Rostock. Math. Kolloq. **50**, 3–8 (1997)

Subject Classification (AMS)
34A37 34E05

S. G. Hristova
M. B. Kapitanova
S. I. Kostadinov

Method of a small parameter for a class of nonautonomous systems of second order differential equations with impulses

Systems with impulses find a wide application to mathematical modelling in ratio engineering, physics, chemistry, biology, etc. . The development of the qualitative theory of these systems is of particular interest since there solutions can be found in an explicit form only in exceptional cases.

An efficient method by means of which the periodic solutions of differential equations are investigated successfully is the method of a small parameter. In the present paper by means of this method a periodic solution of a nonautonomous system of second order impulsive differential equations is constructed. The case when the characteristic equation has some complex conjugate roots is studied.

Consider the system of impulsive differential equations

$$A\ddot{x} = Bx + f(t) + \sum_{k=1}^{\infty} \varepsilon^k f^{(k)}(t, x, \dot{x}, \ddot{x}) \tag{1}$$

$$\Delta x\big|_{t=\pi i} = Sx(\pi i) \tag{2}$$

$$\Delta \dot{x}\big|_{t=\pi i} = T\dot{x}(\pi i)\ , \tag{3}$$

where $x \in R^n$, A and B are $n \times n-$dimensional matrices, A is non-singular matrix, f and $f^{(k)}$ $(k = 1, 2, \dots)$ are $n-$dimensional vectors, $\Delta x|_{t=\pi i} = x(\pi i + 0) - x(\pi i - 0)$, $S = \operatorname{diag}(s_{11}, s_{22}, \dots, s_{nn})$, $T = \operatorname{diag}(t_{11}, t_{22}, \dots, t_{nn})$, $\varepsilon > 0$ is a small parameter, $i = 0, \pm 1, \pm 2, \dots$.

We will construct a $2\pi-$periodic solution of a system (1)–(3). We will consider the case when $t \in [0, 2\pi]$ and $i = 1$, i.e. the „jump“ condition (2) and (3) are at the piont $t = \pi$.

The equation

$$\det(P - \lambda^2 E) = 0\ , \tag{4}$$

where $P = A^{-1}B = (p_{ij})_1^n$ and E is the $n-$dimensional identity matrix, will be called a characteristic equation of the system (1)–(3).

Let the equation (4) has l complex conjugate roots of the type $\pm iq_j$, where $q_j (j = 1, 2, \dots, l)$ are natural numbers.

We will call a generating system of the system (1)–(3) the following system

$$\ddot{x}^{(0)} = Px^{(0)} + G(t) \tag{5}$$

$$\Delta x^{(0)}\big|_{t=\pi} = Sx^{(0)}(\pi) \tag{6}$$

$$\Delta \dot{x}^{(0)}\big|_{t=\pi} = T\dot{x}^{(0)}(\pi)\ , \tag{7}$$

where $G(t) = A^{-1}f(t)$.

The general solution of the system (5)–(7) is of the following type

$$x^{(0)}(t) = \begin{cases} C^{(0)}\varphi(t) + \overline{x}^{(0)}(t) & \text{for } t \in [0, \pi] \\ \overline{C}^{(0)}\varphi(t) + \overline{x}^{(0)}(t) & \text{for } t \in (\pi, 2\pi] \end{cases}\ , \tag{8}$$

where $C^{(0)} = (C_1^{(0)}, c_2^{(0)}, \dots, C_{2n}^{(0)})$, $\overline{C}^{(0)} = (\overline{C}_1^{(0)}, \dots, \overline{C}_{2n}^{(0)})$. $\varphi(t)$ is $2n \times n-$dimensional matrix of fundamental solutions $\varphi_{j_k}(t)$ $(j = 1, \dots, 2n)$ of the homogeneous system $\ddot{x} = Px$, and the function $\overline{x}^{(0)}(t)$ is a particular solution of (5) and ([1]). According to ([1]) we have

$$\overline{x}^{(0)}(t) = \int\limits_0^t G(t) \cdot \Xi(t - \lambda) d\lambda\ , \tag{9}$$

$\Xi(t)$ is a matrix of fundamental solutions $\xi_{s_j}(t) (s = \overline{1, n}; j = \overline{1, 2n})$ of the homogeneous system, corresponding to (5), for which

$$\begin{aligned} &\xi_{s_j}(0) = \delta_{s_j}\ , \quad \dot{\xi}_{s,n+j}(0) = \delta_{s_j} \quad \text{for} \quad s \neq j \\ &\xi_{s_j}(0) = 0\ , \quad \dot{\xi}_{s,n+j}(0) = 0 \quad \text{for} \quad s = j\ ,\ (s, j = \overline{1, n}) \end{aligned} \tag{10}$$

and δ_{s_j} are the symbols of Kronecker.

In (8) $x^{(0)}(t) = (x_1^{(0)}(t), \dots, x_n^{(0)}(t))$ and

$$x_s^{(0)}(t) = \begin{cases} C_1^{(0)}\varphi_{s1}(t) + \cdots + C_{2l}\varphi_{s,2l}(t) + \cdots + C_{2n}\varphi_{2n}^{(0)}(t) \text{ for } t \in [0, \pi] \\ \overline{C}_1^{(0)}\varphi_{s1}(t) + \cdots + \overline{C}_{2l}^{(0)}\varphi_{s,2l}(t) + \cdots + \overline{C}_{2n}^{(0)}\varphi_{2n}(t) \text{ for } t \in [\pi, 2\pi] \end{cases} \tag{11}$$

where $C_j^{(0)}$ and $\overline{C}_j^{(0)}$ $(j=\overline{1,n})$ are unknown constants and

$$\begin{aligned}
\varphi_{s1}(t) &= L_s^{(1)}\,\omega s\cos q_1 t\ ,\\
\varphi_{s2}(t) &= L_s^{(1)}\,\sin q_1 t\ ,\\
\varphi_{s3}(t) &= L_s^{(2)}\,\omega s\cos q_2 t\ ,\\
\varphi_{s4}(t) &= L_s^{(2)}\,\sin q_2 t\ ,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
\varphi_{s,2l-1}(t) &= L_s^{(l)}\,\omega s\cos q_l t\ ,\\
\varphi_{s,2l}(t) &= L_s^{(l)}\,\sin q_l t\ ,\quad (s=\overline{1,n})\ .
\end{aligned} \tag{12}$$

In the notations (12) the constants $L_s^{(j)}$ $(j=\overline{1,l})$ are nonzero solutions of the systems

$$p_{s1}L_1^{(j)}+p_{s2}L_2^{(j)}+\cdots+(p_{ss}+q_j)L_s^{(j)}+\cdots+p_{sn}L_n^{(j)}=0\ ,\quad (s=\overline{1,n})\ .$$

The conditions for periodicity and the „jump" conditions of the solution $x^{(0)}(t)$ of the system (5)–(7) are of the type

$$\begin{cases}
x_k^{(0)}(2\pi)-x_k^{(0)}(0)=0\\
\dot{x}_k^{(0)}(2\pi)-\dot{x}_k^{(0)}(0)=0\\
x_k^{(0)}(\pi+0)-x_k^{(0)}(\pi)=s_{kk}\cdot x^{(0)}(\pi)\\
\dot{x}_k^{(0)}(\pi+0)-\dot{x}_k^{(0)}(\pi)=t_{kk}\cdot\dot{x}^{(0)}(k)\ ,\quad (k=\overline{1,n})\ .
\end{cases} \tag{13}$$

From conditions (11) and (13) it follows that

$$\begin{cases}
L_k^{(1)}\left(\overline{C}_1^{(0)}-C_1^{(0)}\right)+L_k^{(2)}\left(\overline{C}_3^{(0)}-C_3^{(0)}\right)+\cdots+\\
+L_k^{(l)}\left(\overline{C}_{2l-1}^{(0)}-C_{2l-1}^{(0)}\right)+\left(\overline{C}_{2l+1}^{(0)}-C_{2l+1}^{(0)}\right)\varphi_{k,2l+1}(0)+\\
+\cdots+\left(\overline{C}_{2n}^{(0)}-C_{2n}^{(0)}\right)\varphi_{k,2n}(0)+\overline{x}_k^{(0)}(2\pi)=0\qquad (k=\overline{1,n})
\end{cases} \tag{14}$$

$$\begin{cases}
p_1L_k^{(1)}\left(\overline{C}_2^{(0)}-C_2^{(0)}\right)+p_2L_k^{(2)}\left(\overline{C}_4^{(0)}-C_4^{(0)}\right)+\cdots+\\
+p_lL_k^{(l)}\left(\overline{C}_{2l}^{(0)}-C_{2l}^{(0)}\right)+\left(\overline{C}_{2l+1}^{(0)}-C_{2l+1}^{(0)}\right)\cdot\dot{\varphi}_{k,2l+1}(0)+\\
+\cdots+\left(\overline{C}_{2n}^{(0)}-C_{2n}^{(0)}\right)\dot{\varphi}_{k,2n}(0)+\dot{\overline{x}}^{(0)}(2\pi)=0\qquad (k=\overline{1,n})
\end{cases} \tag{15}$$

$$\begin{cases}
(-1)^{p_1}\cdot L_k^{(1)}\overline{C}_1^{(0)}+(-1)^{p_2}\cdot L_k^{(2)}\cdot\overline{C}_3^{(0)}+\cdots+(-1)^{p_l}L_k^{(2)}\overline{C}_{2l-1}^{(0)}-\\
-\overline{C}_{2l+1}^{(0)}\cdot\varphi_{k,2l+1}(\pi)-\cdots-\overline{C}_{2n}^{(0)}\cdot\varphi_{k,2k}(\pi)=\\
=(s_{kk}+1)\Big[(-1)^{p_1}\cdot L_k^{(1)}\cdot C_1^{(0)}+(-1)^{p_2}L_k^{(2)}\cdot C_3^{(0)}+\cdots+\\
\qquad+(-1)^{p_l}\cdot L_k^{(l)}\cdot C_{2l-1}^{(0)}\cdot\varphi_{k,2l+1}(\pi)-\cdots-C_{2n}^{(0)}\cdot\varphi_{k,2n}(\pi)\Big]\ ,\qquad (k=\overline{1,n})
\end{cases} \tag{16}$$

$$\begin{cases} p_1\cdot(-1)^{p_1}L_k^{(1)}\overline{C}_2^{(0)}+p_2(-1)^{p_2}\cdot L_k^{(2)}\cdot\overline{C}_4^{(0)}+ \\ +p_l\cdot(-1)^{p_l}\cdot L_k^{(2)}\overline{C}_{2l}^{(0)}-\overline{C}_{2l+1}^{(0)}\cdot\dot{\varphi}_{k,2l+1}(\pi)- \\ -\overline{C}_{2n}^{(0)}\cdot\dot{\varphi}_{k,2k}(\pi) \\ =(t_{kk}+1)\Big[p_1\cdot(-1)^{p_1}L_k^{(1)}C_2^{(0)}+\cdots+p_l(-1)^{p_l}L_k^{(l)}\cdot C_{2l}^{(0)}- \\ -C_{2l+1}^{(0)}\cdot\dot{\varphi}_{k,2l+1}(\pi)-\cdots-C_{2n}^{(0)}\cdot\dot{\varphi}_{k,2\pi}(\pi)\Big]\ , \qquad (k=\overline{1,n}) \end{cases} \tag{17}$$

The determinant D of the system (14) - (17) with $4n$ linear equations and unknown $C_1^{(0)}$, $\dots, C_{2n}^{(0)}, \overline{C}_1^{(0)}, \dots, \overline{C}_{2n}^{(0)}$ is nonzero when S and T are nonzero matrices. Then this system has a unique solution $C_j^{(0)}, \overline{C}_j^{(0)} (j=\overline{1,2n})$. If we substitute the values of these constants in (11) then we find a 2π-periodic solution $x^{(0)}(t)=(x_1^{(0)}(t),\dots,x_n^{(0)}(t))$ in the generating system (5) - (7). The periodic solution of the system (1) - (3) can be developed in a power series and it coincides for $\varepsilon=0$ with the solution $x^{(0)}(t)=(x_1^{(0)}(t),x_2^{(0)}(t),\dots,x_n^{(0)}(t))$ of the generating system (5) - (7). So we seek the solution in the form

$$x(t)=x^{(0)}(t)+\varepsilon x^{(1)}(t)+\varepsilon^2x^{(2)}(t)+\dots, \tag{18}$$

where $x^{(s)}(t)=(x_1^{(s)}(t),x_2^{(s)}(t),\dots,x_n^{(s)}(t)),(s=\overline{1,n})$.
Let the functions $x^{(0)}(t),x^{(1)}(t),\dots,x^{(k-1)}(t)$ are found.
We substitute the expansion (18) into equations (1)-(3) and calculate the coefficients before ε of the power k. So we obtain the following system of differential equations for the functions $x^{(k)}(t)$:

$$\ddot{x}^{(k)}=P\cdot x^{(k)}+G(t) \tag{19}$$

$$\Delta x^{(k)}\big|_{t=\pi}=S\cdot x^{(k)}(t) \tag{20}$$

$$\Delta\dot{x}^{(k)}\big|_{t=\pi}=T\cdot\dot{x}^{(k)}(t)\ , \tag{21}$$

where $G^{(k)}=A^{-1}(\Theta^{(k)}+\overline{G}^{(k)})$, and

$$\Theta^{(k)}=\left(\frac{\partial f^{(1)}(\xi)}{\partial x}\right)\cdot x^{(k-1)}+\left(\frac{\partial f^{(1)}(\xi)}{\partial\dot{x}}\right)\cdot\dot{x}^{(k-1)}+\left(\frac{\partial f^{(1)}(\xi)}{\partial\ddot{x}}\right)\cdot\ddot{x}^{(k-1)}, \tag{22}$$

$\overline{G}(k)$ contains $x^{(0)}(t),\dots,x^{(k-2)}(t)$ and their first and second derivatives,

$$\xi=(t,x^{(0)}(t),\dot{x}^{(0)}(t),\ddot{x}^{(0)}(t)).$$

The general solution of the system (19) - (21) is of the type

$$x^{(k)}(t)=\begin{cases} C^{(k)}\varphi(t)+\overline{x}^{(k)}(t) & \text{for} \quad t\in[0,\pi] \\ \overline{C}^{(k)}\varphi(t)+\overline{x}^{(k)}(t) & \text{for} \quad t\in(\pi,2\pi] \end{cases}$$

where $\overline{x}^{(k)}(t)$ is a paiticular solution of (19) and

$$\overline{x}^{(k)}(t) = \int\limits_0^t G^{(k)}(t) \cdot \Xi(t-\lambda)\partial\lambda,$$

$$C^{(k)} = (C_1^{(k)}, \dots, C_{2n}^{(k)}),\ \overline{C}^{(k)} = (\overline{C}_1^{(k)}, \dots ; \overline{C}_{2n}^{(k)}),$$

$C_j^{(k)}$ and $\overline{C}_j^{(k)} (j = \overline{1, 2n})$ are unknown constants.
The conditions for periodicity and „jump" conditions of the solution $x^{(k)}(t)$ of system (19)-(21) are of the type (13), where the superscript "0" is replaced by the superscript "k".
The constants $C_1^{(k)}, \dots, C_{2n}^{(k)}, \overline{C}_1^{(k)}, \dots, \overline{C}_{2n}^{(k)}$ are solution of a system of the type (14)-(17) where the superscript "0" is replaced by the superscript "k". The determinant of this system is equal to D. That's why the system has a unique solution $C_1^{(k)}, \dots, C_{2n}^{(k)}, \overline{C}_1^{(k)}, \dots, \overline{C}_{2n}^{(k)}$ if S and T are nonzero matrices.
In this way we can construct a 2π-periodic solution of the system (1)-(3) as a power series.

Theorem 1 *Let the following conditions be fulfilled:*

1. *The functions $f(t)$ and $f^{(k)}(t, x, \dot{x}, \ddot{x})(k = 1, 2, \dots)$ are continous and periodic functions with respect to t with a period 2π and they are analytic functions with respect to all arguments.*

2. *The matrices S and T are nonzero diagonal matrices.*

3. *The matrix A is non-singular.*

4. *The equation (4) has l complex conjugate roots of the type $\pm iq_j$ $(j = \overline{1, l})$ where q_j are natural numbres.*

Then the system with impulses (1)-(3) has a 2π-perodic solution which coincides with a 2π-periodic solution of the generating system when $\varepsilon = 0$.

Remark: The Theorem 1 is valid when all or part of the variables satisfy the „jump" conditions.

References

[1] **Paskalev, G.** : *Periodic solutions of non-autonomous linear systems of second order differential equations with constant coefficients.* Trav. sci. , Univ. de Plovdiv, vol. **12**, fasc. 1, 1974 (in Bulgarien)

received: November 24, 1993

Authors:

S. G. Hristova
University of Plovdiv
„Paissii Hilendarski"
Math. Fac.
Zar Assen 24
4000 Plovdiv
Bulgaria

M. B. Kapitanova
University of Plovdiv
„Paissii Hilendarski"
Math. Fac.
Zar Assen 24
4000 Plovdiv
Bulgaria

S. I. Kostadinov
University of Plovdiv
„Paissii Hilendarski"
Math. Fac.
Zar Assen 24
4000 Plovdiv
Bulgaria