Rostock. Math. Kolloq. **48**, 53–58 (1995)

Subject Classification (AMS)
**54H25**

Zeqing Liu

# Common fixed points of multivalued mappings

**Abstract.** *Common fixed point theorems for contractive type multivalued mappings are proved which generalize some results of Fisher* [1,2,3].

**Key words and phrases.** *Multivalued mappings, common fixed point, bounded metric space, compact metric space.*

## 1. Introduction

Let $(X, d)$ be a complete metric space, $B(X)$ denote the set of all nonempty, bounded subsets of $X$. For $A, B$ in $X$, define $\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$ and $\delta(A, A) = \delta(A)$. If $A$ is a singleton, we write $\delta(A, B) = \delta(a, B)$ and if $B = \{b\}$, then we put $\delta(A, B) = d(a, b)$.

Now let $\{A_n : n = 1, 2, \dots\}$ be a sequence of nonempty subsets of $X$. We say that the sequence $\{A_n\}$ converges to the subset $A$ of $X$ if

(i) each point $a$ in $A$ is the limit of a convergent sequence $\{a_n \in A_n : n = 1, 2 \dots\}$, and

(ii) for arbitrary $\epsilon > 0$, there exists an integer $k$ such that $A_n \subseteq A_\epsilon$, for $n > k$, where $A_\epsilon$ is the union of all open spheres with centres in $A$ and radius $\epsilon$.

Fisher [3] proved the following result.

**Lemma 1** *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of a complete metric space $(X, d)$ which converges to the bounded subsets $A$ and $B$, respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

Let $F$ and $G$ be mappings of $(X, d)$ into $B(X)$ and $f$ a self mapping of $(X, d)$. For $A \subseteq X$ and $n \geq 2$, let $FA = \cup_{a\in A} Fa$, $GFA = G(FA)$ and $F^n A = F(F^{n-1}A)$. $F$ and $G$ are said to be commuting if $FGx = GFx$ for all $x$ in $X$; $F$ and $f$ are said to be commuting if $Ffx = fFx$ for all $x$ in $X$. We say that $F$ is continuous at the point $x$ in $X$ if whenever $\{x_n\}$ is a sequence of points in $X$ converging to x, the sequence $\{Fx_n\}$ in $B(X)$ converges to $Fx$ in $B(X)$. We say that $F$ is a continuous mapping of $X$ into $B(X)$ if $F$ is continuous at each point $x$ in $X$. Define $C_F = \{f : f$ is a self mapping of $X$ and commutes with $F\}$.

Let $R^+$ denote the set of all nonnegative real numbers, $\Phi$ the family of mappings $\varphi$ from $(R^+)^5$ into $R^+$ such that each $\varphi$ is upper semicontinuous, nondecreasing in each coordinate variable, and for any $t > 0, ht = \varphi(t,t,t,t,t) < t$. $N$ denote the set of all positive integers.

The followings lemma was proved in [4].

**Lemma 2** *For every $t > 0, ht < t$ if and only if $\lim_{n\to\infty} h^n t = 0$, where $h^n$ denotes the $n$-times composition of $h$.*

Fisher $[1,2,3]$ established common fixed point theorems for two commuting multivalued mappings. The purpose of this paper is to extend Fisher's results to a more general case.

## 2. Common fixed point theorems.

We now prove the following theorem.

**Theorem 1** *Let $F$ and $G$ be multivalued mappings of a complete bounded metric space $(X,d)$ into $B(X), f$ and $g$ continuous mappings of $X$ into itself such that $F$ and $G$ are commuting, $f,g$ in $C_F \cap C_G$. If there exists $\varphi$ in $\Phi$ satisfying*

$$\delta(Fx,Gy) \le \varphi(\delta(fx,Fx),\delta(gy,Gy),\delta(fx,Gy),\delta(gy,Fx),d(fx,gy)) \tag{1}$$

*for all $x,y$ in $X$, then $f,g,F$ and $G$ have a unique common fixed point $z$ in $X$. Further, $Fz = Gz = \{z\}$.*

**Proof:** Let $A,B$ in $B(X)$. By (1) we have

$$\delta(Fa,Gb) \le \varphi(\delta(fa,Fa),\delta(gb,Gb),\delta(fa,Gb),\delta(gb,Fa),d(fa,gb))$$

for all a in $A, b$ in $B$, which implies

$$\delta(FA,GB) \le \varphi(\delta(fa,FA),\delta(gB,GB),\delta(fA,GB),\delta(gB,FA),\delta(fA,gB)). \tag{2}$$

Since $X$ is bounded, then $r = \delta(X) < \infty$. We first show by induction that

$$\delta(F^nG^nX) \le h^n r \tag{3}$$

for $n$ in $N$. Note that $F$ and $G$ commute and $f,g$ in $C_F \cap C_G$: Using (2)

$$\begin{aligned}\delta(FGX,GFX) &\le \varphi(\delta(GfX,FGX),\delta(FgX,GFX),\delta(GfX,GFX),\\ &\qquad \delta(FgX,FGX),\delta(GfX,FgX))\\ &\le hr\end{aligned}$$

and hence (3) is true when $n = 1$. Now assume that (3) holds for some $n$ in $N$. Again using (2)

$$\begin{aligned}\delta(F^{n+1}G^{n+1}X, G^{n+1}F^{n+1}X) &\leq \varphi(\delta(F^nG^nGfX, G^nF^nFGX), \delta(F^nG^nFgX, G^nF^nGFX),\\ &\qquad \delta(F^nG^nGfX, G^nF^nGFX), \delta(F^nG^nFgX, G^nF^nFGX),\\ &\qquad \delta(F^nG^nGfX, G^nF^nFgX))\\ &\leq h\delta(F^nG^nX) \leq h^{n+1}r\end{aligned}$$

by our assumption. Now (3) follows by induction.

For each $n$ in $N$, choose a point $x_n$ in $F^nG^nX$. It follows that

$$fx_n \in fF^nG^nX \subseteq F^nG^nfX \subseteq F^nG^nX.$$

Similarly, $gx_n \in F^nG^nX$. From (3) we get

$$d(x_n, x_m) \leq \delta(F^nG^nX, G^nF^nG^{m-n}F^{m-n}X) \leq h^nr$$

for $m, n$ in $N$ with $m \geq n$. By Lemma 2 and the above inequality it follows that $\{x_n\}$ is a Cauchy sequence. Since $X$ is complete, there exists a point $z$ in $X$ such that $x_n \to z$ as $n \to \infty$. Note that $f$ and $g$ are continuous. Thus $fx_n \to fz$, $gx_n \to gz$ as $n \to \infty$. Consequently, we have

$$d(fz, gz) \leq d(fz, fx_n) + d(fx_n, gx_n) + d(gx_n, gz) \leq d(fz, fx_n) + h^nr + d(gx_n, gz).$$

Letting $n$ tend to infinity, we obtain $d(fz, gz) \leq 0$; i. e., $fz = gz$.

We next show that $z$ is a common fixed point of $f, g, F$ and $G$. Clearly, we have

$$d(z, fz) \leq d(z, x_n) + h^nr + d(fx_n, fz).$$

As $n \to \infty$, we have $d(z, fz) \leq 0$; i. e., $z = fz$ and hence $z = gz$. We now assert that $\delta(z, Gz) = 0$. Otherwise $\delta(z, Gz) > 0$. It follows from (2) and (3) that

$$\begin{aligned}\delta(z, Gz) &\leq d(z, gx_n) + \delta(F^nG^nX, Gz)\\ &\leq d(z, gx_n) + \varphi(\delta(F^{n-1}G^nfX, F^nG^nX), \delta(z, Gz), \delta(F^{n-1}G^nfX, Gz),\\ &\qquad \delta(z, F^nG^nX), \delta(F^{n-1}G^nfX, z))\\ &\leq d(z, gx_n) + \varphi(h^{n-1}r, \delta(z, Gz), h^{n-1}r + d(gx_{n-1}, z) + \delta(z, Gz),\\ &\qquad d(gx_n, z) + h^nr, h^{n-1}r + d(gx_{n-1}, z)).\end{aligned}$$

Letting $n$ tend to infinity, we obtain

$$\delta(z, Gz) \leq \varphi(0, \delta(z, Gz), \delta(z, Gz), 0, 0) \leq h\delta(z, Gz) < \delta(z, Gz)$$

which is a contradiction and hence $\delta(z, Gz) = 0$. Consequently, $Gz = \{z\}$. Similarly, $Fz = \{z\}$.

We finally show that $z$ is the only common fixed point of $f, g, F$ and $G$. Suppose that $f, g, F$ and $G$ have a second common fixed point $w$. From (1) have

$$\delta(Fw, Gw) \leq \varphi(\delta(w, Fw), \delta(w, Gw), \delta(w, Gw), \delta(w, Fw), 0) \leq h\delta(Fw, Gw)$$

which implies $\delta(Fw, Gw) = 0$. Note that $w \in Fw \cap Gw$. Consequently, $Fw = Gw = \{w\}$. Using (1) again

$$d(z, w) = \delta(Fz, Gw) \leq \varphi(0, 0, d(z, w), d(w, z), d(z, w)) \leq hd(z, w).$$

Therefore $z = w$; i. e., $z$ is the unique common fixed point of $f, g, F$ and $G$. This completes the proof.

**Corollary 1** *Let $S, T, f$ and $g$ be self mappings of a complete bounded metric space $(X, d)$ such that $S$ and $T$ are commuting and $f, g$ are in $C_S \cap C_T$. If there exists $\varphi$ in $\Phi$ satisfying*

$$d(Sx, Ty) \leq \varphi(d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx), d(fx, gy))$$

*for all $x, y$ in $X$, then $f, g, S$ and $T$ have a unique common fixed point.*

**Proof:** Define mappings $F$ and $G$ of $X$ into $B(X)$ by $Fx = \{Sx\}, Gx = \{Tx\}$ for $x$ in $X$. Then Corollary 1 follows from Theorem 1.

**Remark 1** If $f$ and $g$ are the indentity mapping on $X$ and

$$\varphi(u, v, w, x, y) = c \cdot \max\{u, v, w, x, y\},$$

then Theorem 1 is reduced to Theorem 2 of [1] and Theorem 3 of [2]; Corollary 1 is reduced to the Corollary of [1].

We next prove a theorem for compact metric spaces.

**Theorem 2** *Let $F$ and $G$ be continuous commuting mappings of a compact metric space $(X, d)$ into $B(X)$, $f$ and $g$ continuous mappings of $X$ into itself satisfying*

$$\delta(Fx, Gy) < \max\{\delta(fx, Fx), \delta(gy, Gy), \delta(fx, Gy), \delta(gy, Fx), d(fx, gy)\} \tag{4}$$

*for all $x, y$ in $X$ for which the right-hand side of (4) is positive. If $f, g$ are in $C_F \cap C_G$, then $f, g, F$ and $G$ have a unique common fixed point $z$ in $X$. Further, $Fz = Gz = \{z\}$.*

**Proof:** Let $s(x, y) = \max\{\delta(fx, Fx), \delta(gy, Gy), \delta(fx, Gy), \delta(gy, Fx), d(fx, gy)\}$. We now assert that $s(x, y) = 0$ for some $x, y$ in $X$. Otherwise $s(x, y) > 0$ for all $x, y$ in $X$. Define the real-valued function $t(x, y)$ on $X \times X$ by $t(x, y) = \delta(Fx, Gy)/s(x, y)$. Then if $\{(x_n, y_n)\}$ is an arbitrary sequence in $X \times X$ converging to $(x, y)$, it follows Lemma 1 that the sequence $\{t(x_n, y_n)\}$ converges to $t(x, y)$. The function $t$ is therefore a continuous function defined on the compact metric space $X \times X$ and so achieves its maximum value $c$. It is easy to see that (4) implies $c < 1$. Choose $\varphi(u, v, w, m, n) = c \cdot \max\{u, v, w, m, n\}$. Then the conditions of Theorem 1 are satisfied. Consequently, $f, g, F$ and $G$ have a unique common fixed point $z$ and $Fz = Gz = \{z\}$. It follows that $s(z, z) = 0$, giving a contradiction. Since $s(x, y) = 0$ for some $x, y$ in $X$, then $Fx = Gy = \{fx\} = \{gy\}$ is a singleton $\{z\}$ and so $F^2x = Ffx = fFx = \{fz\}$ is also a singleton. Suppose that $fz \neq z$. By (4) we have

$$\begin{aligned} d(fz, z) = \delta(Ffx, Gy) &< \max\{\delta(f^2x, fFx), \delta(gy, Gy), \delta(f^2x, Gy), \delta(gy, Ffx), d(f^2x, gy)\} \\ &= \max\{0, 0, d(fz, z), d(z, fz), d(fz, z)\} = d(fz, z) \end{aligned}$$

which is impossible. Hence $\{z\} = \{fz\} = Fz$. Similarly, we can show that $gy = z$ is also a common fixed point of $G$ and $g$ and $Gz = \{z\}$.

Now suppose that $f, g, F$ and $G$ have a second distinct common fixed point $w$. Then if $Fw \neq \{w\}$ or $Gw \neq \{w\}$, by (4) we get

$$\delta(Fw, Gw) < \max\{\delta(w, Fw), \delta(w, Gw), \delta(w, Gw), \delta(w, Fw), 0\} \leq \delta(Fw, Gw)$$

since $fw = gw = w$ is in $Fw$ and $Gw$, giving a contradiction. Thus $Fw = Gw = \{w\}$ and so

$$d(z, w) = \delta(Fz, Gw) < \max\{0, 0, d(z, w), d(z, w), d(z, w)\} = d(z, w)$$

which is a contradiction. The fixed point $z$ must therefore be unique. This completes the proof.

**Remark 2** Theorem 2 of Fisher [3] is a special case of our Theorem 2. The following example reveals that our Theorem 2 extends properly Theorem 2 of Fisher [3].

**Example** Let $X = \{1, 2, 5, 9\}$ with the usual metric. Define a self mapping $f$ of $X$ and a multivalued mapping $F$ of $X$ into $B(X)$ by $f1 = f2 = f5 = 2$, $f9 = 1$, $F1 = \{2, 5\}$, $F2 = \{2\}$, $F5 = \{1, 2\}$, $F9 = \{5\}$. Then Theorem 2 of Fisher [3] is not applicable since

$$\delta(F2, F5) = 1 = \max\{\delta(f2, F2), \delta(f5, F5), \delta(f2, F5), \delta(f5, F2), d(f2, f5)\}.$$

Take $G = F$ and $g = i$ — the identity mapping of $X$. It is easy to check that $f, g, F$ and $G$ satisfy the conditions of our Theorem 2.

**Acknowledgement** The author is grateful to Dr. Brain Fisher for providing him with reprints of his papers which motivated the present work.

## References


[1] **Fisher, B.** : *Set-valued mappings on bounded metric spaces.* Indian J. Pure Appl. Math. **11**, 8-12 (1980)

[2] **Fisher, B.** : *Common fixed points of set-valued mappings on bounded metric spaces.* Math. Sem. Notes Kobe Univ. **11**, 307-311 (1983)

[3] **Fisher, B.** : *Common fixed points of mappings and set-valued mappings.* Rostock. Math. Kolloq. **18**, 69-77 (1981)

[4] **Singh, S. P.** and **Meade, B. A.** : *On common fixed point theorems.* Bull. Austral. Math. Soc. **16**, 49-53 (1977)


**received:** July 22, 1994

**Author:**

Dr. Zeqing Liu
Department of Mathematics
Liaoning Normal University
Dalian
Liaoning, 116022
P. R. China