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On a theorem of Deaconescu[1]

ABSTRACT. Consider finite groups having all nontrivial elements of prime order as a subclass of CN–groups in which the centralizers of all nontrivial elements are nilpotent. Then we give in generalization of a Theorem of DEACONESCU [2] necessary and sufficient conditions to be such a group.

1 Introduction

Let CP_1, CP and CN be the class of finite groups in which the centralizers of all nontrivial elements contain only elements of prime order, of prime power order and are nilpotent, respectively.

Clearly $CP_1 \subset CP \subset CN$, and CP_1, CP consists of exactly those groups having all nontrivial elements of prime order and prime power order, respectively.

DEACONESCU [2] had shown

Theorem 1 *Let G be a CP_1–group. Then one of the following cases occurs:*

I. G is a p–group of exponent p.

II. (a) $|G| = p^a q, \quad 3 \le p < q, a \ge 3, \quad |F(G)| = p^{a-1}, \quad |G : G'| = p.$

(b) $|G| = p^a q, \quad 3 \le p, q, a \ge 1, \quad |F(G)| = |G'| = p^a.$

(c) $|G| = 2^a p, \quad p \ge 3, \quad a \ge 2, \quad |F(G)| = |G'| = 2^a.$

(d) $|G| = 2p^a, \quad p \ge 3, \quad a \ge 1, \quad |F(G)| = |G'| = p^a$ and $F(G)$ is elementary abelian.

III. $G \cong A_5$.

Remark: In the main theorem of [2] (IIb) the condition $q < p$ is false because 2.8 of [2] is only true if H is a normal subgroup.

[1] Ausgearbeitete Fassung eines Vortrags, gehalten auf dem 3. Bremen – Rostocker Kolloquium, Rostock, 22. und 23. Mai 1992

Deaconescu asks if the conditions in Theorem 1 (II) are also sufficient. In this note we show that this is not the case in generally. Furthermore we give a new characterization of CP_1–groups.

2 Result and Examples

In [3] and [1] we can find a classification of all solvable CN–groups and nonsolvable CP–groups. From this we can easily deduce:

Theorem 2 *A group G is a CP–group iff one of the following holds:*

(i) G is isomorphic with $PSL(2,q)$ with $q = 4,7,8,9,17$; $PSL(3,4)$, $Sz(8)$, $Sz(32)$ or M_{10}.

(ii) G has a nontrivial normal 2–subgroup P and G/P is isomorphic with $PSL(2,4)$, $PSL(2,8)$, $Sz(8)$ or $Sz(32)$. Moreover, P is elementary abelian and isomorphic with a direct sum of natural modules.

(iii) G is a p–group.

(iv) G is a FROBENIUS *group whose kernel is a p–group and the complement is either a cyclic q–group ($q \neq p$) or a generalized quaternion group.*

(v) G is a 3–step group of order $p^a q^b$ (p, q primes, $q > 2$) i.e. $G = O_{pp'p}(G)$ and $G \supset O_{pp'}(G)$ with

a) $O_{pp'}(G)$ is a FROBENIUS *group with kernel $O_p(G)$ and cyclic complement.*

b) $G/O_p(G)$ is a FROBENIUS *group with kernel $O_{pp'}(G)/O_p(G)$.*

Now we prove

Theorem 3 *A group G is a CP_1–group if G is isomorphic with a group of the following list:*

(i) A_5,

(ii) a p–group of exponent p,

(iii) a group of type (iv) in Theorem 2 having only Sylow subgroups of exponent p and q.

Proof. By Theorems 1 and 2 we have only to show that there is no group of type II a) in Theorem 1 in which the Sylow p–subgroup is of exponent p. Assume there is a minimal counterexample G with a Sylow p–subgroup of exponent p and $|G| = p^a q$, $|F(G)| = p^{a-1}$, $|G'| = p^{a-1}q$. Hence G has p–length 2. This is a contradiction to [5] 1.13 Theorem p. 413.

Now we search if the statements of Theorem 1, II are sufficient to be a CP_1–group. Clearly it is necessary that all Sylow p– and 2–groups must have exponent p or 2, respectively. We cannot omit this supposition:

Example 1. Let G be an abnilpotent group of index system $(3, 13, 3^3)$ (see [7] Theorem 4.2), that means $G = \langle a \rangle \ltimes \langle b \rangle \ltimes GF(3^3)^+$ with $a^3 = b^{13} = 1, a^{-1}ba = b^3, a^{-1}ca = c^3, b^{-1}cb = cw$, where w is a primitive 13th root of unity in $GF(3^3)$. Then we have $|F(G)| = |GF(3^3)^+| = 3^3$ and $|G : G'| = 3$, but a Sylow 3–group of G is not of exponent 3.

Now we add the above supposition in Theorem 1, II. Then statement (a) cannot occur (see Theorem 3). The following example shows that statement (b) is not sufficient for CP_1–groups.

Example 2. Let $p > 2$ be a prime. Take a natural number n such that there exists a prime $q > 2$ with $q | p^n - 1$. Then by [4] (9.24 Satz a), S. 228) there is an extra special p–group P with:

a) $\exp P = p$,

b) $|P| = p^{2n+1}$,

c) $\exists \alpha \in \mathrm{Aut}\ P :$ (i) α acts trivially on $Z(P)$
(ii) α acts fixed–point–freely on $P/Z(P)$.

Let G be the semidirect product $< \alpha > \ltimes P$. It is easy to check that the assertion of II(b) in Theorem 1 is valid. But G has elements of order pq namely $\alpha^i z (i = 1, \ldots, q-1, z \in Z(P))$.

But if we know that the Sylow p–subgroup P of G in statement II(b) of Theorem 1 is abelain then the condition is also sufficient: By [6] (Satz 7.13) we have then $G' = P = O_P(G) = [P, G] \times C_P(G) = G' \times C_P(G)$ from which $C_P(G) = 1$ follows and therefore $C_G(x) = \langle x \rangle$ for all q–elements x.

Consequently, the conditions II(c), (d) of Theorem 1 with the above supposition are sufficient to be a CP_1–group.

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