Rostock. Math. Kolloq. **50**, 47–52 (1997)

Subject Classification (AMS)
05A99 06A07 68R15

TRAN-NGOC DANH; DAVID E. DAYKIN

Sets of 0,1 vectors with minimal sets of subvectors

ABSTRACT. Let $\mathcal{C}(n)$ be the 0,1 vectors $\underset{\sim}{a} = a_1 \dots a_n$. To get a subvector of $\underset{\sim}{a}$ delete any a_i. If $\mathcal{A} \subseteq \mathcal{C}(n)$ then $\Delta\mathcal{A}$ is the set of all subvectors of members of $\mathcal{A}$, so $\Delta\mathcal{A} \subseteq \mathcal{C}(n-1)$. Put $W\underset{\sim}{a} = a_1 + \dots + a_n$. We order $\mathcal{C}(n)$ by $\underset{\sim}{a} < \underset{\sim}{b}$ if (i) $W\underset{\sim}{a} < W\underset{\sim}{b}$ or (ii) $W\underset{\sim}{a} = W\underset{\sim}{b}$ and $1 = a_i > b_i = 0$ for the least i with $a_i \neq b_i$. We present a completely new proof of our <u>Theorem</u>. If $\mathcal{A} \subseteq \mathcal{C}(n)$ and $\mathcal{I}$ is the first $|\mathcal{A}|$ members in $\mathcal{C}(n)$ then $|\Delta\mathcal{I}| \leq |\Delta\mathcal{A}|$.

1 Introduction

Let $\mathcal{C}(n)$ be the set of all vectors $\underset{\sim}{a} = a_1 \dots a_n$ having $a_h = 0$ or $a_h = 1$ for $1 \leq h \leq n$. Further let $\delta_h\underset{\sim}{a}$ be the subvector in $\mathcal{C}(n-1)$ obtained from $\underset{\sim}{a}$ by deleting coordinate a_h. The shadow $\Delta\underset{\sim}{a}$ of $\underset{\sim}{a}$ is the set $\{\delta_1\underset{\sim}{a}, \dots, \delta_n\underset{\sim}{a}\}$. The shadow $\Delta\mathcal{A}$ of any $\mathcal{A} \subseteq \mathcal{C}(n)$ is $\cup\{\underset{\sim}{a} \in \mathcal{A}\}\Delta\underset{\sim}{a}$. We are concerned with this problem. Given $0 \leq k \leq 2^n$ find $\mathcal{A} \subseteq \mathcal{C}(n)$ with cardinality $|\mathcal{A}| = k$ and shadow size $|\Delta\mathcal{A}|$ minimal. We solved the problem in [1] 1993 by Theorem 1 below, where $\mathcal{C}(n)$ is V-ordered. If $\underset{\sim}{a}, \underset{\sim}{b} \in \mathcal{C}(n)$ and $\underset{\sim}{a} \neq \underset{\sim}{b}$ then $\alpha(\underset{\sim}{a}, \underset{\sim}{b})$ is the first i in $1 \leq i \leq n$ with $a_i \neq b_i$, also $W\underset{\sim}{a} = a_1 + \dots + a_n$.

Definition 1 *(V-order). Let $\underset{\sim}{a}, \underset{\sim}{b} \in \mathcal{C}(n)$ with $\underset{\sim}{a} \neq \underset{\sim}{b}$. Then $\underset{\sim}{a} < \underset{\sim}{b}$ if (i) $W\underset{\sim}{a} < W\underset{\sim}{b}$ or (ii) $W\underset{\sim}{a} = W\underset{\sim}{b}$ and $a_j > b_j$ with $j = \alpha(\underset{\sim}{a}, \underset{\sim}{b})$.*

By an initial section IS we mean the first so many members of $\mathcal{C}(n)$ in V-order. In particular $IS(\underset{\sim}{a}) = \{\underset{\sim}{x} \in \mathcal{C}(n) : \underset{\sim}{x} \leq \underset{\sim}{a}\}$.

Theorem 1 *If $\mathcal{I}, \mathcal{A} \subseteq \mathcal{C}(n)$ have $|\mathcal{I}| = |A|$ and $\mathcal{I}$ is an IS the $|\Delta\mathcal{I}| \leq |\Delta\mathcal{A}|$.*

In this note we present a completely new proof of Theorem 1. Any statement not proved in full detail is easy to verify. We use condensed notation. If $\underset{\sim}{u} = u_1 \dots u_r$ and $\underset{\sim}{v} = v_1 \dots v_s$ are vectors then $\underset{\sim}{u}\,\underset{\sim}{v}$ is the vector $u_1 \dots u_r v_1 \dots v_s$. If $\mathcal{A} \subseteq \mathcal{C}(n)$ and $\underset{\sim}{e} \in \mathcal{C}(1)$ then $\mathcal{A}\underset{\sim}{e} = \{\underset{\sim}{a}\underset{\sim}{e} :$

$\underset{\sim}{a} \in \mathcal{A}\}$. By $\mathcal{A}+\mathcal{B}$ we mean the union $\mathcal{A}\cup\mathcal{B}$ and are saying that $\mathcal{A}\cap\mathcal{B}=\emptyset$, the empty set. Each $\mathcal{A}\subseteq\mathcal{C}(n+1)$ has a partion $\mathcal{A}=A0+B1$ where $A,B\subseteq\mathcal{C}(n)$. When A,B are IS we say $\mathcal{A}$ is *part compressed* PC.

We tacitly use the fact that for $\underset{\sim}{x},\underset{\sim}{y}\in\mathcal{C}(n)$ and any vectors $\underset{\sim}{a},\underset{\sim}{b}$ we have $\underset{\sim}{a}\underset{\sim}{x}\underset{\sim}{b}<\underset{\sim}{a}\underset{\sim}{y}\underset{\sim}{b}$ iff $\underset{\sim}{x}<y$.

Given $\underset{\sim}{a}\in\mathcal{C}(n)$, if $\underset{\sim}{a}=1\dots1$ put $V\underset{\sim}{a}=1$, ortherwise let $V\underset{\sim}{a}$ denote the last h in $1\le h\le n$ with $a_h=0$. Then $\underset{\sim}{a}^*=\delta_j\underset{\sim}{a}$ where $j=V\underset{\sim}{a}$. It is important that $\underset{\sim}{a}^*=\max\{\Delta\underset{\sim}{a}\}$. Of course $\mathcal{A}^*=\{\underset{\sim}{a}^*:\underset{\sim}{a}\in\mathcal{A}\}$.

2 Initial sections of V-order

Let $\mathcal{I}$ be an IS. Put $\mathcal{I}=G0+H1$ and $G=G_00+G_11$ and $H=H_00+H_11$.
Then

(1) G,H are IS,

(2) $H\subseteq G$ and $H_0\subseteq G_1$ so $H_1\subseteq H_0\subseteq G_1\subseteq G_0$,

(3) $\Delta\mathcal{I}=G=\mathcal{I}^*$,

(4) $\Delta(\mathcal{I}0)=\mathcal{I}$,

(5) $\Delta(\mathcal{I}1)=G0+G1$ not usually IS.

G_1	$G0$	$H1$	H_0
	0000		
	1000		
	0100		
00	0010		
		0001	00
	1100		
10	1010		
		1001	10
01	0110		
		0101	10
		0011	
11	1110		
		1101	11
		1011	
		0111	
		1111	

Figure 1. The V-order of $\mathcal{C}(4)$

In Figure 1 we see the V-orders (i) of $\mathcal{C}(2)$ in G_1, H_0, (ii) of $\mathcal{C}(3)$ in G, H and (iii) of $\mathcal{C}(4)$ in $G0 + H1$. In the figure $\mathcal{C}(4)$ is cut into slices by the last coordinates of its vectors. Imagine we have this figure for every $\mathcal{C}(n)$.

Consider $IS(\utilde{a})$ growing with $\utilde{a}$. We only get an increase in G_1 (resp. H_0) when $\utilde{a} = \utilde{x}10$(resp. $\utilde{a} = \utilde{y}01$) at the end of a 0-slice (resp. at the beginning of a 1-slice) and the increase is $\utilde{x}$ (resp. $\utilde{y}$). Moreover if the 0-slice is immediately before the 1-slice then $\utilde{x} = \utilde{y}$. All ohter vectors in a 0-slice (resp. 1-slice) end 00 (resp.11). Therefore (6) if $\mathcal{I} = IS(\utilde{x}10)$ then $G_1 = H_0 + \utilde{x}$ otherwide $G_1 = H_0$.

Let $n \geq 3$ and S be the 0-slice just before a 1-slice T. If $|T| \geq 2$ then $|S| = 1$, but if $|T| = 1$ then $|S| \geq 2$.

3 Part compressed families

Let D, E be any IS of $\mathcal{C}(n)$ and put $\mathcal{B} = D0+E1$ so $\mathcal{B}$ is PC. Next Γ is the set of $\utilde{a} \in \mathcal{C}(n+1)$ of the form $\utilde{a} = \utilde{u}\utilde{z}1$, where $\utilde{z} = 0\ldots0$ of $\dim \utilde{z} \geq 1$, and either $\utilde{u} = \emptyset$ or $\utilde{u}$ ends 1. Also $\gamma\utilde{a} = \utilde{u}1\utilde{z}$ if $\utilde{a} = \utilde{u}\utilde{z}1 \in \Gamma$. We can now define a map $\psi : \mathcal{B} \to \mathcal{C}(n+1)$ by

$$\psi\utilde{a} = \begin{cases} \gamma\utilde{a} & \text{if } \utilde{a} \in \Gamma \text{ and } \gamma\utilde{a} \notin \mathcal{B}, \\ \utilde{a} & \text{otherwise} \end{cases}$$

Theorem 2 *If $\mathcal{B}$ is PC then $\Delta\psi\mathcal{B} \subseteq \psi\Delta\mathcal{B}$.*

Proof: Observe that the two ψ are different. We can have $\mathcal{B} = D0 + E1$ as above. Let $\utilde{x} \in \Delta\psi\mathcal{B}$. There is a $\utilde{b} \in \psi\mathcal{B}$ with $\utilde{x} \in \Delta\utilde{b}$ and an $\utilde{a} \in \mathcal{B}$ with $\utilde{b} = \psi\utilde{a}$. We distinguish three possibilities. Case 1. $\utilde{a} \notin \Gamma$. Here $\utilde{b} = \utilde{a}$ and $\utilde{a}$ ends 0 or 11. Case 2. $\utilde{a} \in \Gamma$ and $\utilde{b} \neq \utilde{a}$. So $\utilde{b} = \utilde{u}1\utilde{z}$ and $\utilde{a} = \utilde{u}\utilde{z}1$. Case 3. $\utilde{a} \in \Gamma$ and $\utilde{b} = \utilde{a}$. Here $\utilde{a} = \utilde{u}\utilde{z}1$ and $\utilde{u}1\utilde{z}, \utilde{a}$ are both in $\mathcal{B}$.

Please look at the last line of Case 1 in Figure 2. There we are given $\utilde{a} = \utilde{w}11$, $\utilde{x} = \utilde{w}1$ and $\utilde{w}$ ends 0. So $\utilde{w} = \utilde{u}\utilde{z}$ where $\utilde{u} = \emptyset$ or $\utilde{u}$ ends 1. As shown $\utilde{u}1\utilde{z}1 \in \mathcal{B}$ yielding $\utilde{u}1\utilde{z}, \utilde{u}\utilde{z}1 \in \Delta\mathcal{A}$ and $\utilde{x} = \psi\utilde{x} \in \psi\Delta\mathcal{B}$ as required. In the alst line of Case 2 we are given $\utilde{a}, \utilde{b}, \utilde{x}, \utilde{u}'$ as shown for some $\utilde{w}$. Now $\utilde{u}$ ends 1 so $\utilde{x} = \utilde{w}01\utilde{z} \in \Delta\mathcal{B}$ and $\utilde{x} = \psi\utilde{x} \in \psi\Delta\mathcal{B}$ again. On the final line we are in Case 3 with $\utilde{x} = \utilde{u}'\utilde{z}1$ and $\utilde{u}' = \utilde{w}z_1$ where $\utilde{w}$ ends 1 or $\utilde{w} = \emptyset$. Hence $\utilde{u} = \utilde{w}\utilde{z}_1 1$ and $\utilde{c} = \utilde{w}\utilde{z}_1 11\utilde{z} \in \mathcal{B}$ and $\utilde{c} > \utilde{d} = \utilde{w}1\utilde{z}_1 1\utilde{z}$. Because D is IS this gives $\utilde{a}, \utilde{d} \in \mathcal{B}$ and $\utilde{w}1\utilde{z}_1\utilde{z}$, $\utilde{w}\utilde{z}_1\utilde{z}1 = \utilde{x} \in \Delta\mathcal{B}$ so $\utilde{x} \in \psi\Delta\mathcal{B}$. $\Diamond$

	$\underset{\sim}{x}$	$\psi\underset{\sim}{a}$ $\quad$ $\underset{\sim}{a}$	$\in \Delta\mathcal{B}$	Comment
Case 1.	$\underset{\sim}{w}'0$	$\underset{\sim}{w}0$	$\underset{\sim}{w}'0$	
	$\underset{\sim}{w} \notin \Gamma$	"	$\underset{\sim}{w}$	
	$\underset{\sim}{w} \in \Gamma$	"	$\underset{\sim}{u}1\underset{\sim}{z},\ \underset{\sim}{u}\underset{\sim}{z}1$	$\underset{\sim}{a} = \underset{\sim}{u}\underset{\sim}{z}10 > \underset{\sim}{u}1\underset{\sim}{z}0 \in \mathcal{B}$ as $D = IS$
	$\underset{\sim}{w}'11$	$\underset{\sim}{w}11$	$\underset{\sim}{w}'11$	
	$\underset{\sim}{w}1,\ w_n = 1$	"	$\underset{\sim}{w}1$	
	$\underset{\sim}{w}1,\ w_n = 0$	"	$\underset{\sim}{u}1\underset{\sim}{z},\ \underset{\sim}{u}\underset{\sim}{z}1$	$\underset{\sim}{a} = \underset{\sim}{u}\underset{\sim}{z}11 > \underset{\sim}{u}1\underset{\sim}{z}1 \in \mathcal{B}$ as $E = IS$
Case 2.	$\underset{\sim}{u}1\underset{\sim}{z}'$	$\underset{\sim}{u}1\underset{\sim}{z}$ $\quad$ $\underset{\sim}{u}\underset{\sim}{z}1$	$\underset{\sim}{u}\underset{\sim}{z}'1$	
	$\underset{\sim}{u}\underset{\sim}{z}$	" $\quad$ "	$\underset{\sim}{u}\underset{\sim}{z}$	
	$\underset{\sim}{u}'1\underset{\sim}{z}$	" $\quad$ "	$\underset{\sim}{u}'\underset{\sim}{z}1$	$\underset{\sim}{u}$ ends 1 or $\underset{\sim}{u}' = \emptyset$
	$\underset{\sim}{u}'1\underset{\sim}{z}$	" $\quad$ "	$\underset{\sim}{w}01\underset{\sim}{z}$	$\underset{\sim}{u}' = \underset{\sim}{w}0, \underset{\sim}{u} = \underset{\sim}{w}01, \underset{\sim}{a} = \underset{\sim}{w}01\underset{\sim}{z}1$
Case 3	$\underset{\sim}{u}\underset{\sim}{z}$	$\underset{\sim}{u}\underset{\sim}{z}1$	$\underset{\sim}{u}\underset{\sim}{z}$	
	$\underset{\sim}{u}\underset{\sim}{z}'1$	"	$\underset{\sim}{u}1\underset{\sim}{z},\ \underset{\sim}{u}\underset{\sim}{z}'1$	
	$\underset{\sim}{u}'\underset{\sim}{z}1$	"	$\underset{\sim}{u}'1\underset{\sim}{z},\ \underset{\sim}{u}'\underset{\sim}{z}1$	$\underset{\sim}{u}'$ ends 1 or $\underset{\sim}{u}' = \emptyset$
	$\underset{\sim}{u}'\underset{\sim}{z}1$	"	" $\quad$ "	$\underset{\sim}{u}'$ ends 0. See text.

Figure 2. Proof table for Theorem 2.

Theorem 3 [1] *Let D, E be IS of $C(n)$ with $E \subseteq D$. Put $D = D_0 0 + D_1 1$ and $E = E_0 0, + E_1 1$ and $\mathcal{B} = D0 + E1$. Then*

$$\Delta\mathcal{B} = \begin{cases} D & \text{if } E_0 \subseteq D_1, \\ D_0 0 + E_0 1 & \text{if } D_1 \subseteq E_0. \end{cases}$$

Proof: The result holds as $\Delta\mathcal{B} = D_0 0 \cup D_1 1 \cup E_0 1$ by (3).$\Diamond$

Example: To show that $D_1 = E_0$ does not imply $\mathcal{B} = IS$, let $D = IS(1110)$ and $E = IS(0010)$ then $D_1 = E_0 = IS(001)$ but $\mathcal{B} = IS(11100)\backslash 00011$ and $\mathcal{B} \neq IS$.

[1] Remark on Theorem 3. Let f be the final vector in $\mathcal{B}$. Let $\mathcal{I}$ be the largest IS in $\mathcal{B}$ and put $\mathcal{F} = \mathcal{B}\backslash\mathcal{I}$. If $D_1 \subseteq E_0$ then $\mathcal{B} \neq IS$ and $\underset{\sim}{f}$ ends 1 and $\Delta\mathcal{B} = \mathcal{I}^* + \mathcal{F}^*$. If $E_0 \subseteq D_1$ and $\mathcal{B} \neq IS$ then $\underset{\sim}{f}$ ends 0. We do not use these facts in this note.

4 Proof of Theorem 1.

The result holds trivially for $n = 1, 2$. To use induction we assume it holds for $n \le m$. Let $\mathcal{A} \subseteq C(m+1)$ and put $\mathcal{A} = A0 + B1$ so

$$\Delta A = A \cup (\Delta A)0 \cup B \cup (\Delta B)1.$$

By exchanging 0 and 1 we may assume $|A| \ge |B|$. Let D, E be IS of $\mathcal{C}(m)$ with $|D| = |A|$ and $|E| = |B|$ so $E \subseteq D$. By induction $|\Delta D| \le |\Delta A|$ and $|\Delta E| \le |\Delta B|$. Put $D = D_0 0 + D_1 1$ and $E = E_0 0 + E_1 1$ and $\mathcal{B} = D0 + E1$ as usual so $\mathcal{B}$ is PC.

Case 1. $E_0 \subseteq D_1$. By Theorem 3 we have $|\Delta\mathcal{B}| = |D| = |A| \le |\Delta\mathcal{A}|$. We forget $\mathcal{A}$ and study the slices of $\mathcal{B}$ in Figure 1. Let R, S, T, U be the consecutive slices with S the last 0-slice containing a vector of $\mathcal{B}$. Thus R, T are 1-slices and $U \cap \mathcal{B} = \emptyset$. Let $\underset{\sim}{x}10$ be the last vector of S, then $\underset{\sim}{x}01$ is the first vector of T.

Case 1.1 $\underset{\sim}{x}01 \in \mathcal{B}$. Here $R, S \subseteq \mathcal{B}$ and $\mathcal{B} = IS$.

Case 1.2 $\underset{\sim}{x}01 \notin \mathcal{B}$. Here $T \cap \mathcal{B} = \emptyset$. If $\mathcal{B} = IS$ we are done, so assume $\mathcal{B} \ne IS$. Let $\underset{\sim}{f}$ be the final vector of $\mathcal{B}$ so $\underset{\sim}{f} \in S$. There is a first $\underset{\sim}{e} \notin \mathcal{B}$ and $\underset{\sim}{e} < \underset{\sim}{f}$ and $\underset{\sim}{e}$ ends 1. We exchange $\mathcal{B}$ for $\mathcal{D} = (\mathcal{B} \backslash \underset{\sim}{f}) + \underset{\sim}{e}$. Then $\mathcal{D}$ is in Case 1. Also $|\Delta\mathcal{D}|| = |\Delta\mathcal{B}| - 1$ by Theorem 3, so we start on $\mathcal{D}$. Repetition proves Theorem 1 holds in Case 1.

Case 2. $D_1 \subseteq E_0$. By Theorem 3 and induction

$$|\Delta\mathcal{B}| = |D_0| + |E_0| = |\Delta D| + |\Delta E| \le |\Delta A| + |\Delta B| \le |\Delta\mathcal{A}|.$$

So we deal with $\mathcal{B}$. Let T be the last 1-slice containing a vector of $\mathcal{B}$. Again let R, S, T, U be consecutive slices. Further let $\underset{\sim}{x}10, \underset{\sim}{x}01$ be as before. Then $\underset{\sim}{x}01 \in \mathcal{B}$ but by definition of this case $\underset{\sim}{x}10 \notin \mathcal{B}$. Let $\underset{\sim}{g}$ be the first vector of S.

Case 2.1 $\underset{\sim}{g} \in \mathcal{B}$. Here $E_0 = D_1 + \underset{\sim}{x}$ and $|\Delta\mathcal{B}| = |D_0| + |E_0| = |D| + 1$ by Theorem 3. We have $|S| \ge 2$ and this implies $|T| = 1$. If $\underset{\sim}{e}$ is the first vector not in $\mathcal{B}$, then $\underset{\sim}{e} \in S$, and $\mathcal{D} = (\mathcal{B} \backslash \underset{\sim}{x}01) + \underset{\sim}{e}$ is an IS. Again using Theorem 3 we get $|\Delta\mathcal{D}| = |D_0| = |D| + 1 = |\Delta\mathcal{B}|$. Theorem 1 holds in this Case 2.1.

Case 2.2 $\underset{\sim}{g} \notin \mathcal{B}$. Put $\mathcal{D} = \psi\mathcal{B}$ and note that $\mathcal{D} \neq \mathcal{B}$ because $\underset{\sim}{g} = \psi(\underset{\sim}{x}01)$, except in the trivial case $\underset{\sim}{g} = \underset{\sim}{z}$. But $|\mathcal{D}| = |\mathcal{B}|$ and Theorem 2 tells that $|\Delta\mathcal{D}| = |\Delta\mathcal{B}|$. If $\mathcal{D} = A0 + B1$ then $B \subset E \subseteq D \subset A$ so we can only enter this Case 2.2 a finite number of times. We forget $\mathcal{B}$ and work on $\mathcal{D}$. Since $\mathcal{D}$ may not be PC we must start at the very beginning $\Diamond$

References

[1] **Danh, T.-N.** and **Daykin, D. E.** : *Ordering integer vectors for coordinate deletions.* (Submitted to J. London Math. Soc., 12. Feb 1994.)

[2] **Danh, T.-N.** and **Daykin, D. E.** : *Set Casades and vector Valleys in Pascal's triangle.* (Submitted to Theta, 14 July 1994.)

[3] **Danh, T.-N.** and **Daykin, D. E.** : *Bezrukov-Gronau Order is Not Optimal.* Rostock Math. Kolloq. **50**, 45-46 (1997)

received: April 19, 1995

Authors:

T.-N. Danh
University of HoCiMinh City
VietNam
227. Nguyen von Cu
HoCiMinh City

D. E. Daykin
University of Reading
England
RG6-2AX