Rostock. Math. Kolloq. **50**, 89–94 (1997)

Subject Classification (AMS)
47H09

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On the nearest point projection in Hilbert spaces with application to Nonlinear Ergodic Theory

ABSTRACT. Let P be the nearest point projection of a Hilbert space H onto a closed and convex subset K and $\Sigma\lambda_i x_i$ any convex combination of the points x_i in H. We give an estimate for $||P(\Sigma\lambda_i x_i) - \Sigma\lambda_i P x_i||$. A consequence of this estimate is the following: Let T be a nonexpansive selfmapping of a bounded, closed and convex subset C of H, let $F(T)$ denote the fixed point set of T which is nonempty, closed and convex, and for $x \in C$ let be $S_n x = (x + Tx + \ldots + T^{n-1}x)/n$. Then the sequence $\{\mathrm{Proj}_{F(T)} S_n x\}$ converges to a fixed point p, which is the weak limit of the sequence $\{S_n x\}$ too.

Keywords and phrases: Nearest point projection, nonexpansive mapping, inequalities, Nonlinear Ergodic Theorem, Hilbert space

Let K be a nonempty, closed and convex subset of a (real) Hilbert space H. Then it is well known that for any x in H the nearest point projection $p = \mathrm{Proj}_K x$ is well defined as the unique point of K which is nearest to x, i.e. $||x - p|| \le ||x - z||$ for all z in K (cf. [4]). Further it is known that the nearest point projection in Hilbert space is nonexpansive. In this note we shall apply the identity

$$||\lambda x + (1-\lambda)y||^2 = \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)||x-y||^2 \tag{1}$$

for all x, y in H and $0 \le \lambda \le 1$. For any n points $x_1, x_2, \ldots, x_n$ in H the following generalized identity holds

$$\Big|\Big|\sum_{i=1}^n \lambda_i x_i\Big|\Big|^2 = \sum_{i=1}^n \lambda_i ||x_i||^2 - \sum_{i<j} \lambda_i\lambda_j ||x_i - x_j||^2 \tag{2}$$

where $\lambda_i \ge 0$ and $\Sigma\lambda_i = 1$.

Lemma 1 *If P is the nearest point projection of H onto the closed convex subset K, then holds it for all z in K*

$$||x - Px||^2 + ||z - Px||^2 \le ||x - z||^2.$$

Proof: Let z be any point in K. Since K is convex the point $[\lambda z + (1-\lambda)Px]$ lies in K too. By (1), we have

$$||x - [\lambda z + (1-\lambda)Px]||^2 = \lambda||x-z||^2 + (1-\lambda)||x-Px||^2 - \lambda(1-\lambda)||z-Px||^2.$$

Because of $||x - Px|| \le ||x - [\lambda z + (1-\lambda)Px]||$ it follows

$$\lambda||x-Px||^2 \le \lambda||x-z||^2 - \lambda(1-\lambda)||z-Px||^2.$$

Dividing by λ and let tends $\lambda \to 0$ the lemma is proved.

Theorem 2 *If P is the nearest point projection of the Hilbert space H onto the closed convex subset K, then it holds for any convex combination of points $x_1, x_2, \dots, x_n$ in H*

$$\left|\left|\sum_{i=1}^n \lambda_i P x_i - P\left(\sum_{i=1}^n \lambda_i x_i\right)\right|\right|^2 \le \sum_{i<j} \lambda_i\lambda_j \langle Px_i - Px_j, \, (I-P)x_i - (I-P)x_j\rangle.$$

Proof: Let be $x = \Sigma\lambda_i x_i$. By lemma 1, we have

$$||x-Px||^2 + ||\sum \lambda_i P x_i - Px||^2 \le ||x - \sum \lambda_i P x_i||^2 = ||\sum \lambda_i (x_i - Px_i)||^2.$$

From (2) it follows

$$||\sum \lambda_i(x_i - Px_i)||^2 = \sum \lambda_i ||x_i - Px_i||^2 - \sum_{i<j} \lambda_i\lambda_j ||(x_i - Px_i) - (x_j - Px_j)||^2.$$

Again by lemma 1 it is

$$\sum \lambda_i ||x_i - Px_i||^2 \le \sum \lambda_i ||x_i - Px||^2 - \sum \lambda_i ||Px_i - Px||^2.$$

Because of the identity (2) we have

$$||x-Px||^2 = ||\sum \lambda_i (x_i - Px)||^2 = \sum \lambda_i ||x_i - Px||^2 - \sum_{i<j} \lambda_i\lambda_j ||x_i - x_j||^2$$

and

$$||\sum \lambda_i P x_i - Px||^2 = ||\sum \lambda_i (Px_i - Px)||^2 = \sum \lambda_i ||Px_i - Px||^2 - \sum_{i<j} \lambda_i\lambda_j ||Px_i - Px_j||^2.$$

Altogether, we obtain

$$2||\sum \lambda_i P x_i - Px||^2 \le \sum_{i<j} \lambda_i\lambda_j (||x_i - x_j||^2 - ||Px_i - Px_j||^2 - ||x_i - x_j - (Px_i - Px_j)||^2).$$

By definition of the inner product the theorem is proved.

Corollary 3 *If P is a nearest point projection in a Hilbert space onto a closed and convex subset K, then for any two points x and y in H it holds*

$$\langle Px - Py, (x-y) - (Px - Py)\rangle \geq 0.$$

From this one can conclude several properties of a nearest point projection P in a Hilbert space, e.g. P is nonexpansive (cf. [5], [6]). Since P is nonexpansive, it is by the inequality of Schwarz

$$\begin{aligned}\langle Px - Py, (x-y) - (Px - Py)\rangle &= \langle Px - Py, x - y\rangle - ||Px - Py||^2 \\ &\leq ||Px - Py||\,||x - y|| - ||Px - Py||^2 \\ &\leq ||x - y||^2 - ||Px - Py||^2.\end{aligned}$$

So from theorem 2 the inequality of Zarantonello

$$\left|\left|\sum_{i=1}^{n} \lambda_i P x_i - P\left(\sum_{i=1}^{n} \lambda_i x_i\right)\right|\right|^2 \leq \sum_{i<j} \lambda_i\lambda_j(||x_i - x_j||^2 - ||Px_i - Px_j||^2)$$

follows which is valid for each nonexpansive mapping in Hilbert space.

Corollary 4 *Let be P the nearest point projection in a Hilbert space onto a closed and convex subset K, then for any convex combination of points $x_1, x_2, \ldots, x_n$ in H and any p in K*

$$||P(\sum \lambda_i x_i) - \sum \lambda_i P x_i||^2 \leq d \sum_{i=1}^{n} \lambda_i ||Px_i - p||$$

where

$$d = \sup_{i,j}(||x_i - x_j|| - ||Px_i - Px_j||).$$

Proof: By theorem 2 we get

$$\begin{aligned}||P(\sum \lambda_i x_i) - \sum \lambda_i P x_i||^2 &\leq \sum_{i<j} \lambda_i\lambda_j(\langle Px_i - Px_j, x_i - x_j\rangle - ||Px_i - Px_j||^2) \\ &\leq \sum_{i<j} \lambda_i\lambda_j ||Px_i - Px_j||\,(||x_i - x_j|| - ||Px_i - Px_j||) \\ &\leq d \sum_{i<j} \lambda_i\lambda_j \|Px_i - Px_j\| \\ &\leq d \sum_{i<j} \lambda_i\lambda_j(||Px_i - p|| + ||Px_j - p||) \\ &\leq d \sum_{i<j} \lambda_i ||Px_i - p||.\end{aligned}$$

Corollary 5 *The nearest point projection P in a Hilbert space onto a closed and convex subset K is demiclosed,i.e. if $x_n \rightharpoonup x$ (weakly) and $Px_n \to p$ then $Px = p$.*

Proof: To given $\varepsilon > 0$ there exists an integer n_0 with $||Px_n - p|| \leq \varepsilon$ for $n \geq n_0$. Because of a theorem of Mazur there exists a finite convex combination of the points $x_n (n = n_0, \dots, m)$ so that

$$\left|\left| \sum_{i=n_0}^{m} \lambda_i x_i - x \right|\right| < \varepsilon.$$

Since P is nonexpansive, we obtain by corollary 4

$$\begin{aligned} ||Px - p|| &\leq \left|\left| Px - P\left(\sum_{i=n_0}^{m} \lambda_i x_i\right) \right|\right| + \left|\left| P\left(\sum_{i=n_0}^{m} \lambda_i x_i\right) - \sum_{i=n_0}^{m} \lambda_i P x_i \right|\right| + \left|\left| \sum_{i=n_0}^{m} \lambda_i P x_i - p \right|\right| \\ &\leq \left|\left| x - \sum_{i=n_0}^{m} \lambda_i x_i \right|\right| + \sqrt{d \sum_{i=n_0}^{m} \lambda_i ||Px_i - p||} + \sum_{i=n_0}^{m} \lambda_i ||Px_i - p|| \\ &\leq 2\varepsilon + \sqrt{d\varepsilon}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the corollary is proved.

Corollary 6 *Let P be the nearest point projection in a Hilbert space onto a closed and convex subset K, let (x_n) be a bounded sequence with $Px_n \to p$. For any regular matrix (a_{nk}), i.e.*

$$a_{n,k} \geq 0, \qquad \sum_{k=0}^{\infty} a_{n,k} = 1, \qquad \lim_{n\to\infty} a_{n,k} = 0 \tag{3}$$

let be

$$y_n = \sum_{k=0}^{\infty} a_{n,k} x_k$$

Then $Py_n \to p$.

Proof: By corollary 4, we have

$$\begin{aligned} ||Py_n - p|| &\leq ||P(\sum a_{n,k} x_k) - \sum a_{n,k} P x_k|| + ||\sum a_{n,k} P x_k - p|| \\ &\leq \sqrt{d \sum a_{n,k} ||Px_k - p||} + \sum a_{n,k} ||Px_k - p||. \end{aligned}$$

Now the conclusion follows by a limit theorem due to Toeplitz (cf. [6]).

Remark: Theorem 2 is not valid in an arbitrary Banach space X, as the following example shows: Let be $X = l_3^p, x_1 = (1,0,0), x_2 = (0,1,0)$ and $K = \{x = t(1,1,1) : |t| \leq 1\}$.

Let P be the nearest point projection of X onto K, then $Px_1 = Px_2 = y$ where $y = (2^{1/p-1}+1)^{-1}(1,1,1)$ but $P((x_1+x_2)/2) = 2^{-1+1/(p-1)}y = y$ if and only if $p = 2$.

Finally we give an application for the ergodic theory of nonexpansive mappings in Hilbert space. Let C be a nonempty bounded closed and convex subset of a Hilbert space H and $T : C \to C$ a nonexpansive mapping, i.e. $||Tx - Ty|| \leq ||x - y||$ for all x, y in C. In 1975 Baillon [1] has proved the first ergodic theorem for nonexpansive mappings, namely that for each x in C the sequence of CESARO-means

$$S_n x = \frac{x + Tx + \ldots + T^{n-1}x}{n}$$

converges weakly for $n \to \infty$ to a fixed point of T. Brezis and Browder [2] generalized Baillons result to the more general summation method

$$A_n x = \sum_{k=0}^{\infty} a_{n,k} T^k x,$$

where $(a_{n,k})$ is any strongly regular matrix, i.e. it holds (3) and additionally

$$s_n = \sum_{k=0}^{\infty} |a_{n,k+1} - a_{n,k}| \to 0 \qquad \text{for } n \to \infty.$$

Theorem 7 *Let C be a bounded, closed, convex subset of a Hilbert space H, let $T : C \to C$ be a nonexpansive mapping and for $x \in C$ let $A_n x$ be a summation method with any strongly regular matrix. Then the fixed point set $F(T)$ is nonempty, closed, convex and there exists a fixed point p in $F(T)$ such that holds:*

(i) $Proj_{F(T)} T^n x$ converges to p.

(ii) $Proj_{F(T)} A_n x$ converges to p.

(iii) $A_n x$ converges weakly to p.

Proof: 1. That the fixed point set $F(T)$ is nonempty, closed and convex is firstly proved by Browder [3].
2. Point (i) is proved by Baillon [1] and Pazy [7].
3. By point (i) and corollary 6 it follows (ii).
4. Point (iii) is the mean ergodic theorem in the form of Brezis and Browder [2].

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received: June 17, 1996

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