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The $3n+1$ Collatz Problem and Functional Equations

Abstract. *This paper reports on some functional equations, which arise in connection with the famous $3n+1$ Collatz problem. In particular, it reports on two analytic versions of the corresponding Collatz conjecture, which are contained in* [1].

1 Preliminaries

We start with the modified Collatz mapping $t : \mathbb{Z} \to \mathbb{Z}$, defined by

$$t(n) = \begin{cases} \frac{1}{2}n & \text{for} \quad n \equiv 0 \mod 2 \quad , \\ \frac{1}{2}(3n+1) & \text{for} \quad n \equiv 1 \mod 2 \quad , \end{cases} \tag{1}$$

with $n \in \mathbb{Z}$, and consider the iterates t_m, defined recursively by

$$t_m(n) = t(t_{m-1}(n)),\ t_0(n) = n \tag{2}$$

for $m \in \mathbb{N}$. Already as a student, about 1930, L. Collatz has stated the following conjecture, cf. J. C. Lagarias [3] and U. Eckhardt [2]:

Conjecture 1. *For every number $n \in \mathbb{N}$ there exists a number $m(n)$ with the property*

$$t_{m(n)}(n) = 1 \quad . \tag{3}$$

A general proof of this Collatz conjecture is still open. After introducing the functions $x_m(n)$ with the only values 0 and 1 by

$$t_m(n) \equiv x_m(n) \mod 2 \quad , \tag{4}$$

the iterates t_m can be represented explicitly in the following way, cf. J. C. Lagarias [3]:

Lemma . *The iterates t_m possess the representation*

$$t_m(n) = \frac{1}{2^m}(a_{m\nu}n + b_{m\nu}) \ \text{for}\ n \equiv \nu \mod 2^m \tag{5}$$

and $0 \leq \nu < 2^m$ with the coefficients

$$a_{m\nu} = 3^{x_0(\nu)+\cdots+x_{m-1}(\nu)},\ b_{m\nu} = \sum_{j=0}^{m-1} x_j(\nu)2^j \cdot 3^{x_{j+1}(\nu)+\cdots+x_{m-1}(\nu)} \tag{6}$$

and the initial values $a_{m0} = 1,\ b_{m0} = 0$.

In what follows, we give a report on the main results of [1], but we also make some new considerations. Concerning the corresponding predecessor sets, cf. G. Wirsching [4].

2 Generating functions

For the mapping (1), we introduce the three generating functions

$$f_m(z) = \sum_{n=0}^{\infty} t_m(n)z^n \quad , \tag{7}$$

$$g_n(w) = \sum_{m=0}^{\infty} t_m(n)w^n \quad , \tag{8}$$

$$F(z,w) = \sum_{m,n=0}^{\infty} t_m(n)z^n w^m \tag{9}$$

of the complex variables z and w. In view of the estimation

$$t_m(n) \leq \left(\frac{3}{2}\right)^m (n+1) - 1$$

for all integers $m, n \geq 0$, the first series converges for $|z| < 1$, the second for $|w| < \frac{2}{3}$, and the third in the polydisk $\left\{(z,w)\,\middle|\,|z| < 1, |w| < \frac{2}{3}\right\}$. Hence, the three generating functions are holomorphic in the just mentioned domains.

Theorem 1. *The functions $f_m(z)$ are rational functions of the form*

$$f_m(z) = \frac{p_m(z)}{(1 - z^{2^m})^2} \tag{10}$$

with the numerator

$$p_m(z) = \sum_{\nu=1}^{2^{m+1}-1} c_{m\nu} z^\nu \tag{11}$$

and the coefficients

$$c_{m\nu} = \begin{cases} t_m(\nu) & \text{for } \nu = 1, 2, \ldots, 2^m \quad , \\ -t_m(\nu - 2^{m+1}) & \text{for } \nu = 2^m + 1, \ldots, 2^{m+1} - 1 \quad . \end{cases} \tag{12}$$

It can be proved that Conjecture 1 is equivalent to

Conjecture 2. *The functions $g_n(w)$ from (8) are rational functions of the form*

$$g_n(w) = \frac{q_n(w)}{1 - w^2} \quad , \tag{13}$$

where $q_n(w)$ are polynomials with integer coefficients.

The functions (9) can also be represented as

$$F(z, w) = \sum_{m=0}^{\infty} f_m(z) w^m = \sum_{n=0}^{\infty} g_n(w) z^n \quad .$$

3 Functional equations

Now, we introduce the notation

$$\lambda = e^{2\pi i/3} \quad . \tag{14}$$

Theorem 2. *The generating functions (7),(8) and (9) satisfy the linear functional equations*

$$f_{m+1}(z^3) = f_m(z^6) + \frac{1}{3z} \sum_{\nu=0}^{2} \lambda^\nu f_m(\lambda^\nu z^2) \quad , \tag{15}$$

$$g_{2n}(w) = 2n + w g_n(w), \quad g_{2n-1}(w) = 2n - 1 + w g_{3n-1}(w) \quad , \tag{16}$$

$$F(z^3, w) = \frac{z^3}{(1 - z^3)^2} + w F(z^6, w) + \frac{w}{3z} \sum_{\nu=0}^{2} \lambda^\nu F(\lambda^\nu z^2, w) \quad . \tag{17}$$

In view of the factor w at the right-hand side, the homogeneous equation

$$H(z^3, w) = w H(z^6, w) + \frac{w}{3z} \sum_{\nu=0}^{2} \lambda^\nu \, H(\lambda^\nu z^2, w)$$

corresponding to (17) cannot have a nontrivial solution, which is holomorphic in w in the neighbourhood of $w = 0$, and for a fixed $z \neq 0$. However, for $w = 1$ and $H(z, 1) = h(z)$ we get the functional equation

$$h(z^3) = h(z^6) + \frac{1}{3z} \sum_{\nu=0}^{2} \lambda^\nu h(\lambda^\nu z^2) \quad , \tag{18}$$

and this equation has the solution

$$h(z) = h_0 + h_1 \frac{z}{1-z} \tag{19}$$

for arbitrary constants h_0 and h_1.

It can be proved that Conjecture 1 is also equivalent to

Conjecture 3. *Besides of* (19), *there exist no further solutions of* (18), *which are holomorphic for* $|z| < 1$.

A first step to an analytic discussion of the functional equation (18) is given by the following

Theorem 3. *Every entire solution of the functional equation* (18) *is constant.*

Proof. Let f be an entire solution of (18) and $M(r)$ its maximum modulus, defined by

$$M(r) = \max_{|z| \leq r} |f(z)| \ .$$

The equation (18) yields for $r > 1$:

$$M(r^6) \leq M(r^3) + \frac{1}{r} M(r^2) \leq 2M(r^3) \ ,$$

since $M(r)$ is non-decreasing, and the maximum is attained at the boundary $|z| = r$.

Hence, for every integer $k \geq 0$, we obtain

$$M(r^6) \leq 2^{k+1} M(r^{3 \cdot 2^{-k}}) \ .$$

Now, let $r > 2^{1/3}$ and choose

$$k = 1 + [\log(3 \log r)] \ ,$$

where log denotes the logarithm of basis 2. Then

$$3 \log r < 2^k \leq 6 \log r \ ,$$

and therefore

$$r^{3 \cdot 2^{-k}} < 2 \ .$$

From this, we conclude

$$M(r^6) \leq 12M(2) \log r$$

for all $r > 2^{1/3}$, or, equivalently,

$$M(r) \leq 2M(2) \log r$$

for all $r > 4$, and Cauchy's estimation for the coefficients of the power series of the function f at $z = 0$ yields the assertion. ■

Next, we want to set up some further functional equations, which are closely connected with (18). After replacing z^3 by z and using Cauchy's integral formula, equation (18) can also be written in form of the integral equation

$$h(z) = \frac{1}{2\pi i} \int \frac{(w^2 + (w+1)z + z^2)(w - z)}{(w^3 - z^2)(w - z^2)} h(w)\, dw \quad , \tag{20}$$

where we have to integrate over the circle $|w| = r$ with $|z|^{\frac{2}{3}} < r < 1$ in the positiv direction.

Replacing z in (18) by $-z$, we easily find that (18) is equivalent to the system of functional equations

$$h(z) + h(-z) = 2h(z^2) \quad , \tag{21}$$

$$h(z^3) - h(-z^3) = \frac{2}{3z} \sum_{\nu=0}^{2} \lambda^\nu h(\lambda^\nu z^2) \tag{22}$$

with $|z| < 1$. Equation (21) has the general solution

$$h(z) = c_0 + \sum_{n=1}^{\infty} \frac{c_n z^{2n-1}}{1 - z^{2n-1}}$$

with arbitrary constants c_n, so far as the series converges for $|z| < 1$. Another form of the general solution of (21) was given in [1], and there was also constructed the general solution of (22).

It may be useful to know that the right–hand side of (22), if we denote it by $2Th(z)$, has the property

$$T \frac{z^{2n-1}}{1 - z^{2n-1}} = \begin{cases} \dfrac{z^{4n-3}}{1 - z^{12n-6}} & \text{for} \quad n \equiv 0 \mod 3 \quad , \\ \dfrac{z^{8n-5}}{1 - z^{12n-6}} & \text{for} \quad n \equiv 1 \mod 3 \quad , \\ 0 & \text{for} \quad n \equiv 2 \mod 3 \end{cases}$$

for $n \geq 1$. In any case, it is a holomorphic function of z^3 for $|z| < 1$.

The integral $\Phi(z) = \int_o^z h(w)dw$ of a solution of (22) satisfies the functional equation with constant coefficients

$$\Phi(z^3) + \Phi(-z^3) = \sum_{\nu=0}^{2} \Phi(\lambda^\nu z^2) \quad . \tag{23}$$

After the substitutions $z = \lambda^t$ and $\varphi(t) = \Phi(\lambda^t)$, we obtain the functional equation

$$\varphi(3t) + \varphi\left(3t + \frac{3}{2}\right) = \varphi(2t) + \varphi(2t+1) + \varphi(2t+2) \quad , \tag{24}$$

where we have to look for a solution $\varphi(t)$ with period 3. In view of (14) and $|z| < 1$, equation (24) must be solved for Im $t > 0$. But it also may be interesting to study it for real t.

4 The function $f_m(z)$

Introducing the notation

$$d_m(z) = f_{m+1}(z) - f_m(z) - \frac{1}{4(1-z)} \quad , \tag{25}$$

the function $f_m(z)$ from (7) can be written in the form

$$f_m(z) = \frac{z}{(1-z)^2} + \frac{m}{4(1-z)} + \sum_{\nu=0}^{m-1} d_\nu(z) \quad , \tag{26}$$

where it turns out that

$$d_m(z) = \frac{r_m(z)}{(1+z^{2^m})^2} \tag{27}$$

with polynomials $r_m(z)$ of degree $2^{m+1} - 1$ (in [1] we have written $\frac{1}{4}r_m$ instead of r_m). The functions (10) and (27) can also be written in the form

$$f_m(z) = \frac{p_{2m}(z)}{(1-z^{2^m})^2} + \frac{p_{1m}(z)}{1-z^{2^m}}, \; d_m(z) = \frac{r_{2m}(z)}{(1+z^{2^m})^2} + \frac{r_{1m}(z)}{1+z^{2^m}} \tag{28}$$

with polynomials of degree $2^m - 1$ in the numerators, in particular,

$$p_{2m}(z) = \sum_{\nu=0}^{2^m-1} a_{m\nu} z^\nu, \; r_{2m}(z) = -\frac{1}{2} \sum_{\nu=0}^{2^m-1} (-1)^{x_m(\nu)} a_{m\nu} z^\nu \quad . \tag{29}$$

The representation of $p_{2m}(z)$ follows immediately from (5), (10), (11), (12) and (18). The proof for the representation of $r_{2m}(z)$ was only scetched in [1], so that we give it here in detail. For this reason we introduce the notation $p(z) \sim q(z)$, if $p(z) = q(z) + O(1 + z^{2^m})$ for $z^{2^m} \to -1$. Then $1 - z^{2^m} \sim 2(1 + z^{2^m})$, and (25) and (28) imply

$$r_{2m}(z) \sim \frac{1}{4} p_{2,m+1}(z) \quad .$$

Expressing $p_{2,m+1}(z)$ by the first equation of (29), we obtain

$$r_{2m}(z) \sim \frac{1}{4} \sum_{\nu=0}^{2^m-1} (a_{m+1,\nu} - a_{m+1,2^m+\nu}) z^\nu \quad , \tag{30}$$

and since $r_{2m}(z)$ is a polynomial of degree $2^m - 1$, formula (30) must be an equality. On the other side, we find from (4) and (5) that $x_j(2^m + \nu) = x_j(\nu)$ for $0 \le j < m$, and $t_m(2^m + \nu) = a_{m\nu} + t_m(\nu)$ with $a_{m\nu} \equiv 1 \bmod 2$ in view of (6). Hence, (4) and (6) imply

$$a_{m+1,\nu} - a_{m+1,2^m+\nu} = \left(3^{x_m(\nu)} - 3^{\overline{x}_m(\nu)}\right) a_{m\nu} \quad ,$$

where $\overline{x}$ denotes the complement value of x, i. e. $\overline{x} = 1$ for $x = 0$, and $\overline{x} = 0$ for $x = 1$. But for these two values we have

$$3^x = 2 - (-1)^x, \quad 3^{\overline{x}} = 2 + (-1)^x \quad ,$$

so that

$$a_{m+1,\nu} - a_{m+1,2^m+\nu} = -2(-1)^{x_m(\nu)} a_{m\nu} \quad ,$$

and also the second equation of (29) is proved.

Finally, if we introduce the notation

$$r_m(z) = \sum_{\nu=0}^{2^{m+1}-1} e_{m\nu} z^\nu$$

and compare the Laurent-series of both sides of (25) for $|z| < 1$ and $|z| > 1$, respectively, the formulas (10), (11), (12) and (27) imply the representations

$$e_{m\nu} = t_{m+1}(\nu) - t_m(\nu) - \tfrac{1}{4} \qquad \text{for } \nu = 0, \ldots, 2^m - 1 \ ,$$

$$e_{m,2^{m+1}-\nu} = t_m(-\nu) - t_{m+1}(-\nu) + \tfrac{1}{4} \quad \text{for } \nu = 1, \ldots, 2^m \ .$$

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