamma \tilde{F}) \rightharpoonup X.$$

2) *The typical fiber F of α is birationally equivalent to $\mathbb{P}_k \times \tilde{F}$ and $\mathfrak{M}(\tilde{F}) = \mathfrak{M}(F)^B$.*

Proof: Note that $G = CI$ by Lemma 2. The center $C(I) = C \cap I$ of I is a linear algebraic group and every Borel subgroup B of I^0 contains $C(I^0) = C \cap I^0 = C \cap B$. Applying Proposition 9 to the group CB we get a birational splitting $\tau : \mathbb{P}_k \times Y \rightharpoonup X$ and a meromorphic action of the abelian variety $T = CB/B = C/C \cap B$ on Y. Now Proposition 11 follows from Lemma 10 in connection with statement 2) of Proposition 9. ∎

Corollary *Let X be a projective algebraic manifold with G-induced bundle $\alpha : X \to A(X)$ with typical fiber F.*

1) *A suitable finite unramified covering $\tilde{X}$ of X splits birationally into a product $F \times A(\tilde{X})$.*

2) *X is birationally equivalent to a projective algebraic manifold $T \times_\Gamma F$, where T is an abelian variety and $\Gamma \subset T$ a finite subgroup with $T/\Gamma = A(X)$.*

Summing up we have a sufficient criterion for the birational splitting of Albanese bundles in terms of linear algebraic groups:

Proposition 12 *Let X be a projective algebraic manifold with G-induced Albanese bundle $\alpha : X \to A(X)$ and typical fiber F. Assume that at least one of the following conditions is fullfilled for $I := \ker \alpha_* \subset Aut\ F$, $C := C^0(Aut^0 X)$, and the center $C(I) = C \cap I$ of I:*

1) *$C(I) \subset I^0$ (i.e. $C(I) = C(I^0)$)*

2) *$C(I) \subset Aut^0 F$ and $C(I)$ contains a regular semisimple element of $Aut^0 F$.*

3) *$C(I)$ consists only of semisimple elements and is contained in a Borel subgroup B of $AutF$.*

4) *A Borel subgroup of $Aut\ X$ operates with an open orbit on X.*

Then X splits birationally into a product $A(X) \times F$.

Proof: 1) By Proposition 11 there is a birational splitting $X \rightharpoonup \mathbb{P}_k \times A(X) \times \tilde{F}$, and $\mathbb{P}_k \times \tilde{F}$ is birationally equivalent to F.
2) The centralizer in $\mathrm{Aut}^0 F$ of a regular semisimple element is a maximal torus S in $\mathrm{Aut}^0 F$. Therefore $C \cap I \subset S \subset I^0$ since S is abelian.
3) Every subgroup of B consisting only of semisimple elements is contained in a maximal torus S of B. As above we get $C \cap I \subset S \subset I^0$.
4) Every Borel subgroup B of I has an open orbit on F. By Proposition 11 it follows that X is birationally equivalent to $\mathbb{P}_k \times A(X)$. ■

Corollary *Let X be an almost homogeneous projective algebraic manifold with $\dim X \leq 2 + \dim A(X)$. Then $X = A(X)$ or X is birationally equivalent to $\mathbb{P}_k \times A(X)$.*

Proof: A Borel subgroup of $\mathrm{Aut}X$ operates with an open orbit on X. ■

Remark: Note that in general the group $C \cap I$ is not contained in a Borel subgroup of Aut F, even if $C \cap I$ is finite and Aut F is connected. Take an abelian variety T and $\Gamma := \mathbb{Z}_2 \times \mathbb{Z}_2 \subset T$ and identify Γ with the subgroup $\{z \mapsto \pm\frac{1}{z}\}$ of Aut $\mathbb{P}_1$. Then $X := T \times_\Gamma \mathbb{P}_1$ is a holomorphic $\mathbb{P}_1$-bundle over $A(X) = T/\Gamma$ with $\mathrm{Aut}^0 X = T$ and $I = C \cap I = \Gamma$, which is not contained in a Borel subgroup of Aut $\mathbb{P}_1$. ■

We know that every holomorphic $\mathbb{P}_1$-bundle over a compact Riemann surface Y splits birationally into a product $\mathbb{P}_1 \times Y$, but for G-induced Albanese bundles with onedimensional fiber we have only the following result:

Proposition 13 *Let $\alpha : X \to A(X)$ be a holomorphic and G-induced $\mathbb{P}_1$-bundle over the abelian variety $A(X)$. Then X is birationally equivalent to $\mathbb{P}_1 \times A(X)$ or $Aut^0 X$ is an abelian variety with $Aut^0 X/(\mathbb{Z}_2 \times \mathbb{Z}_2) = A(X)$ and dim $X \geq 3$.*

Proof: If dim ker $\alpha_* \geq 1$ then the Corollary of Proposition 12 applies. If ker α_* is finite, then $\mathrm{Aut}^0 X$ is an abelian variety and ker α_* equals its centralizer in Aut $\mathbb{P}_1$. But then ker $\alpha_* = \mathbb{Z}_2 \times \mathbb{Z}_2$. ■

Finally we generalize the result of Lemma 4 about the splitting of compact almost homogeneous Kähler manifolds.

Proposition 14 *Let X be a compact Kähler manifold and $G \subset Aut\ X$ a connected complex subgroup with an open orbit Ω on X. Assume that Ω cannot be holomorphically fibered by compact complex tori. The following statements are equivalent:*

i) There is a finite unramified covering $\tilde{X}$ of X which splits bimeromorphically into a product $A(\tilde{X}) \times F$.

ii) $A(X)$ is an abelian variety.

iii) X is projective algebraic.

Proof: i) $\Rightarrow$ ii): From Lemma 4 we get $d(A(X)) = 0$, hence $A(X)$ is an abelian variety.
ii) $\Leftrightarrow$ iii): Proposition 5.
iii) $\Rightarrow$ i): Corollary of Proposition 11. ■

3 Flag–fixing structure groups

Let us assume that we have a projective algebraic manifold X with G-induced Albanese bundle $\alpha : X \to A(X)$ with projective rational fiber F. For $G := \mathrm{Aut}^0 X$, $I := \ker \alpha_*$ and C the identity component of the center of G we know that X is birationally equivalent to a holomorphic fiber bundle $(T \times_\Gamma \tilde{F}) \times \mathbb{P}_k$, where $T = C/C \cap I^0$ is an abelian variety and $\Gamma := C \cap I/C \cap I^0$ is a finite subgroup of T (Proposition 11). Therefore the problem of the birational splitting of the Albanese bundle of X reduces to the problem of the birational splitting of the holomorphic bundle $T \times_\Gamma \tilde{F}$ into a product $A(X) \times \tilde{F}$. Let B be a Borel subgroup of I. We assume that B fixes a flag in $\mathfrak{M}(F)$, i.e. that $\mathfrak{M}(F)^B = \mathfrak{M}(\tilde{F})$ is a rational function field.

Theorem 15 *If Γ is of odd order and fixes a flag in $\mathfrak{M}(\tilde{F})$, then X splits birationally into a product $A(X) \times \mathbb{P}_r$.*

Let $K = \mathbb{C}(z_1, \ldots, z_n)$ be a rational function field with transcendence basis $(z_1, \ldots, z_n)$ and Γ a finite subgroup of $\mathrm{Aut}_{\mathbb{C}}K$ with $K_i := \mathbb{C}(z_1, \ldots, z_i) = \Gamma(K_i)$, $1 \leq i \leq n$. In general the invariant field K^Γ of Γ is not a rational function field even if Γ is cyclic. There is an example (see [24]) for $n = 3$ and an involution $\sigma \in \mathrm{Aut}_{\mathbb{C}}K$ with $\sigma(z_1) = z_1$, $\sigma(z_2) = -z_2$, where K^σ is not a rational function field. This example is based on results of Artin, Mumford and Deligne about unirational hypersurfaces in $\mathbb{P}_4$ which are not rational. But if Γ is of odd order, then K^Γ is always rational. This is a consequence of the following result which was proved in [24]:

If Γ is of odd order, then K^Γ_{i+1} is a purely transcendental extension of K^Γ_i, $0 \leq i \leq n-1$, $K_0 := \mathbb{C}$.

This result yields the induction step for the proof of the following splitting theorem, which implies Theorem 15 by the above remarks.

Theorem 16 *Let T be an abelian variety and $\Gamma \subset T$ a finite subgroup of odd order operating holomorphically on a k-dimensional projective rational manifold F. If Γ fixes a flag in $\mathfrak{M}(F)$, then $X := T \times_\Gamma F$ is birationally equivalent to $A(X) \times \mathbb{P}_k$.*

Let Γ and $K = \mathbb{C}(z_1, \ldots, z_n)$ be as above. In the following we assume that Γ is of odd order.

Lemma 17 *The group Γ fixes a flag $(x_1, \ldots, x_n)$ in K with the property*

$$\gamma(x_n) = \alpha(\gamma)x_n, \quad \alpha(\gamma) \in \mathbb{C}(x_1, \ldots, x_{n-1})$$

for all $\gamma \in \Gamma$.
For $\alpha \in \mathbb{C}^*$ define $g_\alpha \in Aut_{\mathbb{C}}K$ by $g_\alpha(x_i) := x_i + \delta_{in}(\alpha - 1)x_i$, $1 \leq i \leq n$. The group $A := \{g_\alpha \mid \alpha \in \mathbb{C}^*\}$ centralizes Γ and $K^A = \mathbb{C}(x_1, \ldots, x_{n-1})$.

Proof: Let $r := \min\{i \in \mathbb{N}_n \mid \Gamma \to \mathrm{Aut}_{\mathbb{C}}K_i \text{ is injective}\}$ and $\tilde{\gamma} \in \Gamma$ with $\tilde{\gamma}_{|K_r} \neq \mathrm{id}$, $\tilde{\gamma}_{|K_{r-1}} = \mathrm{id}$. We may assume that $\tilde{\gamma}(z_r) = \beta z_r$ where β is a root of unity with $\mathrm{ord}\,\beta \geq 3$. For $\gamma \in \Gamma$ we get

$$\gamma(z_r) = \frac{az_r + b}{cz_r + d}, \quad a, b, c, d \in K_{r-1},\ ad - bc \neq 0,$$

and the relation $\tilde{\gamma} \circ \gamma = \gamma \circ \tilde{\gamma}$ yields

$$\frac{a\beta z_r + b}{c\beta z_r + d} = \beta \frac{az_r + b}{cz_r + d}.$$

Then $ac = bc = 0$, since $\beta^2 \neq 1$. In particular $b = c = 0$. By the above quoted result of [24] the invariant field K^Γ of Γ is a purely transcendental extension of K^Γ_r, say $K^\Gamma = K^\Gamma_r(u_{r+1}, \ldots, u_n)$. Then

$$[K_r(u_{r+1}, \ldots, u_n) : K^\Gamma_r(u_{r+1}, \ldots, u_n)] =$$
$$[K_r(u_{r+1}, \ldots, u_n) : K^\Gamma] \geq [K_r : K^\Gamma_r] = \mathrm{ord}\,\Gamma = [K : K^\Gamma]$$

and therefore $K = K_r(u_{r+1}, \dots, u_n) = \mathbb{C}(z_1, \dots, z_{r-1}, u_{r+1}, \dots, u_n, z_r)$.
Define $x_j := z_j$, $1 \le j \le r-1$, $x_k := u_{k+1}$, $r+1 \le k \le n-1$, $x_n := z_r$. ■

Proposition 18 *Let F be a projective rational manifold and $\Gamma \subset Aut\ F$ a finite abelian group of odd order, fixing a flag in $\mathfrak{M}(F)$. There is a projective rational manifold $\tilde{F}$ and a birational map $\psi : F \to \tilde{F}$ such that $\tilde{\Gamma} := \psi\Gamma\psi^{-1}$ is contained in a onedimensional abelian reductive subgroup R of $Aut\ \tilde{F}$ and fixes a flag in the rational function field $\mathfrak{M}(\tilde{F})^{R^0}$.*

Proof: With $K := \mathfrak{M}(F)$ we choose $(x_1, \dots, x_r)$ and A as in Lemma 17 and get a birational map

$$\tau : F \rightharpoonup \mathbb{P}_1 \times \dots \times \mathbb{P}_1 =: \hat{F}, \quad y \mapsto (x_1(y), \dots, x_n(y)).$$

Then $\hat{R} := \tau A\Gamma\tau^{-1}$ is a group of birational transformations of $\hat{F}$. Note that $\hat{R}^0 = \tau A \tau^{-1} = \mathbb{C}^*$ operates holomorphically on $\hat{F}$. The $\hat{R} := \tau A\Gamma\tau^{-1}$ is a group of birational transformations of $\hat{F}$. Note that $\hat{R}^0 = \tau A \tau^{-1} = \mathbb{C}^*$ operates holomorphically on $\hat{F}$. The finite abelian group $\hat{\Gamma} := \tau\Gamma\tau^{-1} = \{\gamma_1 = \mathrm{id}, \gamma_2, \dots, \gamma_r\}$ defines a rational map

$$\Lambda : \hat{F} \rightharpoonup \hat{F} \times \dots \times \hat{F}, \quad u \mapsto (\gamma_2(u), \dots, \gamma_r(u)),$$

and the canonical holomorphic and birational map

$$\pi : \text{ graph } \Lambda \to \hat{F}, \quad (u_1, \dots, u_r) \mapsto u_1,$$

induces a π-equivariant holomorphic action of $\hat{R}$ on graph Λ with the property

$$g(u_1, \dots, u_r) = (g(u_1), \dots, g(u_r)), \quad g \in \hat{R},\ (u_1, \dots, u_r) \in \text{ graph } \Lambda,$$

whenever $g(u_j)$ is defined for $1 \le j \le r$. With respect to this action $\hat{R}$ can be identified with a complex subgroup $\pi^{-1}\hat{R}\pi \subset \mathrm{Aut}(\text{graph } \Lambda)$. Let $\tilde{F} \xrightarrow{p}$ graph Λ be a desingularization of graph Λ which is $\pi^{-1}\hat{R}\pi$-equivariant. Then $\psi := p^{-1} \circ \pi^{-1} \circ \tau$ and $R := \psi(A\Gamma)\psi^{-1}$ have the desired properties. ■

Finally we prove Theorem 16 by induction on $k = \dim F$:
Using the notation of Lemma 17 we know that X is birationally equivalent to $\tilde{X} := T \times_\Gamma \tilde{F}$, where Γ is identified with $\psi\Gamma\psi^{-1} \in$ Aut $\tilde{F}$ and fixes a flag in $\mathfrak{M}(\tilde{F})^{R^0}$. The group $\tilde{G} := T \times R^0$ operates holomorphically on $\tilde{X}$ and by Proposition 11 there is a birational map $\tilde{X} \rightharpoonup (T \times_\Gamma \hat{Y}) \times \mathbb{P}_1$, where $\hat{Y}$ is a normal projective rational variety with $\mathfrak{M} = (\tilde{F})^{R_0}$. Substituting $\hat{Y}$ by a Γ-equivariant projective rational desingularization Y we may apply the induction hypothesis to $T \times_\Gamma Y \simeq T \times_\Gamma \hat{Y}$. This completes the proof. ■

Proposition 19 *Let X be an almost homogeneous compact complex manifold. There is a projective algebraic almost homogeneous manifold $\hat{X}$ with $\mathfrak{M}(\hat{X}) = \mathfrak{M}(X)$. The Albanese torus $A(\hat{X})$ is the algebraic reduction of $A(X)$.*

Proof: The family of fibers of an algebraic reduction of a compact complex manifold V is a compact Moishezon subvariety $A_0(V)$ of the Barlet cycle space $\mathcal{C}(V)$ and $A_0(V)$ is invariant under the action of $\mathrm{Aut}^0 V$ on $\mathcal{C}(V)$ ([4]). Therefore after a suitable $\mathrm{Aut}^0 X$-equivariant desingularization of $A_0(X)$ we get an almost homogeneous Moishezon manifold Z with $\mathfrak{M}(Z) = \mathfrak{M}(X)$, and $A(Z)$ is the algebraic reduction of $A(X)$. Now we study the Albanese bundle $\beta : Z \to A(Z)$ with typical fiber F and show that $b_1(F) = 0$. Since F is almost homogeneous, the Albanese map $\beta_F : F \to A(F)$ is a holomorphic fiber bundle. The group $J := ker\beta_*$ operates on F by restriction $\rho : J \to \mathrm{Aut}\ F$ and on $A(F)$ via $\rho_F := (\beta_F)_* \circ \rho$. Note that $Y := \mathrm{Aut}^0 Z \times_{\rho_F} A(F)$ is a homogeneous complex manifold and a holomorphic fiber bundle over $A(Z)$ with fiber $A(F)$. Moreover Z is a holomorphic fiber bundle over Y, induced by β_F; in particular Y is a Moishezon manifold. A compact homogeneous Moishezon manifold is projective algebraic and is a product of an abelian variety with a projective rational manifold [7]. Therefore Y is an abelian variety and the universality of $A(X)$ yields $Y = A(Z)$, hence $A(F) = \{0\}$. But $\dim_{\mathbb{C}} A(F) = \frac{1}{2} b_1(F)$ for the Moishezon manifold F.

Let ξ be a holomorphic line bundle on Z such that the linear system $H^0(Z, \mathcal{O}(\xi))$ defines a bimeromorphic map of Z onto a projective algebraic variety. Since $\mathrm{Aut}^0 Z$ stabilizes the Chern class $c(\xi)$, the group J stabilizes $c(\zeta_{|F})$. It follows that $\zeta_{|F}$ is J-stable, because $H^1(F, \mathcal{O}) = 0$. We may assume that $H^0(F, \mathcal{O}(\xi_{|F}))$ defines a bimeromorphic map φ of F onto a closed subvariety $W \subset \mathbb{P}_N$, $N + 1 = \dim_{\mathbb{C}} H^0(F, \mathcal{O}(\xi_{|F}))$, such that J-operates as a group of collineations on $\mathbb{P}_N$, stabilizing W.

The representation $\tilde{\rho} : J \to \mathrm{Aut}\ W$, $\tilde{\rho}(h) := \varphi \circ h \circ \varphi^{-1}$, defines a projective algebraic variety $\mathrm{Aut}^0 X \times_{\tilde{\rho}} W$, which is a holomorphic fiber bundle over $A(Z)$, almost homogeneous and bimeromorphically equivalent to Z. After a suitable desingularization we get the desired manifold $\hat{X}$. ■

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