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Order completeness and stationary points

ABSTRACT. We give characterizations of order completeness by using stationary point theorems.

KEY WORDS. Order completeness, stationary point, ordered metric space.

1 Introduction

The notion of order complete metric spaces was introduced in [2]. It is worth emphasizing that the notion of order complete metric spaces is a non trivial extension of the notion of complete metric spaces. Conserva and Rizzo [1] characterized a class of order complete metric spaces as those ones in which every map of a suitable family has at least one fixed point. Turinici [2] obtained a necessary condition of order completeness.

In this paper we prove that the converse of Turinici′s result is true and obtain several criteria of order completeness.

Throughout this paper, let (X, d) be a metric space, $\leq$ an order (that is, a reflexive, antisymmetric and transitive relation) on X. N denotes the set of all positive integers. A sequence $\{x_n\}_{n\in N}$ in X is said to be $\leq$-monotone if and only if $x_n \leq x_m$ for all n, m in N with $n \leq m$. (X, d) is said to $\leq$-complete if and only if every $\leq$-monotone Cauchy sequence in$X$ is convergent. For $A \subset X$, let $\delta(A)$ and $\overline{A}$ denote the diameter and closure of A, respectively; $\alpha(A) = \inf\{\varepsilon > 0$: there exists a finite convering of A with sets having a diameter less than $\varepsilon\}$ if $\delta(A) < \infty$ and $\alpha(A) = \infty$ if $\delta(A) = \infty$. $CL(X)$ denotes the family of all nonempty closed subsets of X; $BC(X)$ denotes the family of all nonempty bounded closed subsets of X; 2^X denotes the family of all nonempty subsets of X. Define an order $>$ on 2^X with the aid of the above order $\leq$ on X as follows: for A, B in 2^X, we write $A > B$ if and only if $A \supset B$ and, for every a in A there exits a point b in B with $a \leq b$.

2 Characterizations of order completeness

Our results are as follows:

Theorem 1 *For any metric space (X,d) and any order $\leq$ on X the following statements are equivalent:*

(1) (X,d) *is* $\leq$*-complete;*

(2) *If* $\{F_n\}_{n\in N}$ *is a* $>$*-monotone sequence in* $BC(X)$ *satisfying* $\alpha(F_n)\to 0$ *as* $n\to\infty$, *then* $\bigcap_{n\in N} F_n \neq \Phi$;

(3) *If* $\{F_n\}_{n\in N}$ *is a* $>$*-monotone sequence in* $BC(X)$ *satisfying* $\delta(F_n)\to 0$ *as* $n\to\infty$, *then* $\bigcap_{n\in N} F_n \neq \Phi$.

Proof: (1) $\Rightarrow$ (2) is given in [2].

(2) $\Rightarrow$ (3) Note that $\alpha(F_n)\leq\delta(F_n)$. Then (3) follows immediately from (2).

(3) $\Rightarrow$ (1) Assume that $\{x_n\}_{n\in N}$ is a $\leq$-monotone Cauchy sequence in X. Define

$A_n=\overline{\{x_k : k\geq n\}}$ for all n in N. Then $\{A_n\}_{n\in N}$ is a $>$-monotone sequence and A_n is in $BC(X)$ for each n in N. It is evident that $\delta(A_n)=\delta(\{x_k : k\geq n\})\to 0$ as $n\to\infty$. It follows from (3) that $\bigcap_{n\in N} A_n\neq\Phi$. Let w be in $\bigcap_{n\in N} A_n$. Then $d(x_n,w)\leq\delta(A_n)\to 0$ as $n\to\infty$; i.e., (X,d) is $\leq$-complete.

This completes the proof.

Theorem 2 *For any metric space (X,d) and any order $\leq$ on X, (1) is equivalent to each of the following:*

(4) *If* $\{x_n\}_{n\in N}$ *is a* $\leq$*-monotone sequence in* X *satisfying* $\alpha\left(\overline{\{x_k : k\geq n\}}\right)\to 0$ *as* $n\to\infty$, *then* $\bigcap_{n\in N}\overline{\{x_k : k\geq n\}}\neq\Phi$;

(5) *If* $\{F_n\}_{n\in N}$ *is a sequence in* $CL(X)$ *such that* $\alpha(F_n)\to 0$ *as* $n\to\infty$, *and that* $F_n\supset F_{n+1}$ *for each* n *in* N *and that there exists a* $\leq$*-monotone sequence* $\{x_n\}_{n\in N}$ *in* X *with* $x_n\in F_n$ *for all* n *in* N, *then* $\bigcap_{n\in N} F_n\neq\Phi$;

(6) *If* $\{F_n\}_{n\in N}$ *is a sequence in* $CL(X)$ *such that* $\delta(F_n)\to 0$ *as* $n\to\infty$, *and that* $F_n\supset F_{n+1}$ *for each* n *in* N *and that there exists a* $\leq$*-monotone sequence* $\{x_n\}_{n\in N}$ *in* X *with* $x_n\in F_n$ *for all* n *in* N, *then* $\bigcap_{n\in N} F_n\neq\Phi$.

(7) *If* f *is a map of* X *into* $CL(X)$ *satisfying*

(i) $fy\subset fx$ *for all* x *in* X *and all* y *in* fx;

(ii) *there exists a $\leq$-monotone sequence $\{x_n\}_{n\in N}$ in X such that x_{n+1} is in fx_n for each n in N and that $\delta(fx_n) \to 0$ as $n \to \infty$,*

then f has a stationary point w in X; i.e., $fw = \{w\}$;

(8) *If f is a map of X into $BC(X)$ satisfying* (i) *and* (ii), *then f has a stationary point.*

Proof: (1)$\Rightarrow$(4) From the proof of Theorem 4.1 in [2] it is easy to see that (1) implies (4).

(4)$\Rightarrow$(5) Since $\overline{\{x_k : k \geq n\}} \subset F_n$ for each n in N, it follows that $\alpha\left(\overline{\{x_k : k \geq n\}}\right) \leq \alpha(F_n) \to 0$ as $n \to \infty$.

By (4) we have $\Phi \neq \bigcap_{n\in N} \overline{\{x_k : k \geq n\}} \subset \bigcap_{n\in N} F_n$.

(5)$\Rightarrow$(6) Since $\alpha(F_n) \leq \delta(F_n)$, it follows that (5) implies (6).

(6)$\Rightarrow$(7) Take $F_n = fx_n$ for all n in N. By (6) we conclude that $A = \bigcap_{n\in N} F_n \neq \Phi$. Note that $\delta(A) \leq \delta(F_n) \to 0$ as $n \to \infty$. Consequently $\delta(A) = 0$; i.e., $A = \{w\}$ for some w in X. (i) ensures that $fw \subset fx_n$ for all n in N. Hence $fw \subset \bigcap_{n\in N} fx_n = A = \{w\}$. It is clear that $fw = \{w\}$.

(7)$\Rightarrow$(8) Note that $BC(X) \subset CL(X)$. Hence (8) follows from (7).

(8)$\Rightarrow$(1) Suppose that (X, d) is not $\leq$-complete. Then there exists a $\leq$-monotone Cauchy sequence $\{x_n\}_{n\in N}$ in X such that $\{x_n\}_{n\in N}$ is not convergent. Without loss of generality we may assume that $x_n \neq x_m$ for all distinct n, m in N. Take $A_0 = X$ and $A_n = \{x_k : k \geq n\}$ for each n in N. It is easy to verify that A_n is in $BC(X)$ for all n in N. Define a function $i : X \to N \bigcup \{0\}$ by $i(x) = n$ if $x \in A_n$ and $x \overline{\in} A_{n+1}$. Now we construct a map f of X into $BC(X)$ by $fx = A_{i(x)+1}$ for each x in X. It is clear that f has no stationary point. For any x, y in X with $y \in fx = A_{i(x)+1}$ we have $i(y) \geq i(x) + 1$ by the definition of i. This means that $fy = A_{i(y)+1} \subset A_{i(x)+1} = fx$; i.e., f satisfies (i). Note that $x_n \in A_n$ and $x_n \overline{\in} A_{n+1}$ for every n in N. It follows that $i(x_n) = n$ for all n in N. Consequently $x_{n+1} \in A_{n+1} = A_{i(x_n)+1} = fx_n$ for each n in N and $\delta(fx_n) = \delta(A_{n+1}) \to 0$ as $n \to \infty$; i.e., f satisfies (ii). By (8) f has a stationary point. This is a contradiction.

This completes the proof.

References

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