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Free Commutative Convergence Groups

Dedicated to the professors of mathematics
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Apart from their theoretical interest, free topological objects are also a resource for counterexamples. So the free commutative group over a completely regular, non-normal topological space is an example of a non-normal topological group. In this paper we construct the free commutative convergence group $\mathcal{A}_c(X)$ over a Hausdorff convergence space X. We show that it is a complete, Hausdorff convergence group and that X can be embedded as a closed subspace into $\mathcal{A}_c(X)$. Also we study the question of whether $\mathcal{A}_c(X)$ is a Choquet space, a property which has also been studied by H. Poppe under the name L*-space. As applications we give an example of a compact topological subset K of a Hausdorff convergence group G such that $K - K$ is not topological (not even Choquet) and show that $\Gamma_c(\mathcal{A}(X))$ is in general not Pontryagin c-reflexive. This gives a negative answer to the question of whether all c-character groups of convergence groups are Pontryagin c-reflexive.

In the notation we follow [1], but all convergence spaces are assumed to be Hausdorff and all groups are assumed to be commutative.. Given a convergence space X we denote by $\mathcal{A}(X)$ the free (commutative) group over the underlying set of X and represent it usually as

$$\mathcal{A}(X) = \{\xi : X \longrightarrow \mathbb{Z} \mid \xi(x) \neq 0 \text{ for only finitely many } x\} .$$

If we further define

$$i_X : X \longrightarrow \mathcal{A}(X) \text{ by}$$
$$i_X(x)(p) = \begin{cases} 1 & \text{if } p = x \\ 0 & \text{if } p \neq x \end{cases} ,$$

then i_X is an injection and every mapping $f : X \longrightarrow G$ into a group G can be uniquely extended to a group homomorphism $\hat{f} : \mathcal{A}(X) \longrightarrow G$ such that $\hat{f} \circ i_X = f$. Usually, we will treat i_X as in inclusion therefore will identify an element $x \in X$ with $i_X(x)$. If, finally, A is a subset of X , then the mapping $\sigma_A : X \longrightarrow \mathbb{Z}$ which maps all elements of A to 1 and all other elements to 0, can be extended to a group homomorphism from $\mathcal{A}(X)$ to $\mathbb{Z}$, which we will also denote by all σ_A.

Define a convergence structure on $\mathcal{A}(X)$ by stating that a filter $\mathcal{H}$ on $\mathcal{A}(X)$ converges to an element $\xi \in \mathcal{A}(X)$ if there are filters $\mathcal{F}_1, \dots, \mathcal{F}_n$ which converge in X such that $\mathcal{H} \supseteq \xi + (i_X(\mathcal{F}_1) - i_X(\mathcal{F}_1)) + \dots + (i_X(\mathcal{F}_n) - i_X(\mathcal{F}_n))$.
It is easy to see that one gets in this way a group convergence structure on $\mathcal{A}(X)$ and resulting convergence group is denoted by $\mathcal{A}_c(X)$. Also it is clear that every continuous mapping $T : X \longrightarrow G$ into a convergence group can be uniquely lifted to a continuous group homomorphism $\hat{T} : \mathcal{A}_c(X) \longrightarrow G$ with $\hat{T} \circ i_X = T$, i.e. $\mathcal{A}_c(X)$ is indeed the free commuative convergence group over X. The first result we prove is:

Theorem 1 *For each convergence space X the convergence group $\mathcal{A}_c(X)$ is Hausdorff. The mapping i_X is an embedding onto a closed subspace of X.*

Proof: Since a convergence group is Hausdorff if it is a T_1-space it is sufficient to show that $\dot{\xi}$, the trivial ultrafilter gnerated by ξ, does not converge to 0 if ξ is an element in $\mathcal{A}_c(X) \setminus \{0\}$. So assume to the contrary that there is an element $\xi = \alpha_1 x_1 + \ldots + \alpha_r x_r$ with $\alpha_1 \neq 0$ and $x_i \neq x_1$ for $i \neq 1$, such that $\dot{\xi}$ converges to zero. Then there are convergent filters $\mathcal{F}_1, \dots, \mathcal{F}_n$ on X such that $\dot{\xi} \supseteq (i_X(\mathcal{F}_1) - i_X(\mathcal{F}_1)) + \dots + (i_X(\mathcal{F}_n) - i_X(\mathcal{F}_n))$. Since X is Hausdorff, one can choose sets $F_i \in \mathcal{F}_i$ such that $x_2, \dots, x_n$ do not belong to F_i if $\mathcal{F}_i$ converges to x_1 , $x_1 \notin F_i$ if $\mathcal{F}_i$ does not converge to x_1 and such that $F_i \cap F_j = \emptyset$ if $\mathcal{F}_i$ and $\mathcal{F}_j$ converge to different points. Set $F = \bigcup \{F_i : F_i \text{ converges to } x_1\} \cup \{x_1\}$, then $\sigma_F(\xi) = \alpha_1 \neq 0$ while $\sigma_F((F_1 - F_1) + \dots + (F_n - F_n)) = \{0\}$. Therefore $\xi \notin (F_1 - F_1) + \dots + (F_n - F_n)$ and so $\dot{\xi} \not\supseteq (i_X(\mathcal{F}_1) - i_X(\mathcal{F}_1)) + \dots + (i_X(\mathcal{F}_n) - i_X(\mathcal{F}_n))$. This contradiction shows that $\mathcal{A}_c(X)$ is indeed Hausdorff.
Clearly i_X is continuous, and in order to show that it is an embedding, take filter $\mathcal{F}$ on X and a point $x \in X$ such that $i_X(\mathcal{F})$ converges to $i_X(x)$. Then there are again convergent filters $\mathcal{F}_1, \dots, \mathcal{F}_n$ on X such that $i_X(\mathcal{F}) \supseteq x + (i_X(\mathcal{F}_1) - i_X(\mathcal{F}_1)) + \dots + (i_X(\mathcal{F}_n) - i_X(\mathcal{F}_n))$. We show that:

$$\mathcal{F} \supseteq \bigcap \{\mathcal{F}_i : \mathcal{F}_i \text{ converges to } x\} \cap \dot{x} \ .$$

Take a set G from the right side. For each $i \in \{1, \dots, n\}$ choose a set $F_i \in \mathcal{F}_i$ such that $F_i \cap F_j = \emptyset$ if $\mathcal{F}_i$ and $\mathcal{F}_j$ converge to different points and such that $F_i \subseteq G$ if $\mathcal{F}_i$ converges to x. By assumption, there is a set $F \in \mathcal{F}$ such that

$$F \subseteq x + (F_1 - F_1) + \dots + (F_n - F_n).$$

Set now again $B = \bigcup \{F_i : F_i \text{ converges to } x\} \cup \{x\}$, then $\sigma_B(x + (F_1 - F_1) + \dots + (F_n - F_n)) = \{1\}$ and so $\sigma_B(F) = \{1\}$, implying that $F \subseteq B \subseteq G$ and so $G \in \mathcal{F}$.

Finally, in order to show that X is a closed subset of $\mathcal{A}_c(X)$, choose a filter $\mathcal{F}$ on X such that $i_X(\mathcal{F})$ converges to a point $\xi = \alpha_1\, x_1 + \ldots + \alpha_r\, x_r \in \mathcal{A}_c(X)$. Then

$$i_X(\mathcal{F}) \supseteq \xi + (i_X(\mathcal{F}_1) - i_X(\mathcal{F}_1)) + \cdots + (i_X(\mathcal{F}_n) - i_X(\mathcal{F}_n))$$

for converging filters $\mathcal{F}_1, \ldots, \mathcal{F}_n$ on X. Assume that there is no filter which converges in X and is finer than $\mathcal{F}$, then there are sets $F_0 \in \mathcal{F}$ and $F_i \in \mathcal{F}$ such that $F_0 \cap F_i = \emptyset$ for $i = 1, \ldots, n$. By assumption, there is a set $F \in \mathcal{F}$ with $F \subseteq F_0$ such that

$$F \subseteq \xi + (F_1 - F_1) + \cdots + (F_n - F_n)\ ,$$

implying that $F \subseteq \{x_1, \ldots, x_n\}$. But then $\mathcal{F} \subseteq \dot{x}_i$ for one i , contradicting the assumption. Therefore there exists a filter $\mathcal{G} \supseteq \mathcal{F}$ which converges in X. Since $\mathcal{A}_c(X)$ is Hausdorff this gives $\xi \in X$.

Proposition 1 *$\mathcal{A}_c(X)$ is complete.*

Proof: Let $\mathcal{H}$ be a Cauchy filter on $\mathcal{A}_c(X)$. We first show that there are a natural number k and integers $\tau_1, \ldots, \tau_k$ in $\{-1, 1\}$ such that $\tau_1 X + \cdots + \tau_k X \in \mathcal{H}$ and then prove the Proposition by induction:
Since $\mathcal{H}$ is a Cauchy filter, $\mathcal{H} - \mathcal{H}$ converges to zero and therefore there are convergent filters $\mathcal{F}_1, \ldots, \mathcal{F}_n$ such that

$$\mathcal{H} - \mathcal{H} \supseteq (i_X(\mathcal{F}_1) - i_X(\mathcal{F}_1)) + \cdots + (i_X(\mathcal{F}_n) - i_X(\mathcal{F}_n))\ .$$

Then $Z := (X - X) + \cdots (X - X) \in \mathcal{H} - \mathcal{H}$ and so there is a set $H \in \mathcal{H}$ such that $H - H \subseteq Z$. Choose an element $\xi \in H$, then $H - \xi \subseteq Z$ and therefore $\xi + Z \in \mathcal{H}$.

If now $k = 1$ then $X \in \mathcal{H}$ and we are ready if we can prove that there is a convergent filter on X which is finer than the trace filter $\mathcal{H}|X$ of $\mathcal{H}$ on X. If $\mathcal{H} = \dot{x}$ for some $x \in X$ we are ready, so assume that this is not the case. Then we show, that $\mathcal{H}|X \supseteq \mathcal{F}_1 \cap \ldots \mathcal{F}_n$ holds: Take a set $F \in \mathcal{F}_1 \cap \ldots \mathcal{F}_n$, then there is a set $H \in \mathcal{H}$ such that

$$H - H \subseteq (F - F) + \cdots\ + (F - F)\ .$$

Since H contains more than one point we get $H \subseteq F$ and so $F \in \mathcal{H}$. If now $\mathcal{G} \supseteq \mathcal{H}|X$ is an ultrafilter, then $\mathcal{G} = \mathcal{F}_i$ for some i and so $\mathcal{G}$ converges.

Assume now that the theorem is true for $k - 1$ and that $\tau_1 X + \cdots + \tau_k X \in \mathcal{H}$ for some $\tau_1, \ldots . \tau_k$ in $\{0, 1\}$.
Case 1: There is an element $y \in X$ and an element $H \in \mathcal{H}$ such that $\sigma_y(\xi) \neq 0$ for all $\xi \in H$. Then $\mathcal{H} - y$ or $\mathcal{H} + y$ satisfies the induction hypothesis and we are ready.
Case 2: For all $y \in X$ and all $H \in \mathcal{H}$ there is a point $\xi \in H$ such that $\sigma_y(\xi) = 0$. We then

claim that there are $\tau_1, \ldots, \tau_k$ and $i_1, \ldots, i_k$ in $\{1, \ldots, n\}$ such that $\mathcal{H} \vee (\tau_1 \mathcal{F}_{i_1} + \cdots \tau_k \mathcal{F}_{i_k})$ exists: If this were wrong, one could find sets $H_0 \in \mathcal{H}$ and $F_i \in \mathcal{F}$ such that $H_0 \cap (\tau_1 F_{i_1} + \cdots \tau_k F_{i_k}) = \emptyset$ for all sequences $\tau_1, \ldots \tau_k$ and $i_1, \ldots, i_k$. Choose a set $H \in \mathcal{H}$, $H \subseteq H_0$, such that $H - H \subseteq (F_1 - F_1) + \cdots + (F_n - F_n)$ and an element $\xi = \alpha_1 x_1 + \ldots + \alpha_r x_r \in H$. We then show that $\{x_1, \ldots, x_r\} \subseteq F_1 \cup \ldots \cup F_n$: For each i there is by assumption an element $\eta_i \in H$ such that $\sigma_{x_i}(\eta_i) = 0$. Then $\xi - \eta_i \in (F_1 - F_1) + \cdots + (F_n - F_n)$ and therefore

$$\alpha_i = \sigma_{x_i}(\xi - \eta) \in \sigma_{x_i}((F_1 - F_1) + \cdots + (F_n - F_n))$$

which gives the desired result. Clearly now $\xi \in \tau_1 F_{i_1} + \cdots \tau_k F_{i_k}$ for appropriately chosen $\tau_1, \ldots . \tau_k$ and $i_1, \ldots, i_k$ and therefore $H \cap (\tau_1 F_{i_1} + \cdots + F_{i_k}) \neq \emptyset$. This contradiction finishes the proof of Theorem 1.

We now turn to the question of whether $\mathcal{A}_c(X)$ is a Choquet space. Recall that a convergence space X is called Choquet if a filter $\mathcal{F}$ on X converges to a point $x \in X$ if every finer ultrafilter converges to x. A filter $\mathcal{F}$ converges in the Choquet modification $\chi(X)$ to a point x if every finer ultrafilter converges in X to x. Now in dealing with the free convergence group we will use the following Lemma whose proof is routine:

Lemma 1 *A convergence space X is a Choquet space if and only if a filter $\mathcal{F}$ on X converges to a point $x \in X$ if every net η which is finer than $\mathcal{F}$ contains a subnet which converges to x. A filter $\mathcal{F}$ on X converges in $\chi(X)$ to a point x if every finer net contains a subnet which converges in X to x.*

Call a filter on a convergence space compact, if every finer ultrafilter converges. Then we are going to prove the following theorem:

Theorem 2 *Let X be regular convergence space. Then a filter $\mathcal{H}$ converges in $\chi(X - X)$ to zero if and only if there is a compact filter $\mathcal{K}$ on X such that*

$$\mathcal{H} \supseteq \bigcap \{i_X(\mathcal{F}) - i_X(\mathcal{F}) : \mathcal{F} \supseteq \mathcal{K} \text{ and } \mathcal{F} \text{ converges in } X\}.$$

The proof is a consequence of the following lemmata:

Lemma 2 *Let X be a regular Hausdorff convergence space. Then 0 is the only point of adherence in $X - X$ of the filter*

$$\mathcal{H} := \bigcap \{i_X(\mathcal{F}) - i_X(\mathcal{F}) : \mathcal{F} \text{ converges in } X\}\,.$$

Proof: Assume that there is a filter $\tilde{\mathcal{H}} \supseteq \mathcal{H}$ which converges to a point $x_0 - y_0 \neq 0$. Then there are convergent filters $\mathcal{F}_1, \ldots, \mathcal{F}_n$ such that

$$\tilde{\mathcal{H}} \supseteq x_0 - y_0 + (i_X(\mathcal{F}_1) - i_X(\mathcal{F}_1)) + \cdots + (i_X(\mathcal{F}_n) - i_X(\mathcal{F}_n))\,.$$

Choose a net $\eta - \theta$ in $X - X$ such that $< \eta - \theta >$, the filter generated by $\eta - \theta$, is finer than $\mathcal{H}$ and

$$< \eta - \theta > \supseteq x_0 - y_0 + (\mathcal{F}_1 - \mathcal{F}_1) + \cdots + (\mathcal{F}_n - \mathcal{F}_n) \ .$$

We claim that $\eta \vee \dot{x}_0$ exists or $\eta \vee \mathcal{F}_k$ exists for some k :
If this were false, one could find $\alpha_0, \ldots, \alpha_n$ and $F_1, \ldots, F_n$ such that $\eta(\alpha) \neq x_0$ for all $\alpha \succ \alpha_0$ and

$$\eta(\alpha) \notin F_k \text{ for all } \alpha \succ \alpha_k.$$

Then, if $\beta \succ \alpha_0, \ldots, \alpha_n$ then we have

$$\eta(\alpha) \notin \{x_0\} \cup F_1 \cup \ldots F_n \text{ for all } \alpha \succ \beta.$$

By assumption,

$$x_0 - y_0 + (F_1 - F_1) + \cdots + (F_n - F_n) \in < \eta - \theta >$$

and so there is a γ such that

$$\eta(\alpha) - \theta(\alpha) \in x_0 - y_0 + (F_1 - F_1) + \cdots + (F_n - F_n),$$

for all $\alpha \succ \gamma$, which gives

$$\eta(\alpha) \in \{x_0\} \cup F_1 \cup \ldots \cup F_n \text{ for all } \alpha \succ \gamma.$$

This contradiction gives the desired result and so there is a subnet η' of η which converges in X. By the same argument we get a convergent subnet θ'' of θ' such that θ'' converges in X . Now since $< \eta'' - \theta'' > \supseteq \mathcal{H}$ we get the convergence of η'' to x_0 and θ'' to y_0, respectively. Altogether, we get filters $\mathcal{G}_1$ and $\mathcal{G}_2$ which converge to different points x_1 and y_1 such that $\mathcal{G}_1 - \mathcal{G}_2 \supseteq \mathcal{H}$.
Since X is regular, both $a(\mathcal{G}_1)$ and $a(\mathcal{G}_2)$ converge and since X is Hausdorff, we get sets $F_1 \in \mathcal{G}_1$ and $F_2 \in \mathcal{G}_2$ such that $a(F_1) \cap a(F_2) = \emptyset$. Therefore, $\{X \setminus F_1, X \setminus F_2\}$ is a covering system of X. Putting $G_1 := X \setminus F_1$ and $G_2 := X \setminus F_2$ we get $(G_1 - G_1) \cup (G_2 - G_2) \in \mathcal{H}$ and so

$$((G_1 - G_1) \cup (G_2 - G_2)) \cap (F_1 - F_2) \neq \emptyset,$$

which is clearly impossible.

Lemma 3 *If $\mathcal{K}$ is a compact filter on a regular, Hausdorff convergence space X, then*

$$\mathcal{H} := \bigcap \{\mathcal{F} - \mathcal{F} : \mathcal{F} \supseteq \mathcal{K} \ , \ \mathcal{F} \text{ converges } \}$$

converges to 0 *in* $\chi(X - X)$.

Proof: Choose any net $\eta - \theta$ on X which is finer than $\mathcal{H}$. Then $\eta - \theta$ is finer than $\mathcal{K} - \mathcal{K}$ and so either $\eta - \theta$ contains a constant subnet with the value 0 or η and θ are finer than $\mathcal{K}$. But then we get subnets η' and θ' which converge. Since $< \eta' - \theta' > \supseteq \mathcal{H}$, Lemma 2 implies that $\eta' - \theta'$ converges to 0.

Lemma 4 *Let X be a convergence space and $\mathcal{H} \neq \dot{0}$ be a filter which converges in $\chi(X - X)$ to zero. Then*

$$\mathcal{K} := \{F \subseteq X : F - F \in \mathcal{H}\}$$

is a compact filter on X.

Proof: Choose F_1, F_2 in $\mathcal{K}$, then

$$(F_1 - F_1) \cap (F_2 - F_2) \neq \emptyset, \{0\}$$

and therefore there are $u_0 \neq v_0$ such that

$$u_0 - v_0 \in (F_1 - F_1) \cap (F_2 - F_2),$$

implying $u_0, v_0 \in F_1 \cap F_2$ and therefore $F_1 \cap F_2 \neq \emptyset$.
We now prove

$$(F_1 \cap F_2) - (F_1 \cap F_2) \supseteq (F_1 - F_1) \cap (F_2 - F_2)\,.$$

Take any $u - v \in (F_1 - F_1) \cap (F_2 - F_2)$. If $u - v = 0$ we are ready since $F_1 \cap F_2 \neq \emptyset$. If $u - v \neq 0$ we get $u, v \in F_1, F_2$ and so $u, v \in F_1 \cap F_2$, giving the desired result and therefore $\mathcal{K}$ is a filter.

In order to prove that $\mathcal{K}$ is compact, assume to the contrary that there is an ultrafilter $\mathcal{G} \supseteq \mathcal{K}$ which does not converge in X. Then there is a covering system $\mathcal{C}$ of X such that $C \notin \mathcal{G}$ for all $C \in \mathcal{C}$. The family of all finite unions of $\{C - C : C \in \mathcal{C}\}$ is a local covering system of $X - X$ at zero and therefore there are $C_1, \ldots, C_n$ in $\mathcal{C}$ such that

$$(C_1 - C_1) \cup \ldots \cup (C_n - C_n) \in \mathcal{H}.$$

But

$$(C_1 \cup \ldots \cup C_n) - (C_1 \cup \ldots \cup C_n) \supseteq (C_1 - C_1) \cup \ldots \cup (C_n - C_n)$$

and therefore $C_1 \cup \ldots \cup C_n \in \mathcal{F} \subseteq \mathcal{G}$, implying $C_i \in \mathcal{G}$ for some i since $\mathcal{G}$ is an ultrafilter. This contradiction proves the Lemma.

The last step in the proof of Theorem 2 is now provided by

Lemma 5 *Let $\mathcal{H} \neq \dot{0}$ be a filter which converges in $\chi(X - X)$ to zero and $\mathcal{K} := \{F \subseteq X : F - F \in \mathcal{H}\}$. Then*

$$\mathcal{H}_0 := \bigcap\{\mathcal{F} - \mathcal{F} : \mathcal{F} \textit{ is a convergent filter on } X \textit{ and } \mathcal{K} \subseteq \mathcal{F}\} \subseteq \mathcal{H}.$$

Proof: Assume this to be wrong, then there is a set $H_0 \in \mathcal{H}_0 \setminus \mathcal{H}$ and therefore $H \not\subseteq H_0$ for all $H \in \mathcal{H}$. This gives nets $\theta, \eta : \mathcal{H} \longrightarrow X$ such that $\theta(H) - \eta(H) \in H \setminus H_0$. Clearly $0 \in H_0$ and therefore $\theta(H) - \eta(H) \neq 0$ for all H. Since $< \theta - \eta > \supseteq \mathcal{H}$ there is a subnet $\theta' - \eta'$ which converges to zero. By Lemma 2 there are subnets θ'' and η'' of θ' and η', respectively, which converge to the same point. Therefore $< \theta'' > \cap < \eta'' >$ converges in X and we first claim that $< \theta'' > \cap < \eta'' > \supseteq \mathcal{K}$:
Choose any $F \in \mathcal{K}$, then $F - F \in \mathcal{H}$ and therefore eventually $\theta(H) - \eta(H) \in F - F$. Since $\theta(H) - \eta(H) \neq 0$ we get that eventually $\theta(H), \eta(H) \in F$. From this we get $< \theta >, < \eta > \supseteq \mathcal{K}$, giving the desired result.
Finally, $H_0 \in < \theta'' > \cap < \eta'' > - < \theta'' > \cap < \eta'' >$, implying that eventually $\theta''(H) - \eta''(H) \in H_0$. This contradiction proves the Lemma.

Theorem 2 now enables us to proceed to the announced counterexamles. We first need the following:

Lemma 6 *Let X be a topological space and $K \subseteq X$ a compact subset. Then $\mathcal{U}(K) := \bigcap\{\mathcal{U}(x) : x \in K\}$ is a compact filter.*

Proof: Assume that $\mathcal{U}(K)$ is not compact, then there exists an ultrafilter $\mathcal{G} \supseteq \mathcal{K}$ which does not converge. Therefore there is an open covering $\mathcal{C}$ of X such that $C \notin \mathcal{G}$ for all $C \in \mathcal{C}$. Since K is compact, there are $C_1, \dots, C_n$ in $\mathcal{C}$ such that $K \subseteq C_1 \cup \dots \cup C_n$. But then $C_1 \cup \dots \cup C_n \in \mathcal{U}(K) \subseteq \mathcal{G}$ and so $C_1 \cup \dots \cup C_n \in \mathcal{G}$. Since $\mathcal{G}$ is an ultrafilter we get $C_i \in \mathcal{G}$ for some i and thereby a contradiction.

Example: *Let X be a compact topological space with infinitely many non-discrete points. Then $\mathcal{A}_c(X)$ is not a Choquet space. In particular, $X - X$ is a compact subset of $\mathcal{A}_c(X)$ which is not Choquet and therefore not topological.*

Proof: Evidently $X - X$ is compact. By Lemma 3 and Lemma 6, the filter $\mathcal{H} := \bigcap\{\mathcal{U}(x) - \mathcal{U}(x) : x \in X\}$ converges in $\chi(X - X)$ to zero. Assume that it converges to zero in $X - X$ then there are points $x_1, \dots, x_n$ such that

$$\mathcal{H} \supseteq (\mathcal{U}(x_1) - \mathcal{U}(x_1)) + \cdots + (\mathcal{U}(x_n) - \mathcal{U}(x_n)) .$$

Choose a non-discrete point $z \neq x_1, \dots, x_n$ and neighbourhoods U_i of x_i such that $z \notin U_1 \cup \dots \cup U_n$, then $(U_1 - U_1) + \cdots + (U_n - U_n) \in \mathcal{U}(z) - \mathcal{U}(z)$ and so there is a neighbourhood W of z such that $W - W \subseteq (U_1 - U_1) + \cdots (U_n - U_n)$. Since W contains more than one point this gives $W \subseteq U_1 \cup \dots \cup U_n$, a contradiction.

For a convergence group G we denote by $\Gamma_c(G)$ the group of all continuous group homomorphisms into $\mathbf{T}$, the group of all complex numbers with absolute value 1, endowed with the

continuous convergence structure. Then G is called Pontryagin c-reflexive, if the canonical mapping $\kappa_G : G \longrightarrow \Gamma_c(\Gamma_c(G))$ is a topological isomorphism.
If X is a convergence space, $\nu_X : \mathcal{C}_c(X) \longrightarrow \mathcal{C}_c(X, \mathbf{T})$ denotes the natural projection. We now get:

Proposition 2 *If X is a compact topological space such that ν is not surjective, then $\Gamma_c(\mathcal{A}_c(X))$ is not Pontryagin c-reflexive.*

Proof: The restriction mapping $\Gamma_c(\mathcal{A}_c(X)) \longrightarrow \mathcal{C}_c(X, \mathbf{T})$ is a topological isomorphism. Therefore $\Gamma_c(\Gamma_c(\mathcal{A}_c(X)))$ is isomorphic to $\Gamma_c(\mathcal{C}_c(X, \mathbf{T}))$ which is by Theorem 2 in [2] isomorphic to $\Gamma_c(\nu_X(\mathcal{C}_c(X))) \oplus \Gamma_c(D_X)$, where D_X is a discrete topological group which is non-trivial if ν_X is not surjective. But evidently $\mathcal{A}_c(X)$ is mapped under this isomorphism into $\Gamma_c(\nu_X(\mathcal{C}_c(X)))$.

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