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Basic Properties of Fejer monotone Mappings

Summary. *We consider certain classes of Fejer monotone mappings and their basic properties. These properties are the starting point of a convergence theory for corresponding iterative methods which are widely used to solve convex problems.*

1 Introduction

The present paper continues the paper [6] about basic properties of Fejer monotone sequences. In the applications such a sequence is often generated by a single Fejer monotone mapping g or by a sequence (g_k) of such mappings which can naturally be derived from the considered problem with the solution set M. More precisely we investigate set-valued mappings $g : Q \to \mathbb{P}(Q)$ with domain Q in a Hilbert space H and range in the power set of Q which reduce the distance between an element of Q and each element of the problem subset M. Then iterative methods

$$x_{k+1} \in g_k(x_k) \,, \quad x_0 \in Q$$

arise converging under slight additional assumptions to any element x^* in M (see e.g. [5] and the references cited there). These methods represent Fejer monotone sequences. In [5] a theory of strong convergent Fejer methods was developed using some basic properties of Fejer monotone mappings and sequences given here, in the preceding paper [6] and in the succeeding paper [7].

2 Fejer monotone mappings

Let H be a (real) Hilbert space. We consider a nonempty, convex and closed subset Q of H (set of feasible elements) and set-valued mappings (multioperators) $g : Q \to \mathbb{P}(Q)$, where $\mathbb{P}(Q)$ contains all nonempty subsets of Q. For g we introduce fixed point sets in the weak and in the strong sense, namely

$$F_{-}(g) = \{x \in Q : x \in g(x)\} \ , \ F_{+}(g) = \{x \in Q : \{x\} = g(x)\} \ ,$$

where $F_{+}(g) \subseteq F_{-}(g)$. As usual mappings (operators) $g : Q \to Q$ are integrated as imbeddings. Here both kinds of fixed point sets coincide with $F(g) = \{x \in Q : x = g(x)\}$. Further we introduce some relations and operations between set-valued mappings. Let $g \, , \, g_1 \, , \, g_2 \, : \, Q \to \mathbb{P}(Q)$. Then we define for all $y \in Q$:

$$(co\, g)(y) := co\; g(y) \quad , \quad g_2 \subseteq g_1 \ : \ g_2(y) \subseteq g_1(y) \ ,$$

$$(g_1 \cap g_2)(y) := g_1(y) \cap g_2(y) \quad , \quad (g_1 \cup g_2)(y) := g_1(y) \cup g_2(y) \ ,$$

$$(g_1 \circ g_2)(y) := \{x \in Q : x \in g_1(z) \, , \, z \in g_2(y)\} \ .$$

Now the central concepts are explained.

Definition 2.1 *Let M be a nonempty subset of Q. Then g is said to be M-*Fejer monotone *($g \in \mathbb{F}(M)$) iff*

$$\|z - x\| \le \|y - x\| \quad \forall x \in M \ , \ \forall y \in Q \ , \ \forall z \in g(y) \, .$$

It is said to be regularly M-Fejer monotone *($g \in \mathbb{F}_r(M)$) iff additionally*

$$y \notin g(y) \quad \forall y \in Q \setminus M \quad (or \quad F_{-}(g) \subseteq M) \ .$$

It is said to be strictly M-Fejer monotone *($g \in \mathbb{F}_{<}(M)$) iff additionally*

$$\|z - x\| < \|y - x\| \quad \forall x \in M \ , \ \forall y \in Q \setminus M \ , \ \forall z \in g(y) \, .$$

It is called (regularly, strictly) Fejer monotone *($g \in \mathbb{F} \, , \, g \in \mathbb{F}_r \, , \, g \in \mathbb{F}_{<}$) iff it is (regularly, strictly) M-Fejer monotone for any M.*

The set

$$C(g) = \{x \in Q \ : \ \|z - x\| \le \|y - x\| \quad \forall y \in Q \ , \ \forall z \in g(y)\}$$

is called the Fejer carrier *of g.*

Remarks 2.2 The above definitions are slight modifications of concepts given in [2] and [1] for special situations. In [2] a strictly Fejer monotone mapping g on $\mathbb{R}^n$ is said to be Fejer monotone. In [1] a strictly Fejer monotone continuous operator g on $\mathbb{R}^n$ is called *paracontractive*. But in contrast to our definition $C(g)$ is allowed to be empty.

Remarks 2.3

1. Obviously strictly Fejer monotone mappings g are regularly Fejer monotone and regularly Fejer monotone mappings are Fejer monotone (relative to the same set M), that is

$$\mathbb{F}_<(M) \subseteq \mathbb{F}_r(M) \subseteq \mathbb{F}(M) \quad \forall M \quad , \quad \mathbb{F}_< \subseteq \mathbb{F}_r \subseteq \mathbb{F} \; .$$

2. It is evident that

$$g \in \mathbb{F}_r(M) \iff g \in \mathbb{F}(M) \; , \; z \neq y \; : \; \forall y \in Q \setminus M \; , \; \forall z \in g(y) \; .$$

In the following $\mathbb{F}_\star$ stands either for $\mathbb{F}_r$ or for $\mathbb{F}_<$. If $\mathbb{F}_{(\star)}$ occurs, then the statement holds choicewise with and without $\star$.

Lemma 2.4 *Let (g_k) be a sequence of (regularly, strictly) Fejer monotone mappings all acting on Q . Then the iterative method*

$$x_0 \in Q \quad , \quad x_{k+1} \in g_k(x_k) \quad , \quad k \in \mathbb{N}$$

generates a (regularly, strictly) Fejer monotone sequence in the sense given in [6], respectively.

Proof: The definitions of Fejer monotone sequences in [6] and of Fejer monotone mappings are analogous. The latter are transformed in the first if y and z are replaced by x_k and x_{k+1}, respectively. ■

Theorem 2.5 *The following properties hold:*

a) $g \in \mathbb{F} \iff C(g) \neq \emptyset \iff g \in \mathbb{F}(C(g))$,

b) $g \in \mathbb{F}(M) \implies g \in \mathbb{F}(C(g))$,

c) $g \in \mathbb{F}(M) \, , \, \emptyset \neq N \subseteq M \implies g \in \mathbb{F}(N)$,

d) $g \in \mathbb{F}(M) \quad \Longrightarrow \quad M \subseteq C(g) \subseteq F_+(g)$,

e) $g \in \mathbb{F}_r(M) \quad \Longrightarrow \quad M = C(g) = F_+(g) = F_-(g)$.

Proof: Let be $g \in \mathbb{F}$. Then g is M-Fejer monotone for a certain nonempty set M. Hence $C(g)$ is nonempty, too. Further $C(g)$ supplies the maximal set for which Fejer monotony can be reached. This means $M \subseteq C(g)$ and $g \in \mathbb{F}(C(g))$. But this statement results again in $g \in \mathbb{F}$. Consequently, the assertions a) and b) hold. Assertion c) is obvious by definition. Let be $g \in \mathbb{F}(M)$ such that the estimate $\|z - x\| \leq \|y - x\|$ is fulfilled for $x \in C(g)$, $y \in Q$ and $z \in g(y)$. Then we can replace there y by x which leads to $z = x$ for all $z \in g(x)$, that is $x \in F_+(g)$. Therefore assertion d) is true. Finally we assume $g \in \mathbb{F}_r(M)$ and $x \in F_-(g)$. The second assumption implies $x \in g(x)$ and the first $x \notin g(x)$ for all $x \in Q \backslash M$. So we conclude $x \in M$. Hence, we get the relation chain $F_+(g) \subseteq F_-(g) \subseteq M$, which supplies in connection with d) also assertion e). ■

Now we introduce critical points of Fejer monotone mappings which play an outstanding part in addition to the fixed points. Let $B(x, r)$ denote the closed ball with midpoint x and radius r.

Definition 2.6 *The mapping* $G_M : Q \to \mathbb{P}(Q)$ *with*

$$G_M(y) = \bigcap_{x \in M} B(x, \|y - x\|)$$

is said to be the Fejer zone *of the nonempty set* M*. The derived mapping* ∂G_M *, where* $\partial G_M(y)$ *is the boundary of* $G_M(y)$*, is called the* Fejer boundary *of* M*. Finally*

$$P_g(y) = (g(y) \cap \partial G_{C(g)}(y)) \backslash \{y\}$$

defines a mapping P_g *induced by* $g \in \mathbb{F}$ *whose domain* $D(P_g)$ *contains the so-called* critical points *of* g *and whose range* $R(P_g)$ *contains the so-called* exposed points *of* g*.*

Remarks 2.7 1. Because of $\mathbb{F}_<(M) \subseteq \mathbb{F}_r(M)$ and Theorem 2.5 e) the reference set M is uniquely determined for regularly and all the more for strictly Fejer monotone mappings g. Hence it is not necessary to state M in these cases. Besides we have

$$g \in \mathbb{F}(M) \, , \; g \notin \mathbb{F}_\star(M) \; \Rightarrow \; \nexists N \subset M \; : \; g \in \mathbb{F}_\star(N) \, .$$

2. Without doubt $G_M \in \mathbb{F}(M)$ is satisfied. Moreover, any mapping $g \in \mathbb{F}(M)$ is characterized by the relation $g \subseteq G_M$. In so far the Fejer zone G_M is the maximal

M-Fejer monotone mapping. Additionally we obtain in view of this fact, Definition 2.6 and Theorem 2.5 b)

$$g \subseteq G_{C(g)} \subseteq G_M \ .$$

Obviously the images $G_M(y)$ represent nonempty, bounded and closed convex sets for all y since they arise from ball intersections and contain y. The Fejer boundary ∂G_M has the images

$$\partial G_M(y) = \{z \in Q : \exists x \in M \ : \ \|z - x\| = \|y - x\|\} \ \cap \ G_M(y)$$

and supplies beside its fixed points the exposed points of the Fejer zone G_M. So the critical points of g are the arguments which possess image elements on the Fejer boundary of its Fejer carrier $C(g)$ being no fixed points. These image elements are just the exposed points of g.

3. Theorem 2.5 d) shows that Fejer monotone mappings g have fixed points (in the strong sense). If g is regularly Fejer monotone, all the fixed points ly in $M = C(g)$ by Theorem 2.5 e). But g can possess critical points (outside of M). If g is strictly Fejer monotone, then there are no critical points.

Theorem 2.8 *The Fejer carrier $C(g)$ of g is convex and closed.*

Proof: The verification runs along the same lines as in [6, Theorem 2.6], where the Fejer carrier $C(x_k)$ of a Fejer monotone sequence (x_k) is proven to be convex and closed. The starting point is the reformulation of $C(g)$ as

$$C(g) = \{x \in Q : (y - z, y + z - 2x) \geq 0 \quad \forall y \in Q \ , \ \forall z \in g(y)\} \ . \blacksquare$$

If we define the sets

$$Q(y,z) = \{x \in Q \ : \ \|z - x\| \leq \|y - x\|\}$$

which correspond to halfspaces with normals $y - z$ and elements $\frac{1}{2}(y + z)$ on the boundary, then the Fejer carrier of g can obviously be described in the form

$$C(g) = \bigcap_{y \in Q, z \in g(y)} Q(y,z) \ .$$

The geometrical characterization of $Q(y,z)$ follows immediately from the scalar product representation of the squared defining inequality (compare also with the above reformulation of $C(g)$).

Theorem 2.9 *The following properties hold:*

a) $g \in \mathbb{F}(M) \quad \Longrightarrow \quad \overline{co}\, g \in \mathbb{F}(M)$,

b) $g \in \mathbb{F}_\star(M) \quad \Longrightarrow \quad co\, g \in \mathbb{F}_\star(M)$,

c) $g_1 \in \mathbb{F}_{(\star)}(M)\,,\, g_2 \subseteq g_1 \quad \Longrightarrow \quad g_2 \in \mathbb{F}_{(\star)}(M)$,

d) $g_1 \in \mathbb{F}_{(\star)}(M)\,,\, g_2 \in \mathbb{F}_{(\star)}(M) \quad \Longrightarrow \quad g_1 \cup g_2 \in \mathbb{F}_{(\star)}(M)$.

e) $g_1 \in \mathbb{F}(M)\,,\, g_2 \in \mathbb{F}_{(<)}(M) \quad \Longrightarrow \quad g_1 \circ g_2 \in \mathbb{F}_{(<)}(M)$.

Proof: a) Let be $x \in M$, $y \in Q$, $z \in g(y)$, $z' \in g(y)$ and $\lambda \in (0,1)$. Then $g \in \mathbb{F}(M)$ implies $\|z-x\| \leq \|y-x\|$ and $\|z'-x\| \leq \|y-x\|$. Consequently for $z_\lambda = \lambda\, z + (1-\lambda)\, z'$ the estimate

$$\begin{aligned} \|z_\lambda - x\| &= \|\lambda\,(z-x) + (1-\lambda)\,(z'-x)\| \\ &\leq \lambda\,\|z-x\| + (1-\lambda)\,\|z'-x\| \\ &\leq \lambda\,\|y-x\| + (1-\lambda)\,\|y-x\| \;=\; \|y-x\| \end{aligned}$$

holds. Hence $co\, g \in \mathbb{F}(M)$. Now we consider any $z \in (\overline{co}\, g)(y)$. Then there is a sequence (z_n) with $z_n \in (co\, g)(y)$ for all n and $\lim_{n\to\infty} z_n = z$. Because of $co\, g \in \mathbb{F}(M)$ we have $\|z_n - x\| \leq \|y-x\|$ for all n . So the continuity of the norm supplies

$$\|z-x\| = \lim_{n\to\infty} \|z_n - x\| \leq \|y-x\| \,,$$

that is $\overline{co}\, g \in \mathbb{F}(M)$.

b) Let be $x \in M$, $y \in Q\backslash M$, $z \in g(y)$, $z' \in g(y)$ and $\lambda \in (0,1)$. Then $g \in \mathbb{F}_<(M)$ implies $\|z-x\| < \|y-x\|$ and $\|z'-x\| < \|y-x\|$. Thus we get

$$\begin{aligned} \|z_\lambda - x\| &\leq \lambda\,\|z-x\| + (1-\lambda)\,\|z'-x\| \\ &< \lambda\,\|y-x\| + (1-\lambda)\,\|y-x\| = \|y-x\| \,. \end{aligned}$$

Hence also $co\, g \in \mathbb{F}_<(M)$.

Now we assume $g \in \mathbb{F}_r(M)$. Observing the chain of estimates for $\|z_\lambda - x\|$ in part a) of the proof the equation $\|z_\lambda - x\| = \|y-x\|$ has the consequence

$$\begin{aligned} \|\lambda\,(z-x) + (1-\lambda)\,(z'-x)\| &= \lambda\,\|z-x\| + (1-\lambda)\,\|z'-x\| \,, \\ &= \|y-x\| \,. \end{aligned}$$

The assumptions $\lambda \in (0,1)$, $\|z-x\| \leq \|y-x\|$ and $\|z'-x\| \leq \|y-x\|$ show that

$$\|z-x\| = \|z'-x\| = \|y-x\| \,.$$

Because of $y \in Q \backslash M$ and $x \in M$ we have $y - x \neq 0$ and by the second equation also $z - x \neq 0$ and $z' - x \neq 0$. Since the Hilbert space H is strictly convex, which means that

$$\|u + v\| = \|u\| + \|v\| \implies \exists\, c > 0 \,:\, v = cu$$

holds for elements $u \neq 0$, $v \neq 0$ of H, the first equation gives $z' - x = d\,(z - x)$ for a certain $d > 0$ and therefore $\|z' - x\| = d\,\|z - x\|$. Paying again attention to the second equation we obtain $d = 1$, $z' = z$ and finally $z_\lambda = z$. As $g \in \mathbb{F}_r(M)$ and $y \in Q \backslash M$ leads to $z \neq y$, we have for such y also $z_\lambda \neq y$. But this means $co\, g \in \mathbb{F}_r(M)$.
c) , d) Here the assertions follow immediately from the definition of the inclosed concepts.
e) First we suppose $g_1 \in \mathbb{F}(M)$ and $g_2 \in \mathbb{F}(M)$. Further let be $x \in M$ and $y \in Q$. Then for $u \in (g_1 \circ g_2)(y)$ there is an element $z \in g_2(y)$ with $u \in g_1(z)$ such that the estimates

$$\|u - x\| \leq \|z - x\| \leq \|y - x\|$$

are satisfied. So $g_1 \circ g_2 \in \mathbb{F}(M)$ follows. If g_2 lies even in $F_<(M)$, then for $y \in Q \backslash M$ arises $\|z - x\| < \|y - x\|$ and consequently $\|u - x\| < \|y - x\|$. Hence in this case $g_1 \circ g_2 \in \mathbb{F}_<(M)$. ■

Remarks 2.10 The results in Theorem 2.9 can be combined to get new results. For instance a) and d) supply

$$g_1 \in \mathbb{F}(M)\,,\; g_2 \in \mathbb{F}(M) \implies \overline{co}\,(g_1 \cup g_2) \in \mathbb{F}(M)\,.$$

Further c) leads to

$$g_1 \in \mathbb{F}_{(\star)}(M) \quad \text{or} \quad g_2 \in \mathbb{F}_{(\star)}(M) \implies g_1 \cap g_2 \in \mathbb{F}_{(\star)}(M)\,.$$

Besides e) contains the special case

$$g_1 \in \mathbb{F}_<(M)\,,\; g_2 \in \mathbb{F}_<(M) \implies g_1 \circ g_2 \in \mathbb{F}_<(M)\,.$$

By the way, e) induces the question, wether the strict case can be reached also if only the first term g_1 is strict. But $g_1 \circ g_2 \in \mathbb{F}_<(M)$ is satisfied for $g_1 \in \mathbb{F}_<(M)$ and $g_2 \in \mathbb{F}(M)$ only under additional assumptions. For instance, $g_2(y) \cap M = \emptyset$ implies $\|u - x\| < \|z - x\|$ and hence $\|u - x\| < \|y - x\|$ for $u \in (g_1 \circ g_2)(y)$ since $z \in g_2(y)$ is then in $Q \backslash M$. Besides the question arises what happens in the regular case. Here it does not suffice for $g_1 \circ g_2 \in \mathbb{F}_r(M)$ that both g_1 and g_2 are in $\mathbb{F}_r(M)$. Namely, $\|u - x\| = \|z - x\| = \|y - x\|$, $z \neq y$, $u \neq z$ can lead to $u = y$.

Corollary 2.11 *If* $g \in \mathbb{F}(M)$ *and* $\lambda \in (0,1)$ *, then*

$$g_\lambda = (1-\lambda)I + \lambda\, g \in \mathbb{F}(M) \;, \tag{2.1}$$

where I *represents the identity mapping.*

Proof: By assumption it is $g \in \mathbb{F}(M)$. Obviously we have also $I \in \mathbb{F}(M)$. Using Theorem 2.9 a) and d) we get $\overline{co}(I \cup g) \in \mathbb{F}(M)$ (see also Remark 2.10). Since g_λ belongs to the convex hull of I and g, Theorem 2.9 c) shows the assertion. ■

Remark 2.12 The above result has an important modification. If $g \in \mathbb{F}_r(M)$ holds, then even $g_\lambda \in \mathbb{F}_<(M)$ is true. More precisely, g_λ turns out to be in the subclass $\mathbb{F}^\alpha(M)$ of $\mathbb{F}_<(M)$ with $\alpha = \frac{1-\lambda}{\lambda}$ (for the definition see Example 3.3). This is proven in [7] by a more detailed investigation.

3 Examples

3.1 General examples of Fejer monotone mappings

Example: 3.1 We start with a well-known class of operators $g : Q \to Q$, namely the *nonexpansive* operators ($g \in \mathbb{L}$) defined by

$$\|g(y) - g(x)\| \le \|y - x\| \quad \forall y \in Q \;,\; \forall x \in Q \;. \tag{3.2}$$

Here we have $F(g) = F_+(g) = F_-(g)$. We call g *regularly nonexpansive* ($g \in \mathbb{L}_r$) if g has additionally fixed points ($F(g) \neq \emptyset$). In this case

$$\|g(y) - x\| \le \|y - x\|$$

arises for $x \in F(g)$. Hence such an operator g is Fejer monotone relative to $M = F(g)$. Observing Theorem 2.5 d) we get also $C(g) = F(g)$. So g is even regularly Fejer monotone ($\mathbb{L}_r \subseteq \mathbb{F}_r$). Under the following conditions a nonexpansive operator g is for instance regular:

a) Q is bounded (Browder, Göhde, Kirk, e.g. [8, p. 115]) and therefore weakly compact,

b) $Q = H$ and g is linear ($0 \in F(g)$),

c) g satisfies $\|g(y) - g(x)\| < \|y - x\|$ for all $x, y \in Q$ with $x \neq y$ and g is compact (e.g. [3, p. 46-48], [4, p. 510-512]),

d) g is contractive (Lipschitz-continuous with constant $L < 1$, Banach).

In the linear case b) nonexpansivity means

$$\|g(y)\| \leq \|y\| \quad \forall y \in H$$

because of

$$\|g(y) - g(x)\| = \|g(y - x)\| \leq \|y - x\| \ .$$

This corresponds to Fejer monotony relative to 0 and automatically also relative to $F(g)$, since

$$\|g(y) - x\| = \|g(y) - g(x)\|$$

holds for all $x \in F(g)$. Hence for linear operators all classes $\mathbb{L}$, $\mathbb{L}_r$, $\mathbb{F}$ and $\mathbb{F}_r$ coincide.

Example: 3.2 A subclass of the regularly nonexpansive operators form the *strictly nonexpansive operators* g ($g \in \mathbb{L}_<$) given by $F(g) \neq \emptyset$ and

$$\|g(y) - g(x)\| < \|y - x\| \quad \forall x\,,\, y \in Q \ : \ g(y) - g(x) \neq y - x\ . \tag{3.3}$$

This definition differs from that in [1, p. 306] essentially by the additional demand that $g \in \mathbb{L}_<$ has to possess fixed points. For $x \in F(g)$ and $y \in Q\backslash F(g)$ the condition $g(y) - g(x) \neq y - x$ is fulfilled. So we get

$$\begin{aligned}\|g(y) - x\| &\leq \|y - x\| \quad \forall x \in F(g)\ ,\ \forall y \in Q\ ,\\ \|g(y) - x\| &< \|y - x\| \quad \forall x \in F(g)\ ,\ \forall y \in Q\backslash F(g)\ .\end{aligned}$$

Hence g is strictly Fejer monotone ($\mathbb{L}_< \subseteq \mathbb{F}_<$). The mapping g satisfying (3.3) has for instance fixed points and is therefore strictly nonexpansive under the conditions a) and b) stated in Example 3.1. The assumptions about g cited there in c) and d) lead directly to strictly nonexpansive operators since (3.3) is then fulfilled automatically.

One simple and important example for a strictly Fejer monotone operator is given in [1]. Let M be a nonempty, convex and closed subset of H and let P denote the metric projector onto M. Then the relaxed projector

$$P_\lambda = (1 - \lambda)I + \lambda P \quad , \quad \lambda \in (0, 2) \tag{3.4}$$

is strictly nonexpansive ([1, p. 307]). In [7] this result will be sharpened (see also the remark in Example 3.3 below). Observe that the parameter λ can vary here in a larger interval than in Corollary 2.11 for the relaxation (2.1).

If g is strictly nonexpansive and linear (condition b) above), then the defining relations reduce to

$$\|g(y)\| < \|y\| \quad \forall y \notin F(g)\ .$$

Hence for linear operators strict nonexpansivity means strict Fejer monotony ($\mathbb{L}_< = \mathbb{F}_<$ in the linear case).

Example: 3.3 Let M be a nonempty subset of Q and α a positive number. Then g is said to be *α-strongly M-Fejer monotone* $(g \in \mathbb{F}^\alpha(M))$ iff it is

$$\|y-x\|^2 - \|z-x\|^2 \geq \alpha\|y-z\|^2 \quad \forall x \in M \ , \ \forall y \in Q \ , \ \forall z \in g(y) \tag{3.5}$$

and

$$y \notin g(y) \quad \forall y \in Q\backslash M \ . \tag{3.6}$$

It is called *strongly Fejer monotone* ($g \in \mathbb{F}_s$) iff it is α-strongly M-Fejer monotone for any M and any α.
The condition (3.5) in the above definition ensures that the mapping g is M-Fejer monotone. The condition (3.6) supplies the regularity of g. But g is also strictly M-Fejer monotone because of $z \neq y$ for $y \in Q\backslash M$ and $z \in g(y)$. Namely, such y, z generate a positive right-hand side in the inequality (3.5). Hence we have again $M = C(g)$.
The *basic examples* of Fejer monotone mappings are strongly Fejer monotone. For instance the relaxed projector P_λ in (3.4) turns out to be α-strongly M-Fejer monotone with $\alpha = \frac{2-\lambda}{\lambda}$ (see [7]). Consequently P itself is 1-strongly M-Fejer monotone. A further important example is given by the relaxed operator

$$T_\lambda(y) \quad = \quad \begin{cases} y - \lambda \frac{b(y)-c}{\|b'(y)\|^2} b'(y) & \text{if} \quad b(y) > c \\ y & \text{if} \quad b(y) \leq c \end{cases} \quad , \quad \lambda \in (0,2) \quad ,$$

where b is a convex and continuously differentiable functional on Q. This operator is proven in [1, p. 308] to be strictly Fejer monotone. But we show in [7] that a generalization of T_λ is even α-strongly Fejer monotone with $\alpha = \frac{2-\lambda}{\lambda}$. By the way, in this case we get $C(T_\lambda) = \{x \in Q : b(x) \leq c\}$ for the Fejer carrier.
The subclass $\mathbb{F}_s$ is described in detail in the succeding paper [7].

3.2 Fejer monotone mappings on $\mathbb{R}$

We turn to Fejer monotone mappings with $H = \mathbb{R}$. Let $g : Q \to \mathbb{P}(Q)$ be such a mapping on real intervals $Q = [a,b]$. The reference set M is then a subinterval $[c,d]$. The norm inequality for Fejer monotony specializes to $|z-x| \leq |y-x|$ which is at first considered for $Q = \mathbb{R}$ and fixed $x \in \mathbb{R}$. Then the inequality describes the left and right located domain $\mathbb{G}_x$ between the straight lines with ascents $+1$ and -1 through the point (x,x) in the $y-z-$plane. If x is allowed to vary in the interval M, then the solution set of the inequality reduces to the graph $\mathbb{G}_M(\mathbb{R}) = \bigcap_{x \in M} \mathbb{G}_x$ of G_M (see Definition 2.6) consisting of the two quarter planes (apart from the vertices)

$$\mathbb{G}_1 = \{(y,z) : \; y < c \, , \; y \leq z \leq 2c - y\}$$

$$\mathbb{G}_2 = \{(y,z) : \; y > d \, , \; 2d - y \leq z \leq y\}$$

with the half-line boundaries B_1^+ , B_1^- and B_2^- , B_2^+ , respectively, and the connecting straight line segment

$$\mathbb{G}_3 = \{(y,z) : \; c \leq y \leq d \, , \; z = y\} \; .$$

Hence $g : \mathbb{R} \to \mathbb{P}(\mathbb{R})$ is M-Fejer monotone iff the graph of g is contained in $\mathbb{G}_M(\mathbb{R})$. Especially, the graph has to coincide with $\mathbb{G}_3$ over M. The mapping g is even regularly M-Fejer monotone iff additionally the graph does not meet $B_1^+ \cup B_2^+$, and strictly M-Fejer monotone iff the graph meets no boundary segment of $\mathbb{G}_1$ and $\mathbb{G}_2$. Finally, the graph of α-strongly M-Fejer monotone mappings is additionally restricted (see Example 3.3). Namely, the graph has to be located outside of $\mathbb{G}_3$ in angle domains of $\mathbb{G}_1$ and $\mathbb{G}_2$ arising if the legs with ascent -1 are replaced by legs with ascent $\frac{\alpha-1}{\alpha+1}$. This can easily be derived if you start with the defining inequality

$$(y-x)^2 - (z-x)^2 \geq \alpha \, (y-z)^2 \; ,$$

substitute $u = y - x$, $v = z - x$ and use the third binomial formula to simplify the expression for $y \neq z$ and $u \neq v$, respectively.
If $Q \subset \mathbb{R}$ holds, then $\mathbb{G}_M(\mathbb{R})$ is reduced to $\mathbb{G}_M(Q) := \{(y,z) \in \mathbb{G}_M(\mathbb{R}) \; : \; y \in Q\}$. Obviously the left and right lying subdomains $\mathbb{G}_1$, $\mathbb{G}_2$ are restricted then in the same way.

Now we present for illustration some simple examples which show the relationship between the mentioned classes of mappings. All examples apart from the last relate to the special case $g : \mathbb{R} \to \mathbb{R}$ and $M = C(g) = \{0\}$. Since M is fixed, we will omit M in the following notations. It is easy to modify the mappings in such a way that also the cases $Q \subset \mathbb{R}$ and $\{0\} \subset C(g)$ are reflected. According to Theorem 2.9 it is also no problem to produce examples with $g : \mathbb{R} \to \mathbb{P}(\mathbb{R})$. Starting with $g_1 \, , \, g_2 : \mathbb{R} \to \mathbb{R}$ and $g_1 \, , \, g_2 \in \mathbb{F}_{(\star)}$ the derived mapping $co \, (g_1 \cup g_2)$ is also in $\mathbb{F}_{(\star)}$.

Examples 3.4 1. First we consider the smooth and unbounded function

$$g(y) = y \, \sin \, y \; .$$

The estimate $|g(y)| \leq |y|$ shows $g \in \mathbb{F}$. But in view of

$$F(g) = \{y \in \mathbb{R} : \sin\, y = 1\} \cup \{0\}$$

we have $g \notin \mathbb{F}_r$. By the way, the set

$$D(P_g) = \{y \in \mathbb{R} : \sin\, y = -1\}$$

of critical points (see Definition 2.6) is infinite, too. Since g is not regularly Fejer monotone, g can not be nonexpansive. Naturally, this shows also the derivative $g'(y) = y\, \cos\, y + \sin\, y$, which increases absolutely over all limits if $|y|$ tends to infinity in such a way that the zeros of $\cos\, y$ are avoided.

2. By a slight modification we get the continuous, but not everywhere differentiable function

$$g(y) = -y\, |\sin\, y|\ .$$

Again we have $|g(y)| \leq |y|$ such that $g \in \mathbb{F}$. But because of $F(g) = \{0\}$ it is even $g \in \mathbb{F}_r$, while $g \notin \mathbb{F}_<$ in view of

$$D(P_g) = \{y \in \mathbb{R} : |\sin\, y| = 1\} \neq \emptyset\ .$$

Moreover we have $|g'(y)| = |y\, \cos\, y + \sin\, y|$ for all y with $\sin\, y \neq 0$. Consequently g' is unbounded and g is not nonexpansive.

3. The first function with a reducing scalar factor supplies

$$g(y) = \gamma\, y\, \sin\, y \quad , \quad 0 < |\gamma| < 1\ .$$

Here we get $|g(y)| \leq |\gamma|\, |y| < |y|$ for $y \neq 0$. Hence $g \in \mathbb{F}_<$. In this case $|y| - |g(y)|$ increases over all limits if $|y|$ tends to infinity. This follows from

$$|y| - |g(y)| \geq |y| - |\gamma\, y| \geq (1 - |\gamma|)|y|\ .$$

Nevertheless g is unbounded. It can be shown that g is strongly Fejer monotone ($g \in \mathbb{F}^\alpha$ with $\alpha = \frac{1-|\gamma|}{1+|\gamma|}$, see [7]). But g is again not nonexpansive (compare with the first example above).

Examples 3.5 1. First we define g for nonnegative arguments by

$$g(y) \quad = \quad \begin{cases} \frac{y}{2} & \text{if} \quad 0 \leq y \leq 2 \\ \frac{y}{2} + \frac{1}{2}\sqrt{y^2 - 4} & \text{if} \quad y > 2 \end{cases} \quad .$$

Then we extend g by $g(-y) = -g(y)$ to the negative domain. Here we obtain $|g(y)| < |\frac{y}{2} + \frac{1}{2}\sqrt{y^2}| = |y|$ for $y > 2$ and therefore $|g(y)| < |y|$ for all y such that $g \in \mathbb{F}_<$ holds. By the way, g is even 3-strongly Fejer monotone (see [7]). Further we have $\lim_{|y|\to\infty} (\, y - g(y)\,) = 0$. Because of

$$\lim_{y\to 2+0} \; g'(y) = \lim_{y\to 2+0} \; \frac{1}{2}(1 + \frac{y}{\sqrt{y^2-4}}) = +\infty$$

the function g is far away from being nonexpansive. We get similar examples by choosing $-g$ or by extending g as an even function to the negative domain. But observe that the modified functions remain strictly Fejer monotone while they lose the property of strong Fejer monotony.

2. The smooth and bounded function

$$g(y) = \sin y^2$$

given already in [1, p. 309] satisfies again $|g(y)| < |y|$ for $y \neq 0$. Thus $g \in \mathbb{F}_<$ holds. Moreover, we have even $g \in \mathbb{F}_s$ (see [7]). On the other hand the derivative $g'(y) = 2y \cos y^2$ is unbounded. Hence g is not nonexpansive.

3. The simple example $g(y) = -y$ shows that a regularly nonexpansive function has not to be strictly Fejer monotone. Finally it is easy to check that the continuous and piecewise linear function

$$g(y) \;=\; \begin{cases} n & \text{if} \quad y \in [2n, 2n+1) \\ y - n - 1 & \text{if} \quad y \in [2n+1, 2n+2) \end{cases} \quad , \quad n \in \mathbb{N} \quad ,$$

is strictly nonexpansive and consequently strictly Fejer monotone if we use the extension $g(-y) = -g(y)$ to the negative domain. Moreover, g is obviously 1-strongly Fejer monotone (see [7]). But observe that the opposite function $-g$ is not strictly nonexpansive, although it is regularly nonexpansive and α-strongly Fejer monotone with a certain $\alpha < 1$ (see also [7]).

Example 3.6 We study the function

$$g(y) \;=\; \begin{cases} y & \text{if} \quad -1 \le y \le 1 \\ -1 & \text{if} \quad y < -1 \\ 1 & \text{if} \quad y > 1 \end{cases} \quad .$$

On the one hand g is $\{0\}$-Fejer monotone. On the other hand we have $C(g) = F(g) = [-1,1]$. It is easy to check that g is even 1-strongly $[-1,1]$-Fejer monotone (see [7]). By Remark 2.7.2 we obtain $g \subseteq G_{[-1,1]} \subseteq G_{\{0\}}$. Moreover we can state

$$g(y) \subset G_{[-1,1]}(y) \quad \text{for} \quad |y| > 1 \quad , \quad G_{[-1,1]}(y) \subset G_{\{0\}}(y) \quad \text{for} \quad y \neq 0$$

in this case.

References

[1] **L. Elsner, I. Koltracht**, and **M. Neumann :** *Convergence of sequential and asynchronous nonlinear paracontractions.* Numer. Math., **62**, 305-319 (1992)

[2] **I.I. Eremin**, and **V.D. Mazurov :** *Nestacionarnye Processy Programmirovanija.* Nauka, Moskva 1979

[3] **H. Jeggle :** *Nichtlineare Funktionalanalysis.* Stuttgart 1979

[4] **L.W. Kantorowitsch**, and **G.P. Akilow :** *Funktionalanalysis in normierten Räumen.* Berlin 1964

[5] **D. Schott :** *Iterative solution of convex problems by Fejer monotone methods.* Numer. Funct. Anal. Optimiz. 1323-1357 (1995)

[6] **D. Schott :** *Basic properties of Fejer monotone sequences.* Rostock. Math. Kolloq. **49**, 57-74 (1995)

[7] **D. Schott :** *Basic properties of uniformly Fejer monotone sequences.* (submitted to J. Comput. Appl. Math.)

[8] **E. Zeidler :** *Vorlesungen über nichtlineare Funktionalanalysis I.* Leipzig 1976

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