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Congruences on closed subsets of $\widetilde{P_k}$ and of $P_{k,L}$

Let be $E_k := \{0, 1, \ldots, k-1\}$, $k \geq 2$, $P_k := \bigcup_{n\geq 1} P_k^n$, $P_k^n := \{f^n | f^n : E_k^n \to E_k\}$ and $\boldsymbol{P_k} := (P_k; \zeta, \tau, \triangle, \nabla, *)$, where ζ, τ, $\triangle$, ∇, $*$ denote the Mal'cev operations on P_k (see [6] or [7]). If the arity n of f^n can be seen from the context, we omit the upper index.

Here we consider the algebras

$$\boldsymbol{P(i)} := (P(i), \Omega_i)$$

as a generalization of $\boldsymbol{P_k}$ with $i \in \{0, 1, 2\}$,
$P(0) := \widetilde{P_k} = \bigcup_{n\geq 1} \widetilde{P_k^n}$, $\widetilde{P_k^n} := \{f^n | f^n : E_k^n \to E_k \cup \{\infty\}\}$,
$P(1) = P(2) := P_{k,L} = \bigcup_{n\geq 1} P_{k,L}^n$, $P_{k,L}^n := \{f^n | f^n : E_k^n \to E_L\}$
for $L > k$ and $\Omega_i := \{\zeta, \tau, \triangle, \nabla, *_i\}$.
The operations ζ, τ, $\triangle$, ∇, $*_i$ are defined as follow

$$(\zeta f^n)(x_1, \ldots, x_n) := f(x_2, x_3, \ldots, x_n, x_1),$$

$$(\tau f^n)(x_1, \ldots, x_n) := f(x_2, x_1, x_3, \ldots, x_n),$$

$$(\triangle f^n)(x_1, \ldots, x_{n-1}) := f(x_1, x_1, x_2, \ldots, x_{n-1}),$$

$$(\nabla f^n)(x_1, \ldots, x_{n+1}) := f(x_2, \ldots, x_{n+1}) \text{ for } n \geq 2,$$

$$\zeta f^n = \tau f^n = \triangle f^n = \nabla f^n = f \text{ for } n = 1 \quad (f \in P(i)),$$

$$(f^n * g^m)(x_1, \ldots, x_{n+m-1}) := f\left(g(x_1, \ldots, x_m),\ x_{m+1}, \ldots, x_{m+n-1}\right)\ (f \in P(i),\ g \in P_k).$$

$$(f^n \ *_0\ g^m)\ (x_1, \ldots, x_{m+n-1}) := \begin{cases} f\left(g(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1}\right) \\ \qquad\qquad \text{for } g(x_1, \ldots, x_m) \in E_k\ , \\ \infty \quad \text{else}\ , \end{cases}$$

$$(f, g \in \widetilde{P_k}),$$

$$f^n \ *_1 \ g^m := \begin{cases} f * g \text{ for } g \in P_k \ , \\ f \quad \text{else} \ , \end{cases}$$

and

$$f^n \ *_2 \ g^m := \begin{cases} f * g \text{ for } g \in P_k \ , \\ \text{not defined else} \end{cases}$$

$$(f, g \in P_{k,L}).$$

In [4] and [5] was proved that the algebras $\boldsymbol{P(i)}\,(i \in \{1,2,3\})$ have similar properties. For instance, the number of the maximal classes of $\widetilde{P_k}$ and $P_{k,L}$ is equal and a description of the maximal classes of $\widetilde{P_k}$ (see [3]) gives also a description of the maximal classes of $P_{k,L}$ and conversely. We show now that there exist substantial distinctions concerning the congruences.

Regarding our terminology we mainly follow [4] and [7]. Only the main notations for this paper are introduced below.

$Im\ f$ denotes the range of $f \in P(i)$.

The functions $e_i^n \in P_k^n$ $(1 \le i \le n)$ defined by $e_i^n(x_1, \dots, x_n) = x_i$ are called projections. For any a $\in E_L \cup \{\infty\}$ the symbol c_a^n denotes the mapping $E_k^n \to \{a\}, n = 1, 2, \dots .$

Let be $\alpha^1 := \alpha$ and $\alpha^{j+1} f := \alpha(\alpha^j f)$ for every $\alpha \in \{\zeta, \tau, \triangle, \nabla\}, j \in \mathbb{N}$ and $f \in P(i)$.

A congruence κ on $P(i)$ is an equivalence relation on $P(i)$ which is compatible with the operations in Ω_i, that is

$$\begin{aligned} &\forall f, g, s, t \in P(i) : (f, g) \in \kappa \ \wedge (s, t) \in \kappa \Longrightarrow \\ &(f *_i s, g *_i t) \in \kappa \ \wedge (\forall \alpha \in \{\zeta, \tau, \triangle, \nabla\} : (\alpha f, \alpha g) \in \kappa) \ . \end{aligned} \tag{1}$$

The compatibility property (1) is an obvious condition for introducing an algebraic structure which is inherited from the algebra $\boldsymbol{P(i)}$.

We also write $f \sim g\ (\kappa)$ for $(f, g) \in \kappa$.

It is well–known that each algebra $\boldsymbol{A}$ has two trivial congruences:

$\kappa_0 := \{(f,f)|f \in A\}$ and $\kappa_1 := A \times A$.

A.I. Mal'cev showed in [6] that the algebra $\boldsymbol{P_k}$ moreover has only the congruence

$\kappa_a := \{(f,g) \in P_k^2 \mid af = ag\}$ (af denote the arity of f).

Beside he proved the following theorem:

Theorem 1 [6]. *The algebra $\boldsymbol{P(0)}$ has exactly* 4 *concgruences:*

$$\kappa_0,\ \kappa_1,\ \kappa_a := \left\{(f,g) \in \widetilde{P_k} \times \widetilde{P_k} \mid af = ag\right\} \quad \text{and}$$

$$\kappa_\infty := \left\{(f,g) \in \widetilde{P_k} \times \widetilde{P_k} \mid f = g\ \vee (\exists n,m \in \mathbb{N} : \{f,g\} = \{c_\infty^n, c_\infty^m\}\right\}. \quad \blacksquare$$

A generalization of this theorem can be found in [1] and [2].

Obviously, every congruence on $\widetilde{P_k}$ is isomorphic to a certain congruence on $P(1)$ for $L = k+1$ and a congruence on $P(2)$ is also a congruence on $P(1)$ but not conversely. In particular, $\kappa_k := \{(f,g) \in P_{k,k+1} \times P_{k,k+1} \mid f = g \vee (\exists n,m \in \mathbb{N} : \{f,g\} = \{c_k^n, c_k^m\})\}$ is a congruence on $P(1)$ and $P(2)$ for $L = k+1$ and $\kappa_a := \{(f,g) \in P_{k,L} \times P_{k,L} \mid af = ag\}$ is obviously also a congruence on $\boldsymbol{P(2)}$ but not on $\boldsymbol{P(1)}$.

We need some notations for the discription of all congruences on $\boldsymbol{P(i)}$ with $i \in \{1,2\}$.

Let Z be the set of all partitions of all nonempty subsets of $E_L \backslash E_k$:

$$Z := \bigcup_{\substack{A,t \\ A \subseteq E_L \backslash E_k,\, A \neq \emptyset \\ t = 1,2,\ldots,|A|}} \left\{\{A_1,\ldots,A_t\} | \forall i : A_i \neq \emptyset \wedge \forall i \neq j : A_i \cap A_j = \emptyset \wedge \bigcup_{i=1}^{t} A_i = A\right\}.$$

With EZ we denote the set of all equivalence relations on sets $\mathcal{A} \in Z$. Then for each $\varepsilon \in EZ$ the relation

$$\kappa(\varepsilon) := \{(f,g) \in P_{k,L} \times P_{k,L} \,|\, f = g \vee (\exists (A,B) \in \varepsilon : Im\ f \subseteq A \wedge Im\ g \subseteq B\}$$

is a congruence on $\boldsymbol{P(i)}$, $i \in \{1,2\}$.

Let R be the set

$$\{\varrho \,|\, \exists\, r\ \exists A_{ij} \subseteq E_L : \varrho = \cup_{i=1}^{r} A_{1i} \times A_{2i} \wedge (\forall i \neq j : A_{1i} \cap A_{1j} = \emptyset \wedge A_{2i} \cap A_{2j} = \emptyset)\}$$

and

$$\mathcal{R}(i) := \{\mathcal{M} \subseteq R \mid <\mathcal{M}>_{\circ,\sigma} = \mathcal{M} \ \wedge \ (\cup_{\varrho \in \mathcal{M}} \varrho) \cap X(i) = \emptyset\},$$

where $i \in \{1,2\}$, $\circ$ denotes the relational product, σ is defined for binary relations ϱ as follows: $\sigma_\rho := \{(b,a) \mid (a,b) \in \varrho\}$, and

$$X_i := \begin{cases} \{(a,b) \in E_L \times E_L \mid a \neq b \wedge (a \in E_k \vee b \in E_k\} \text{ for } i = 1, \\ \{(a,b) \in E_k \times E_k \mid a \neq b\} \text{ for } i = 2. \end{cases}$$

It is not hard to verify that for each $\mathcal{M} \in \mathcal{R}(i)\,(i \in \{1,2\})$ the relation

$$\omega(\mathcal{M}) := \{(f,g) \mid af = ag \wedge (f = g \vee (\exists \varrho \in \mathcal{M} : \forall x \in E_k^{af} : (f(x), g(x))) \in \varrho\}$$

is a congruence on $\boldsymbol{P(i)}$.

With the help of the relations of the type $\kappa(\varepsilon)$ and $\omega(\mathcal{M})$ now we can describe all congruences on $\boldsymbol{P(1)}$ and $\boldsymbol{P(2)}$.

Theorem 2. *The algebra $\boldsymbol{P(i)}, i \in \{1,2\}$, has only congruences of the type*

$$\begin{array}{ll} \kappa_0, \kappa_1, & \\ \kappa(\varepsilon) & (\varepsilon \in EZ)\ , \\ \omega(\mathcal{M}) & (\mathcal{M} \in \mathcal{R}(i)), \\ \kappa(\varepsilon') \cup \omega(\mathcal{M}') & (\kappa(\varepsilon') \cup \omega(\mathcal{M}') \textit{ is an equivalence} \\ & \quad \textit{relation and } \varepsilon' \in EZ, \mathcal{M}' \in \mathcal{R}(i))\ , \\ \textit{and } \kappa_a \ (\textit{only for } i = 2). & \end{array}$$

It is easy to see that this theorem is a deduction of the following two lemmas.

Lemma 1. *Let κ be a congruence on $P(\gamma) \in \{P(1), P(2)\}, (f^n, g^n) \in \kappa$ and $T(f,g) := \{(\alpha,\beta) \in E_L^2 \mid \exists a \in E_k^n : \alpha = f(a) \wedge \beta = g(a)\}$. Then the following implications hold:*

(a) $\forall m \in \mathbb{N}\ s,t \in P_{k,L}$:
$(\forall a \in E_k^m : (s(a), t(a)) \in T(f,g) \Longrightarrow (s,t) \in \kappa.$

(b) $T(f,g) \notin \mathcal{R}(\gamma) \Longrightarrow$
$(\exists u,v : (u,v) \in \kappa \wedge T(f,g) \subseteq T(u,v) \wedge T(u,v) \in \mathcal{R}(\gamma)).$

(c) $(\gamma = 1 \wedge (\exists (a,b) \in T(f,g) : \{a,b\} \cap E_k \neq \emptyset \wedge a \neq b))$
$\Longrightarrow \kappa_a \subseteq \kappa.$

(d) $(\gamma = 2 \wedge (\exists (a,b) \in T(f,g) : \{a,b\} \subseteq E_k \wedge a \neq b))$
$\Longrightarrow \kappa_a \subseteq \kappa.$

Proof: If $T(f,g) = \{(\alpha_1,\beta_1), \ldots, (\alpha_t,\beta_t)\}$ there exist $a_1, \ldots, a_t \in E_k^n$ with $a_i = (a_{1i}, \ldots, a_{in})$ and $(f(a_i), g(a_i)) = (\alpha_i, \beta_i), i = 1,2,\ldots,t$. Let be $s,t \in P_{k,L}$ with $(s(a), g(a)) \in T(f,g)$ for every $a \in E_k^m$. Defining functions $h_j \in P_k^m$ by

$$h_j(x) = a_{ij} \quad \text{iff} \quad (s(x), t(x)) = (\alpha_i, \beta_i)$$
$$(i = 1, 2, \dots, t; j = 1, 2, \dots n)$$

we obtain $(s, t) = (f(h_1, \dots, h_n), g(h_1, \dots, h_n)) \in \kappa$.
(b): It is sufficient to show that the following implication holds:

$$\{(a, c), (a, d), (b, c)\} = T(f, g) \Longrightarrow \exists\, u, v : (u, v) \in \kappa \wedge T(u, v) = T(f, g) \cup \{(b, c)\}\ .$$

Let be $u, v \in P_{k,L}$ with $Im\ u = \{a, b\}$, $Im\ v = \{c, d\}$, $u(a_1) = u(a_2) = a$, $u(a_3) = u(a_4) = b$ and $v(a_1) = v(a_3) = c$, $v(a_2) = g(a_4) = d$ for certain $a_1, \dots, a_4 \in E_k^n$. If $\{(a, c), (a, d), (b, c)\} = T(f, g)$ and $(f, g) \in \kappa$ by (a) we get $(u, c_c^n), (c_a^n, c_c^n), (c_a^n, v) \in \kappa$. Thus $(u, v) \in \kappa$.
(c), (d): Let be $f(a) = a \neq b = f(b)$. Then $(c_a^1, c_b^1) \in \kappa$. If $\gamma = 1, a \in E_k$ and without loss of generality $b \notin E_k$ we have:

$$e_1^1 *_1 c_a = c_a \sim e_1^1 *_1 c_b = e_1^1\ (\kappa) \quad \text{and} \quad c_a *_1 t^m = c_a^m \sim e_1^1 *_1 t^m = t^m\ (\kappa)$$

for every $t^m \in P_{k,L}$. Hence $\kappa_a \subseteq \kappa$.
In case $\gamma = 2$ and $\{a, b\} \subseteq E_k$ we obtain $t_1^m = h *_2 c_a \sim h *_2 c_b = t_2^m$, where

$$h(x_1, \dots, x_{m+1}) := \begin{cases} t_1(x_2, \dots, x_{m+1}) \text{ for } x_1 = a, \\ t_2(x_2, \dots, x_{m+1}) \text{ else} \end{cases}$$

and t_1, t_2 denote any functions, which belong to $P_{k,L}$. Therefore, (d) holds. ■

Lemma 2. *Let κ be a congruence on $P(\gamma) \in \{P(1), P(2)\}, (f^n, g^m) \in \kappa$ and $n \neq m$. Then*

(a) $\forall a \in Im\ f\ \forall\, p, q : (c_a^p, c_a^q) \in \kappa$
(b) $Im\ f \cap E_k \neq \emptyset \Longrightarrow \kappa = \kappa_1$
(c) $f = c_a^n \Longrightarrow ((\forall u, v : Im\ u \subseteq Im\ g \wedge Im\ v \subseteq Im\ g) \Longrightarrow (u, v) \in \kappa)$
(d) $(f \notin [\{c_\alpha \mid \alpha \in E_k\}] \wedge n > m) \Longrightarrow$
$(\exists a\ \exists g_1 : a \in Im\ f \wedge (c_a^n, g_1^n) \in \kappa \wedge Im\ g_1 = Im\ g)$
(e) $\forall u, v : Im\ u \subseteq Im\ f \wedge Im\ v \subseteq Im\ g \Longrightarrow (u, v) \in \kappa$.

Proof: Obviously, we have $(c_a^n, c_b^m) \in \kappa$ for certain $b \in$ Im g, if $a \in$ Im f and $(f^n, g^m) \in \kappa$. Consequently, $(\Delta^{r-1} c_a^{r-1}, \Delta^{r-1} c_b^n) \in \{(c_a^1, c_b^1), (c_a^2, c_b^1)\} \cap \kappa$ for $r := \max\{m, n\}$. By this we get $\{(c_a^1, c_b^1), (c_a^2, c_b^2), (c_a^1, c_b^2), (c_b^1, c_a^2), (c_a^1, c_a^2), (c_a^2, c_a^3), \dots\} \subseteq \kappa$, that is (a) holds.

(b): By (a) we get $((P_k \times P_k) \backslash \kappa_a) \cap \kappa \neq \emptyset$. Since P_k has only the congruences κ_0, κ_a and κ_1 and since certain $h, h_1, \dots h_{2q}$ with $h \in P_{k,L}, \{h_1, \dots, h_{2q}\} \subseteq P_k$ and $t_1 = h(h_1, \dots, h_q), t_2 = h(h_{q+1}, \dots, h_{2q})$ exist for every $t_1, t_2 \in P_{k,L}, \kappa = \kappa_1$ follows from this.

(c) is easy to show by the help of (a).

(d): By using of the operations ζ, τ we obtain from $(f, g) \in \kappa$ that there exist functions u^n, v^m with the properties $Im\ f = Im\ u, Im\ g = Im\ v, (u, v) \in \kappa$ and the $(n-m)-$th variable of u is not fictitious. Then

$$\Delta(\zeta^{m+1}(u^n * e_2^2) * c_0^1)) =: f_1 \sim \Delta(\zeta^{m+1}(v^m * e_2^2) * c_0^1)) = v\ (\kappa)$$

and f_1 has one fictitious variable more than f. Iterations of this construction yield (d).

(e) follows immediately from (c) and (d) and the symmetry and transitivity of κ. ■

Examples:
Let $k = 2$ and $L = 3$. Then

1. $P(1)$ has only 3 congruences:
   κ_0, κ_1 and $\kappa_2 = \kappa(\{(\{2\}, \{2\})\})$.

2. $P(2)$ has exactly 20 congruences:
   $\kappa_0, \kappa_a, \kappa_1, \kappa_2$ and $\omega(\mathcal{M})$, where

$$\begin{aligned}\mathcal{M} \in \{\ & <\{a,2\}>_{\circ,\sigma}, <\{(a,2),(\overline{a},\overline{a})\}>,\\ & \{(a,a),(a,2),(2,a),(2,2)\},\\ & \{(a,a),(a,2),(2,a),(2,2),(\overline{a},\overline{a})\},\\ & <\{(a,2),(2,a)\}>, <\{(a,2),(2,a),(\overline{a},\overline{a})\}>,\\ & <\{\{(a,2),(2,a),(\overline{a},\overline{a})\},\{(a,2),(\overline{a},\overline{a})\}\}>,\\ & <\{\{(a,2),(2,a),\}\{(a,2),(\overline{a},\overline{a})\}\}>\ \ |\, a \in \{0,1\}\}.\end{aligned}$$

   The Hasse diagram of the congruences is given in figure 1.

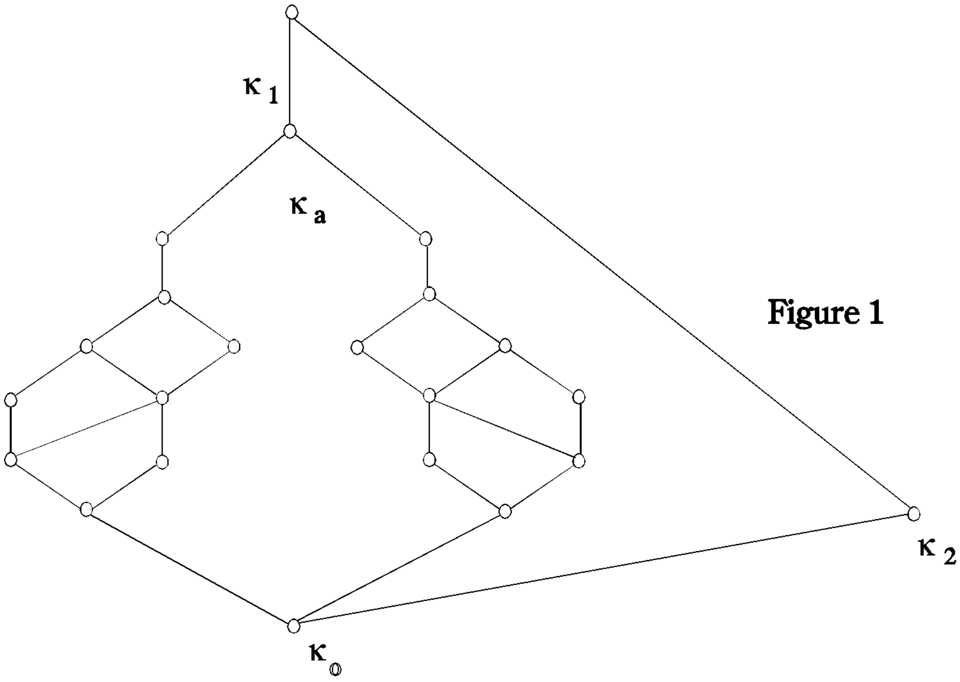

Figure 1

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