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Integral inequalities for piecewise – continuous functions of many variables

1 Introduction

There exist various generalizations of Bellman – Gronwall – Bihari inequalities [1] – [6]. These generalizations have been widely quoted by many mathematicians inproving uniqueness, boundedness, continuous dependence and stability.

In this paper we establish integral inequalities for piecewise – continuous functions of many variables. These generalizations are motivated by specific applications to the theory of impulsive partial differential equations.

2 Preliminaries

Let $x^{(k)} \in \mathbb{R}^n$, $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \dots , x_n^{(k)})$, $k = 0, 1, \dots$ are fixed points so that for $i = \overline{1, n}$ the inequalities $x_i^{(k)} < x_i^{(k+1)}$, $k = 0, 1, \dots$ hold and $\lim\limits_{k\to\infty} x_i^{(k)} = +\infty$

Define the sets

$$D_k = \left\{ x = (x_1, \dots , x_n) : x_i \in \left[x_i^{(k)}, x_i^{(k+1)}\right], i = 1, \dots , n \right\}, k = 0, 1, 2, \dots , \ \Omega = \bigcup_{k=0}^{\infty} D_k .$$

Let $x, y \in \mathbb{R}^n$, $x = (x_1, \dots , x_n)$, $y = (y_1, \dots , y_n)$. We say that $x \geq (\leq y)$ if for $i = \overline{1, n}$ the inequalities $x_i \geq (\leq) y_i$ hold.

Let $x, y \in \mathbb{R}^n$, $x \leq y$. Denote

$$B(x,y) = [x_1, y_1] \times \cdots \times [x_n, y_n] \ ,$$
$$\int\limits_{x_1}^{y_1} \int\limits_{x_2}^{y_2} \cdots \int\limits_{x_n}^{y_n} ds_n ds_{n-1} \ldots ds_1 = \int\limits_{B(x,y)} \mathrm{ds} \ , \quad ds' = ds_n \ldots ds_{k+1} ds_{k-1} \ldots ds_1 \ ,$$
$$B_k(x,y) = [x_1, y_1] \times \cdots \times [x_{k-1}, y_{k-1}] \times [x_{k+1}, y_{k+1}] \times \cdots \times [x_n, y_n] \ ,$$

where $k : 1 \leq k \leq n$ is an arbitrary natural number, $(y_k, x') = (x_1, x_2, \ldots, x_{k-1}, y_k, x_{k+1}, \ldots, x_n)$.

We will use the following lemmas:

Lemma 1 */theorem 1 [5]/. Let for $s \in B(x_0, x)$ and x_0, $x \in R^n$, $x > x_0$ the following conditions be fulfilled:*

1. *The functions $u(x)$, $v(x) : R^n \to R$ and $\mu(x,s) : R^n \times R^n \to R$ are nonegative and continuous.*
2. *The inequality*

$$u(x) \leq a + \beta \int\limits_{B(x_0,x)} \left[v(y)u(y) + \int\limits_{B(x_0,y)} \mu(y,s)u(s)ds \right] dy$$

holds, where a and β are nonnegative constants. Then for $x \geq x_0$ the inequality

$$u(x) \leq a \exp \left\{ \beta \int\limits_{B(x_0,x)} \left[v(y) + \int\limits_{B(x_0,y)} \mu(y,s)ds \right] dy \right\}$$

holds.

Lemma 2 */theorem 2 [6]/. Let the functions $u(x) : R^n \to R^1$ and $f(x) : R^n \to R^1$ are continuous and nonnegative for $x_0 \leq x$, $x_0 \in R^n$. Let the inequality*

$$u(x) \leq c + \int\limits_{B(x_0,x)} f(s)\phi(u(s))ds$$

holds, where $c = \mathrm{const} > 0$,

$$F(t) = \int\limits_c^t \frac{d\tau}{\phi(\tau)} \ , \quad t \in (c, \sigma) \ , \quad \int\limits_{B(x_0,x)} f(s)ds \subset \mathrm{Dom}(F^{-1})$$

and the function $\phi(\cdot) : R \to R$ is positive and increasing.

Then for $x \geq x_0$ the inequality

$$u(x) \leq F^{-1}\left(\int\limits_{B(x_0,x)} f(s)ds \right)$$

holds, where $F^{-1}(\cdot)$ is the inverse function of $F(\cdot)$.

3 Main results

Theorem 1 *Let the following conditions be fulfilled:*

1. *The function $u(x) : R^n \to R^1$ is defined, nonnegative, and piece – continuous for $x \geq x^{(0)}$ with points of interruption in the points $x^{(k)}$, $u(x^{(k)}) = u(x^{(k)} - 0)$ and for $k = 1, 2, \ldots$ the inequalities $u(x^{(k)} + 0) - u(x^{(k)}) < \infty$ hold.*
2. *The functions $v(x) : R^n \to R$ and $w(x, y) : R^n \times R^n \to R$ are continuous and nonnegative for $x, y \geq x^{(0)}$ and the equalities $v(x) = 0$ for $x \in R^n|\Omega$ and $w(x, y) = 0$ for $x, y \overline{\in} \Omega$ hold.*
3. *The constants a, β and γ_k $(k = 1, 2, \ldots)$ are nonnegative.*
4. *For $x \in \Omega$ the inequality*

$$u(x) \leq a + \beta \int\limits_{B(x^{(0)},x)} \left[v(s)u(s) + \int\limits_{B(x^{(0)},s)} w(s,\xi)u(\xi)d\xi \right] ds + \sum_{k:x^{(k)}<x} \gamma_k u(x^{(k)}) \qquad (1)$$

holds.

Then for $x \in \Omega$ the inequality

$$u(x) \leq \prod_{k:x^{(k)}<x} (1 + \gamma_k) \exp\left\{ \beta \int\limits_{B(x^{(0)},x)} \left[v(s) + \int\limits_{B(x^{(0)},s)} w(s,\xi)d\xi \right] ds \right\} \qquad (2)$$

holds.

Proof. Let $x \in \mathcal{D}_0$. Then the inequality (1) can be written in the form

$$u(x) \leq a + \beta \int\limits_{B(x^{(0)},x)} \left[v(s)u(s) + \int\limits_{B(x^{(0)},s)} w(s,\xi)u(\xi)d\xi \right] ds\ . \qquad (3)$$

According to Lemma 1 for $x \in \mathcal{D}_0$ the inequality

$$u(x) \le a \exp\left\{ \beta \int\limits_{B(x^{(0)},x)} \left[v(s) + \int\limits_{B(x^{(0)},s)} w(s,\xi) d\xi \right] ds \right\} \tag{4}$$

holds.

We assume the inequality (2) holds for $x \in \mathcal{D}_{k-1}$. We will prove the inequality (2) holds for $x \in \mathcal{D}_k$. According to the condition 2 of the theorem 1 for $x \in \mathcal{D}_k$ the inequality (1) can be written in the form

$$u(x) \le \alpha + \beta \int\limits_{B(x^{(0)},x)} \left[v(s)u(s) + \int\limits_{B(x^{(0)},s)} w(s,\xi)u(\xi) d\xi \right] ds \ , \tag{5}$$

where

$$\alpha = a + \beta \int\limits_{\bigcup\limits_{i=0}^{k-1} \mathcal{D}_i} \left[v(s)u(s) + \int\limits_{B(x^{(i)},s)} w(s,\xi)u(\xi) d\xi \right] ds + \sum_{i=1}^{k} \gamma_i u(x^{(i)}) \ . \tag{6}$$

By the assuming the inequality

$$\begin{aligned}
\alpha \le\ & a + \beta \sum_{i=0}^{k-1} \int\limits_{\mathcal{D}_i} \Bigg[v(s) a \prod_{j=1}^{i} (1+\gamma_j) \exp\left\{ \beta \int\limits_{B(x^{(0)},s)} \left[v(y) + \int\limits_{B(x^{(0)},y)} w(y,\xi) d\xi \right] ds \right\} + \\
& + \int\limits_{B(x^{(i)},s)} w(s,\xi) a \prod_{j=1}^{i} (1+\gamma_j) \exp\left\{ \beta \int\limits_{B(x^{(0)},s)} \left[v(y) + \int\limits_{B(x^{(0)},y)} w(y,\eta) d\eta \right] dy \right\} d\xi \Bigg] ds + \\
& + \sum_{i=1}^{k} \gamma_i a \prod_{j=1}^{i-1} (1+\gamma_j) \exp\left\{ \beta \int\limits_{B(x^{(0)},x^{(i)})} \left[v(s) + \int\limits_{B(x^{(0)},s)} w(s,\xi) d\xi \right] ds \right\} \le \\
\le\ & a + a\beta \sum_{i=0}^{k-1} \Bigg\{ \int\limits_{\mathcal{D}_i} \Bigg[v(s) \prod_{j=1}^{i} (1+\gamma_j) \cdot \prod_{j=1}^{i-1} f_j \cdot g_i(s) + \\
& + \int\limits_{B(x^{(i)},s)} w(s,\xi) \prod_{j=1}^{i} (1+\gamma_j) \cdot \prod_{j=1}^{i-1} f_j \cdot g_i(\xi) d\xi \Bigg] ds \Bigg\} + \\
& + a \sum_{i=1}^{k} \gamma_i \prod_{j=1}^{i-1} (1+\gamma_j) \cdot \prod_{j=0}^{i-1} f_i \ ,
\end{aligned} \tag{7}$$

holds, where

$$\gamma_0 = 0 \ ,$$

$$f_j = \exp\left\{ \beta \int\limits_{\mathcal{D}_j} \left[v(s) + \int\limits_{B(x^{(j)},s)} w(s,\xi)d\xi \right] ds \right\} \ ,$$

$$g_j(x) = \exp\left\{ \beta \int\limits_{B(x^{(j)},x)} \left[v(s) + \int\limits_{B(x^{(j)},s)} w(s,\xi)d\xi \right] ds \right\} \ , \qquad x > x^{(j)} \ , \quad j = 0, 1, \dots$$

and the symbols $\prod_{i=1}^{0}(1+\gamma_i)$ and $\prod_{i=1}^{-1} f_i$ denote the number 1.

From the inequality (7) and the monotonicity of the exponential function we obtain the inequality

$$\begin{aligned} \alpha \le a \Bigg\{ 1 + \sum_{i=0}^{k-1} \left[\prod_{j=1}^{i}(1+\gamma_j) \cdot \left(\prod_{j=1}^{i-1} f_j \right) \cdot (g_i(x^{(i+1)}) - 1) \right] + \\ + \sum_{i=1}^{k} \gamma_i \left(\prod_{j=1}^{i-1}(1+\gamma_j) \right) \cdot \left(\prod_{j=0}^{i-1} f_j \right) \Bigg\} = \\ = a \prod_{j=1}^{k}(1+\gamma_j) \cdot \left(\prod_{j=0}^{k-1} f_j \right) \ . \end{aligned} \tag{8}$$

From the inequalities (5), (8) and Lemma 1 we obtain the inequality

$$\begin{aligned} u(x) \le \alpha \exp\left\{ \beta \int\limits_{B(x^{(k)},x)} \left[v(s) + \int\limits_{B(x^{(k)},s)} w(s,\xi)d\xi \right] ds \right\} \le \\ \le a \prod_{j=1}^{k}(1+\gamma_j) \cdot \exp\left\{ \beta \int\limits_{B(x^{(0)},x)} \left[v(s) + \int\limits_{B(x^{(0)},s)} w(s,\xi)d\xi \right] ds \right\} \ . \end{aligned}$$

Remark 1. For $n = 1$ and $w(s,\xi) \equiv 0$ theorem 1 yields one result of [7].

Theorem 2 *Let the following conditions be fulfilled:*

1. *Condition 1 and 2 of the theorem 1 are satisfied.*
2. *The function $g(x) : R^n \to R$ is continuous, positive and non–decreasing for $x \in \Omega$.*

3. *For* $x \in \Omega$ *the inequality*

$$u(x) \leq g(x) + \beta \int\limits_{B(x^{(0)},x)} \left[v(s)u(s) + \int\limits_{B(x^{(0)},s)} w(s,\xi)u(\xi)d\xi \right] ds + \sum_{k:x^{(k)}<x} \gamma_k u(x^{(k)}) \ , \tag{9}$$

holds, where the constants β *and* γ_i $(i = 1, 2, \dots)$ *are nonnegative.*

Then for $x \in \Omega$ *the inequality*

$$u(x) \leq g(x) \prod_{k:x^{(k)}<x} (1+\gamma_k) \cdot \exp\left\{ \beta \int\limits_{B(x^{(0)},x)} \left[v(s) + \int\limits_{B(x^{(0)},s)} w(s,\xi)d\xi \right] ds \right\} \tag{10}$$

holds.

Proof We devide the both sides of the inequality (9) by the function $g(x)$ and use the condition 2 of the theorem 2. We obtain

$$\frac{u(x)}{g(x)} \leq 1 + \beta \int\limits_{B(x^{(0)},x)} \left[v(s)\frac{u(s)}{g(s)} + \int\limits_{B(x^{(0)},s)} w(s,\xi)\frac{u(\xi)}{g(\xi)}d\xi \right] ds + \sum_{k:x^{(k)}<x} \gamma_k \frac{u(x^{(k)})}{g(x^{(k)})} \ . \tag{11}$$

From the inequality (11) and the theorem 1 it follows the inequality (10).

Theorem 3 *Let the following conditions be fulfilled:*

1. *The condition* 1 *of the theorem* 1 *is satisfied.*
2. *The function* $v(x) : R^n \to R$ *is continuous, non–negative, and* $v(x) \equiv 0$ *for* $x \in R^n \backslash \Omega$.
3. *The function* $g(t) : R \to R$ *is continuous, positive and non–decreasing for* $t > 0$.
4. *The functions* $f_k(t) : R \to R$ $(k = 1, 2, \dots)$ *are positive and non–decreasing for* $t > 0$.
5. *For every natural number* k *there exists a constant* $M_k > 0$ *so that for* $t \geq s > 0$ *the inequality*

$$G(\tau_k(t)) - G(\tau_k(s)) \leq M_k \left[G(t) - G(s) \right]$$

holds, where

$$\tau_k(t) = t + f_k(t) \ , \quad G(t) = \int\limits_{t_0}^{t} \frac{d\xi}{g(\xi)} \ , \quad t > t_0 \ , \quad t_0 > 0 \ .$$

6. *For $x \geq x^{(0)}$ the inequality*

$$u(x) \leq a + \int\limits_{B(x^{(0)},x)} v(s)g(u(s))ds + \sum_{k:x^{(k)}<x} f_k(u(x^{(k)})) \ . \tag{12}$$

holds, where the constant a is positive.

Then for $x \in \Omega_1$ the inequality

$$u(x) \leq G^{-1}\left[G(R(x)) + \int\limits_{B(x^{(0)},x)} \left(\prod_{s<x^{(j)}<x} M_j\right) v(s)ds\right] \tag{13}$$

holds, where $R_0(x) \equiv a$ for $x \in \mathcal{D}_0$, $R(x) \equiv R_k = \tau_k(R_{k-1})$ for $x \in \mathcal{D}_k$,

$$\Omega_1 = \sup\left\{ x \geq x^0 : G(R(y)) + \right.$$

$$\left. + \int\limits_{B(x^{(0)},y)} \left(\prod_{s<x^{(j)}<y} M_j\right) v(s)ds \in \mathrm{Dom}\,(G^{-1}) \ , \quad y \in B(x^{(0)},x)\right\} \ .$$

Proof Let $x \in \mathcal{D}_0$. Then the inequality (12) can be written in the form

$$u(x) \leq a + \int\limits_{B(x^{(0)},x)} v(s)g(u(s))ds \ . \tag{14}$$

According to Lemma 2 for $x \in \mathcal{D}_0 \cap \Omega_1$ the inequality

$$u(x) \leq G^{-1}\left[G(a) + \int\limits_{B(x^{(0)},x)} v(s)ds\right] \ ,$$

holds.

We assume the inequality (13) holds for $x \in \mathcal{D}_{k-1} \cap \Omega_1$. We will prove that this inequality holds for $x \in \mathcal{D}_k \cap \Omega_1$. According to the condition 2 of theorem 3 the inequality (12) can be written in the form

$$u(x) \leq a + \sum_{i=0}^{k-1} \int\limits_{\mathcal{D}_i} v(s)g(u(s))ds + \int\limits_{B(x^{(k)},x)} v(s)g(u(s))ds + \sum_{i=1}^{k} f_i(u(x^{(i)})) \equiv w(x) \ .$$

The function $w(x)$ satisfies the inequality $u(x) \leq w(x)$. Then

$$w(x) \leq a + \sum_{i=0}^{k-1} \int\limits_{\mathcal{D}_i} v(s)g(w(s))ds + \int\limits_{B(x^{(k)},x)} v(s)g(w(s))ds + \sum_{i=1}^{k} f_i(w(x^{(i)})) \ . \tag{15}$$

We fix an arbitrary natural number $j : 1 \le j \le n$. Then the inequalities

$$\frac{\partial}{\partial x_j} w(x) \le \int\limits_{B_j(x^{(k)},x)} v(x_j, s') g(w(x_j, s')) ds' \ , \tag{16}$$

$$\begin{aligned} w(x_j^{(k)} + 0, (x^{(k)})') &\le w(x_j^{(k)} - 0, (x^{(k)})') + f_k(w(x_j^{(k)} - 0, (x^{(k)})')) = \\ &= w(x^{(k)}) - f_k(w(x^{(k)})) = \tau_k(w(x^{(k)})) \ , \end{aligned} \tag{17}$$

hold.

From the inequalities (16) and (17) and Bihari's lemma [1] it follows that the inequality

$$\begin{aligned} w(x) &\le G^{-1}\left[G(w(x_j^{(k)} + 0, (x^{(k)})')) + \int\limits_{x_j^{(k)}}^{x_j} \int\limits_{B(x^{(k)},x)} v(x_j, s') ds' ds_j \right] = \\ &= G^{-1}\left[G(w(x_j^{(k)} + 0, (x^{(k)})')) + \int\limits_{B(x^{(k)},x)} v(s) ds \right] \end{aligned} \tag{18}$$

holds.

From the properties of the function $G(x)$ and the inequalities (17) and (18) it follows that the inequality

$$G(w(x)) \le G(\tau_k(w(x^{(k)}))) + \int\limits_{B(x^{(k)},x)} v(s) ds \tag{19}$$

holds.

If the inequality $w(x^{(k)}) \le R_{k-1}$ holds then from (18) it follows the inequality (13). Otherwise we obtain

$$\begin{aligned} G(\tau_k(w(x^{(k)}))) - G(R_k) &\le M_k \left[G(w(x^{(k)})) - G(R_{k-1}) \right] \le \\ &\le M_k \int\limits_{B(x^{(0)},x)} \left(\prod_{s < x^{(j)} < x} M_j \right) v(s) ds \ . \end{aligned} \tag{20}$$

According to inequalities (19), (20) and the condition 2 of theorem 3 it follows that the inequality

$$\begin{aligned} G(w(x)) &\le G(R_k) + \int\limits_{B(x^{(k)},x)} v(s) ds + M_k \int\limits_{B(x^{(0)},x)} \left(\prod_{s < x^{(j)} < x} M_j \right) v(s) ds = \\ &= G(R_k) + \int\limits_{B(x^{(0)},x)} \left(\prod_{s < x^{(j)} < x} M_j \right) v(s) ds \end{aligned}$$

holds.

Theorem 4 *Let the following conditions be fulfilled:*

1. *The conditions* 1, 2 *and* 5 *of theorem* 3 *are satisfied.*

2. *For* $x > x^{(0)}$ *the inequality*

$$u(x) \leq a + \int\limits_{B(x^{(0)},x)} v(s)u^m(s)ds + \sum_{i:x^{(i)}<x} \beta_i u(x^{(i)}) \; , \tag{21}$$

holds, where $m > 0$, $m \neq 1$ *and* $\beta_i \geq 0$ $(i = 1, 2, \dots)$ *are constants.*

Then for $x \in \Omega_2$ *the inequality*

$$u(x) \leq a \prod_{k:x^{(k)}<x} (1+\beta_k) \left\{ 1 + (1-m) \int\limits_{B(x^{(0)},x)} \left[\prod_{s<x^{(k)}<x} a(1+\beta_k)^{m-1} \right] v(s)ds \right\} \tag{22}$$

holds, where

$$\Omega_2 = \sup \left\{ x \geq x_0 : (m-1) \int\limits_{B(x^{(0)},y)} \left(\prod_{s<x^{(k)}<y} a(1+\beta_k)^{m-1} \right) v(s)ds < 1 \; , \right.$$
$$\left. y \in B(x^{(0)}, x) \right\} \; .$$

Proof Define the functions $g(s) = s^m$, $\tau_k(s) = (1+\beta_k)s$, $R(x) = a \prod\limits_{x^{(i)}<x} (1+\beta_i)$, $G(t) = \frac{t^{1-m}}{1-m} + \sigma$, $(\sigma > 0)$.
Then $M_i = (1+\beta_i)^{1-m}$. According to (21) and theorem 3 the inequality (22) holds.

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