Rostock. Math. Kolloq. **51**, 65–82 (1997)

Subject Classification (AMS)
65R20, 45M10

Kurt Frischmuth; Neville J. Ford; John. T. Edwards

Volterra Integral Equations with non-Lipschitz Nonlinearity

Dedicated to the professors of mathematics
G. Maeß, H. Poppe, *and* G. Wildenhain

ABSTRACT. In this work we consider equations of the form:

$$y(t) = g(t) + \int_0^t K(t,s,y(s))ds, \quad t \in \mathbb{R}^+, \quad (\dagger)$$

analytically and when solved numerically.

In some recent work the long-term behaviour of numerical solutions of the nonlinear convolution equation

$$y(t) = g(t) + \int_0^t k(t-s)\varphi(y(s))ds, \quad t \in \mathbb{R}^+, \quad (\ddagger)$$

has been considered. In the present paper, we consider examples of the form ($\ddagger$) which do not satisfy classical conditions guaranteeing existence and uniqueness of the exact solutions and suitable behaviour of approximate solutions. We are able to give a strengthened version of a theorem given originally by Corduneanu for certain kernels and nonlinearities. On that basis we consider how certain numerical codes may fail. We explore and test methods known to preserve properties of the solutions in the linear case.

KEY WORDS. Volterra integral equations, stability, quadrature rules.

1 Introduction

We consider Volterra integral equations of the form ($\dagger$), and particularly those which take the form

$$y(t) = g(t) + \int_0^t k(t,s)\varphi(y(s))ds, \quad t \in \mathbb{R}^+ \quad (1)$$

Of equations of this type, the nonlinear convolution equation

$$y(t) = g(t) + \int_0^t k(t-s)\varphi(y(s))ds, \quad t \in \mathbb{R}^+, \tag{2}$$

is particularly amenable to analysis. For certain assumptions about the kernel k, input g, and nonlinearity φ, known analytical results ensure existence, uniqueness, boundedness and transience of the exact solutions (see, for example the book [12]). Many results have been given for the case where φ is a linear function. The behaviour of solutions to nonlinear equations is less well understood. It has also been shown that for certain quadrature rules the properties of the analytical solutions apply also to the numerical ones ([13], [9], [10]). However, for certain cases of practical interest, the assumptions mentioned above are either not met or are nearly violated. For example, the Lipschitz constants typically used in theorems relating to existence and uniqueness of solutions may be very large, or the nonlinearity may be unbounded. In such cases, it seems from the existing literature that the numerical solutions may lose stability in their long-term behaviour.

In this paper, we explore the situation where the function $\varphi(y)$ is not Lipschitz continuous. By example we demonstrate that the solutions obtained in this case may not be smooth. We select some cases for analysis that can be reduced to ordinary differential equation form and we compare solutions obtained by standard numerical methods for ordinary differential equations with those obtained using direct quadrature rules. We are able to exhibit (and explain) the problems that arise with methods of each type and we compare the effectiveness of the implicit Euler rule, the repeated trapezoidal rule and the second order backward difference rule, all of which might be expected to perform reasonably well for the examples selected for analysis.

A simple example of the type of problem that might arise with a numerical method is shown in Figure 1. Here we have applied a quadrature rule to solve approximately a nonlinear integral equation whose true solution is known to converge to zero for large positive values of the independent variable. One may easily observe that the approximate solution does not exhibit this property, but rather displays oscillations of increasing amplitude.

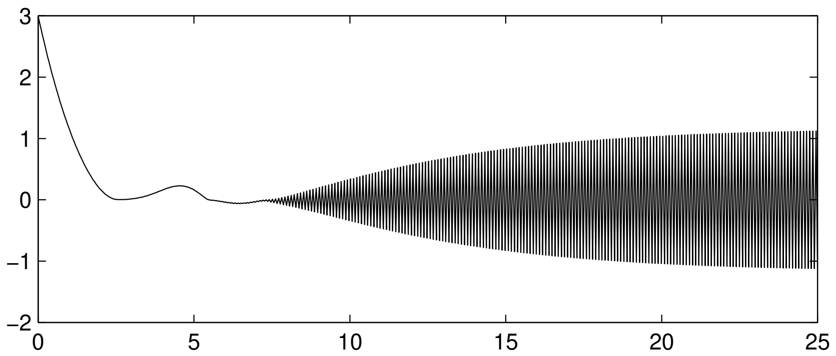

Fig. 1: Numerical solution exhibiting spurious oscillatory behaviour

2 Background and Applications

Systems found in ecology, genetics and the life sciences in general are often complicated to model mathematically. The systems may be poorly understood and this leads to the adoption of a phenomenological approach. In practice, very often we can observe that the information needed in order to determine the future evolution of the system is contained in the past records of the observable parameters together with data concerning external influences (such as the weather, human activity etc.).

In the present paper, we study the evolution of the state of a system described by the history of a certain (vector-valued) quantity y whose solution is assumed to lie within some set Y. The function y is assumed to be under the influence of certain external factors g from a set of permitted external influences G.

In practice, equations modelling ecological systems frequently take the form

$$y(t) = g(t) + \int_0^t K(t,s)\varphi(y(s))ds, \quad t \in \mathbb{R}^+. \tag{3}$$

In (3), the kernel function $K(t,s)$ is said to be a convolution kernel when $K(t,s) = k(t-s)$. This form arises commonly in applications both from the biological and from the physical and engineering sciences, in models of situations where there is no ageing or seasons (and therefore no explicit dependence upon time) and it turns out that convolution equations can be particularly convenient for analysis. In the case of the linear equation $\varphi(s) = \lambda s$, equation (3) is, under certain natural conditions on k, g, amenable to solution through the use of Laplace or Fourier transforms. This, in particular, has led to a great deal of discussion of equations of the form

$$y(t) = g(t) + \int_0^t k(t-s)y(s)ds, \quad t \in \mathbb{R}^+. \tag{4}$$

(We refer the reader to the work of [19],[17],[8], [12], [4].)

A detailed analysis of the behaviour of numerical methods for solving equation (4) has also proved possible. The interested reader is referred to the work of, for example, [3], [12], [13].

The *nonlinear* convolution equation

$$y(t) = g(t) + \int_0^t k(t-s)\varphi(y(s))ds, \quad t \in \mathbb{R}^+. \tag{5}$$

cannot be analysed quite so satisfactorily. For most general functions φ it is not possible to solve (5) to give a closed-form solution. The usual approach is to apply a numerical method (based on a family of quadrature rules – see below) to obtain an approximate solution. A natural requirement is that the numerical method should converge and that it should preserve the qualitative behaviour that would be exhibited by the true solution if it could be found. Work by (for example) Corduneanu (see [5]) and recent work by Ford and Baker (see [9] and [10]) has demonstrated how results on qualitative behaviour of both exact and numerical solutions may be obtained for equation (5) even though the exact solution is not known (and may not be expressible in closed form).

For motivation, we give two examples of problems in Mathematical Biology that lead to equations of the type (3). Many more example applications may be found, for example in Chapter 4 of the book [4]. We also refer to the works [1], [25], [24], and the classical paper [27].

Example 1 An ecosystem model

We consider an ecosystem modelled by the feedback loop illustrated in the figure below under the following assumptions:

- $\phi y(t) = \varphi(y(t))$ for some function $\varphi(x)$
- $\tilde{g}(t) = 0$ for $t < 0$.

Then, for $t \geq 0$, we can write $y = K(\tilde{g} - \phi y)$ (in operator notation) or, alternatively,

$$y(t) = \int_0^t k(t-s)\tilde{g}(s)ds - \int_0^t k(t-s)\varphi(y(s))ds, \quad t \geq 0. \tag{6}$$

As can be easily seen, equation (6) takes the form of (5) with $g(t) = \int_0^t k(t-s)\tilde{g}(s)ds$. In this work, we shall be most interested in the case of nonlinear functions φ.

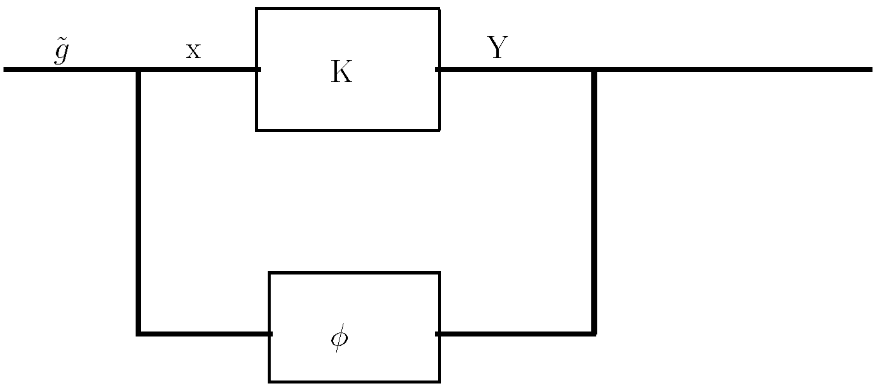

Fig. 2: Ecosystem Model

Example 2 A nonlinear problem of soil contamination

We start from the boundary value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, \quad x > 0, \quad t > 0 && (7) \\ u(x,0) &= 0, \quad x > 0 && (8) \\ -u_x(0,t) &= \varphi\left(u(0,t)\right), \quad t > 0 && (9) \end{aligned}$$

where the function φ is assumed continuous, monotonically decreasing and $\varphi(0) = 0$.

The underlying problem is to find the unknown function u, the concentration of some contaminant in the soil, (the half space $x \geq 0$). The concentration of the uncontaminated medium is taken to be $u \equiv 0$. The concentration in the semi-infinite domain is determined by the boundary values $u(0,t) = y(t)$. We can derive the Volterra equation

$$u(0,t) = \int_0^t \frac{\varphi\left(u(0,s)\right)}{\sqrt{\pi(t-s)}} ds \tag{10}$$

This equation again takes the form of (5). Here $\varphi(u)$ describes the leakage of the contaminant through the boundary (for example into some body of water or into the river bed) dependent upon the initial concentration.

3 Analytical Results

3.1 Existence and Uniqueness

For the equation (5) there is a well-established existence and uniqueness theory for solutions on a bounded interval $[0,T)$ for some $T > 0$. We give here the basic theorem from [12], and further results can be found, for example, in [12], [4], [17] and [5]. A fuller discussion of available theoretical results is given in the more recent work [11].

Theorem 3.1 *In the equation*

$$y(t) = g(t) + \int_0^t K(t,s,y(s))ds, \quad 0 \leq t \leq T \tag{11}$$

assume that the functions $g(t)$ and $K(t,s,u)$ are continuous in $0 \leq s \leq t \leq T < \infty$, $-\infty < u < \infty$ and that the kernel K satisfies a Lipschitz condition of the form

$$|K(t,s,y) - K(t,s,z)| \leq L|y-z| \tag{12}$$

where L is independent of t,s,y,z. Then (11) has a unique continuous solution on $[0,T]$.

The equation (5) is a special case of equation (11). In particular, if we assume in (5) that $k(t)$ is continuous on $\mathbb{R}$ and that $\varphi(\sigma)$ is Lipschitz continuous on $\mathbb{R}$, then the conditions of Theorem 3.1 are satisfied, and we can conclude that (5) has a unique continuous solution on $[0, \infty)$. We observe that the latter condition on φ is satisfied if φ has a bounded derivative on $[0, \infty)$.

In practical cases, we are often interested in considering the behaviour of solutions to equations where the standard existence and uniqueness theory fails to hold. For example, we may need to analyse the case of an equation where $K(t, s, z)$ is not Lipschitz continuous in its third argument.

3.2 Stability and Transience

Certain long-term properties may be of particular interest. Particular applications may demand that a solution remains bounded or tends to zero as $t \to \infty$. This is a natural requirement in applications relating to mathematical biology, where populations are to be modelled or infections are to be kept under control. There can be similar natural requirements in applications to control theory. Further, consideration of questions of *stability* may require boundedness or transience of solutions to a homogeneous equation.

We make the following definition (see [10]):

Definition Let $y(t)$ be a solution of an equation E and assume $\lim_{t\to\infty} y(t) = 0$, then y is termed a *transient* solution to E.

The analysis is well known for the linear form of (5):

$$y(t) = g(t) + \int_0^t k(t-s)y(s)ds, \quad t \in \mathbb{R}^+. \tag{13}$$

In (13), the simplest analysis applies to the case when $k(t) = \lambda$. In this case, for $\Re\lambda < 0$, all solutions of (13) are transient if g is transient. If $\Re\lambda \leq 0$ then all solutions of (13) are bounded if g is bounded. The basic results have been extended through the use of Laplace transforms ([19], [13]) for more general convolution kernels $k \in L^1[0, \infty)$.

In this paper, we assume that the function $\varphi(s)$ is nonlinear in s, and so the results on linear equations do not apply. However, certain results for nonlinear equations have been given (see, for example, [17], [14], [5], [15]).

A good example of a result of this type is given in the following:

Theorem 3.2 **(Based on Corduneanu (1973) (Theorem 2.2, p. 87))** *Under the following assumptions:*

1. *that* $g(t), g'(t) := \frac{d}{dt}g(t) \in L^1(\mathbb{R}^+)$ *and define* $g(t) = g'(t) = 0$ *for* $t < 0$,
2. *that* $k(t), k'(t) := \frac{d}{dt}k(t) \in L^1(\mathbb{R}^+)$ *and define* $k(t) = k'(t) = 0$ *for* $t < 0$,
3. *that* $\varphi(\sigma)$ *is a continuous bounded function from* $\mathbb{R}$ *into itself, which satisfies* $\sigma\varphi(\sigma) > 0$ *for* $\sigma \neq 0$,
4. *that there is a* $q \geq 0$ *such that* $\Re\{(1 - isq)\widehat{k}(s)\} \leq 0$; $s \in \mathbb{R}$, *where* $\widehat{k}(s)$ *denotes the Fourier transform of* $k(t)$;

then any solution $y(t)$ *of (5) which is continuous and bounded on* $\mathbb{R}^+$ *is uniformly continuous and* $y(t) \to 0$ *as* $t \to \infty$. *Further, if* g *is assumed continuous and bounded on* $\mathbb{R}$ *then the continuity and boundedness of* y *are guaranteed.*

Remark: We note that, under the conditions of Theorem 3.2, we do not, in general, have the conditions of Theorem 3.1 satisfied. So, the assumptions of Theorem 3.2 do not guarantee a unique solution to (5).

The linear equation ($\varphi(y) = y$) is also excluded by the conditions of Theorem 3.2. However, if we are given the boundedness of the solution y by $M < \infty$ we can proceed as follows. Define the function φ_M by

$$\varphi_M(s) = \varphi(s), -M \leq s \leq M, \varphi_M(s) = \varphi(M), s > M, \varphi_M(s) = \varphi(-M), s < -M$$

If φ is continuous, φ_M satisfies the condition 3 of Theorem 3.2, and any solution of equation (5) with φ modified in this way is also a solution of the original equation.

However, these assumptions do not imply the uniqueness of the solution of (5). For uniqueness we also require that φ_M satisfies a Lipschitz condition. For this to be the case, it is sufficient that φ satisfies a Lipschitz condition on $[-M, M]$.

For clarity, we express this as a corollary to Theorem 3.2.

Corollary 3.3 *Under the assumptions* 1, 2, *and* 4 *of Theorem* 3.2 *and with the additional assumption*

3' *that* $\varphi(\sigma)$ *is a continuous real-valued function on* $[-M, M]$ *which satisfies* $\sigma\varphi(\sigma) > 0$ *for* $\sigma \neq 0$, *any solution $y(t)$ of (5) which is continuous and satisfies $|y| \leq M$ on $I\!R^+$ is uniformly continuous and $y(t) \to 0$ as $t \to \infty$.*

Remark: Since the choice of M in Corollary 3.3 is arbitrary, any bounded continuous y satisfying (5) satisfies $y(t) \to 0$ as $t \to \infty$.

4 Numerical methods

4.1 Quadrature Methods

A natural approach to the numerical solution of equations of the form (3), (5) is through the use of a quadrature method (cf. [16]) for the approximation of the integral term on the right hand side of the expression. For a fixed step length, h, the number of ordinates used in the quadrature will grow as the upper limit on the integral increases. Accordingly, we need to discretise (5) by using a *family* of quadrature rules $\{w_j^{(n)}\}$.

$$\int_0^{nh} f(s)ds \simeq h \sum_{j=0}^{n} w_j^{(n)} f(jh). \tag{14}$$

This approach is adopted widely in the literature (see [12], [28], [3]).

On application of the quadrature rules (14) to (5) we obtain the equation

$$y_n = g_n + h \sum_{j=0}^{n} w_j^{(n)} k_{n-j} \varphi_j; \quad n \geq 0 \tag{15}$$

where

$$h > 0, y_n \simeq y(nh), g_n = g(nh), k_n = k(nh), t_n = nh, \varphi_n = \varphi(y_n); \tag{16}$$

It is natural, for two reasons, to consider the special case where the weights from the quadrature rules chosen take a convolution form $w_j^{(n)} = w_{n-j}$. The first reason is pragmatic: by preserving the convolution structure in the problem it turns out to be possible to develop a theory for the discrete equations. This follows because the transform methods for analysing the equation (5) then have a discrete analogue based on the use of Z-transforms and generating functions for the analysis of (15). The second reason is that it has been shown (see [28], [3]) that the application of a linear multistep method to a first order ordinary differential equation is equivalent to the use of a quadrature which has weights of this type. Under a convolution quadrature, the discrete equation takes the form:

$$y_n = g_n + h \sum_{j=0}^{n} w_{n-j} k_{n-j} \varphi_j; \tag{17}$$

If $w_0 \neq 0$, the method will be implicit, equation (17) is then (for general φ) a nonlinear equation for y_n.

However, not all families of quadratures yield the convenient convolution form. For example, Simpson's rule has a sequence of weights that varies for different members of the family of quadratures and which requires (for example) the use of a single trapezium rule step at one end of the interval on applications with an even number of ordinates.

4.2 Linear theory

Results have been given that relate to the long-term behaviour of solutions when numerical methods, such as the ones we describe here, are applied to (13) and the aim has been to replicate the results for the exact solution described in the previous section insofar as this has proved possible.

We summarise the key results (see, for example, [18], [13], [3]):

1) Solutions of the linear equation (13) are transient (bounded) whenever g is transient (bounded) if $-k$ is a positive definite L^1 kernel.

2) Convolution quadratures applied to solve (13) preserve the convolution structure in the discrete equation.

3) A convolution quadrature may be constructed which corresponds to the use of a linear multistep method for the solution of an ordinary differential equation. A-stable linear multistep methods lead to convolution quadratures which are positive, and which preserve the long-term behaviour of solutions to (13).

4.3 Nonlinear theory

For the equation (15), [9] gives a corresponding analysis which leads to the following theorem that imposes a condition on the *Z–transform* of the sequence $\{wk\}_n$, a discrete analogue of the Fourier transform:

Theorem 4.1 *Under the following assumptions:*

1^* $(g_{n\geq 0})$, $(\Delta g_{n\geq 0}) \in l^1$,

2^* $(\{wk\}_{n\geq 0})$, $(\{\Delta(wk)\}_{n\geq 0}) \in l^1$,

3^* *$h>0$ is fixed and φ is a bounded real–valued function with $|\varphi| \leq \Phi$, which satisfies $y\varphi(y) > 0$ for $y \neq 0$;*

4* *there exists* $q \geq 0$ *such that*

$$\Re\{(h + q[e^{-i\theta} - 1])Z(\{wk\}_n)(e^{-i\theta})\} \leq 0 \text{ for } \theta \in [0, 2\pi],$$

all solutions of (4.1) *are bounded and any solution* $\{y_n\}$ *of* (4.1) *converges to zero.*

This theorem is a discrete analogue of Theorem 3.2. It is already known that, for the linear equation ($\varphi(y) = y$) the use of a positive convolution quadrature preserves the qualitative behaviour of the solution, as $t \to \infty$, to the integral equation in the numerical solution (see [13]). It is further known that positive convolution quadratures form the largest class of quadrature rules that have this property. Theorem 4.1 was applied in [10] to give corresponding results for the nonlinear equations considered in this paper. We are interested in what happens when we apply a numerical method to solve the original equation. The main result given in [10] is the following Theorem. The theorem demonstrates that for a negative definite kernel k, the qualitative behaviour of the solution is preserved when the equation is solved using a positive convolution quadrature.

Theorem 4.2 *Under the assumptions:*

1. *that* $g, g' \in L^1(I\!R^+) \cap C(I\!R^+)$;
2. *that* $k, k' \in L^1(I\!R^+) \cap C(I\!R^+)$;
3. *that* $\varphi(\sigma)$ *is a continuous bounded function from* $I\!R$ *into itself, which satisfies* $\sigma\varphi(\sigma) > 0$ *for* $\sigma \neq 0$;
4. *that* $-k(s)$ *is positive definite;*
5. *that the quadrature weights* (w_{n-j}) *are derived from an A–stable linear multistep method;*

any solution (y_n) *of* (15) *satisfies* $y_n \to 0$ *as* $n \to \infty$.

Remark: The use of the term *positive definite* in the statement of Theorem 4.2 follows the use of the term introduced by Bochner (see [2]). The key to the application of positive definite functions in the present context is contained in Bochner's characterisation of positive definite functions:

Theorem 4.3 **(Bochner [2])** *The continuous function f:* $I\!R \to \mathbf{C}$ *is positive definite if and only if*

$$f(x) = \int_{-\infty}^{\infty} e^{iux} dp(u)$$

for some increasing bounded function p.

We apply the Bochner characterisation in its alternative form ([6]):

Theorem 4.4 **(Fourier transform characterisation)** *Let $f \in L^1(\mathbb{R})$ and define*

$$\widehat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)dx,$$

the Fourier transform of f. Then f is positive definite if and only if

$$\Re(\widehat{f}(s)) \geq 0 \ \forall s \in \mathbb{R}.$$

The functions g, k, φ satisfying the conditions of Theorem 4.2 necessarily satisfy the conditions of Theorem 3.2. It then follows that the use of quadrature weights derived from an A–stable linear multistep method preserves the qualitative behaviour of the solution predicted by Theorem 3.2.

The report [10] extends the analysis further and considers the treatment of kernel functions $-k$ which are not positive definite. In the present paper, we have confined ourselves to the case of $-k$ positive definite.

(Strict) positive definiteness of $-k$ is implied if k is negative, increasing and (strictly) concave [11].

4.4 Special kernels

For certain kernel functions k, it proves possible to derive an ordinary differential equation which is satisfied by all smooth solutions of the equation (5).

We consider example equations of the form:

$$y(t) = g(t) - \int_0^t \kappa e^{-\lambda(t-s)} \varphi(y(s))ds. \tag{18}$$

For $\lambda > 0$ and $\kappa > 0$ the equation satisfies the conditions on k imposed in Theorem 3.2. We assume also that g and φ satisfy the conditions in Theorem 3.2.

We can differentiate (18) to give a first order ordinary differential equation. There is a whole class of integral equations that can be reduced in a corresponding (or more sophisticated) way: some yield first order equations, others yield equations of higher order.

$$\begin{aligned} y'(t) &= g'(t) - \kappa\varphi(y(t)) + \lambda \int_0^t \kappa e^{-\lambda(t-s)} \varphi(y(s))ds && (19) \\ &= g'(t) - \kappa\varphi(y(t)) + \lambda(g(t) - y(t)). && (20) \end{aligned}$$

For the case when $g(t) \equiv 0$ and so $g'(t) = 0$ it follows that

$$y'(t) = -\kappa\varphi(y(t)) - \lambda y(t), \quad y(0) = 0. \tag{21}$$

Assuming that φ is Lipschitz continuous on an interval $[0, T]$ it follows that equation (21) has the unique solution $y(t) = 0$ on $[0, T]$. That the zero solution to (21) is stable for a suitable choice of φ can be seen by Lyapunov's direct method:

in fact, let $V(t, y) = y^2 \geq 0$. With this definition for V it follows that

$$\begin{aligned} \dot{V} &= 2y(t)y'(t) && (22) \\ &= 2y(t)(-\kappa\varphi(y(t)) - \lambda y(t)) && (23) \\ &= -2(\kappa y(t)\varphi(y(t)) + \lambda(y(t))^2). && (24) \end{aligned}$$

This last expression is negative if $y(t)\varphi(y(t)) > 0$ which is condition 4 of Theorem 3.2. Thus we see that the condition imposed on φ in the Corduneanu work to guarantee transience of the solution is precisely the condition we have needed here to establish our Lyapunov function for the solution.

From this point, we choose as our standard nonlinearity $\varphi(y) = y|y|^{-2/3}$, and we consider the effect of small perturbations from the equilibrium state of equation (21).

The equation can now be solved directly to give

$$y(t) = \left(\frac{-\kappa}{\lambda} + Ce^{-2\lambda t/3}\right)^{3/2}$$

where C is a constant of integration, provided that y is not the zero function.

Using the initial condition $y(0) = \epsilon \neq 0$ we can derive the solution

$$y(t) = \left(\frac{-\kappa}{\lambda} + (\epsilon^{(2/3)} + \kappa/\lambda)e^{-2\lambda t/3}\right)^{3/2} \tag{25}$$

In this case $y(t)$ exhibits surprising behaviour: equilibrium is regained after finite time $t_0 = \frac{3}{2\lambda} \ln\left(1 + \frac{\epsilon^{(2/3)}\lambda}{\kappa}\right)$. Further $y'(t_0) = \infty$ and so the intersection with the equilibrium solution is at right angles. A non-Lipschitz φ does not in itself guarantee the existence of multiple solution trajectories, but in this case we do see two solutions of (21) intersecting at t_0. While the differential equation has a unique solution for each given initial value, integration *backwards* in time is non-unique and hence unstable. Although the initial value problem for the ordinary differential equation has a solution only over a finite time period, the integral equation has a solution for all time $t \geq 0$. This solution is formed by concatenating the transient branch and the equilibrium one. We conclude that such a solution has a singularity in the first derivative.

Now we are able to strengthen Theorem 3.2.

Theorem 4.5 *Let k and φ be as above, then the following holds: if Support(g) is compact then Support(y) is compact.*

Remark: The same can be proved for all φ which satisfy assumption (3) of Theorem 3.2 and also have the property that $y' = -\varphi(y)$ has a bifurcation at $y = 0$.

We can draw the following provisional conclusions from these observations:

We should not expect explicit methods to work well

The problem is stiff and therefore the most appropriate numerical methods are likely to be those that handle stiff problems effectively

Although we have no analytical results for general kernels, we expect that any non-Lipschitz φ will have a similar effect. The use of, for example, BDF methods would be indicated.

The analytical solutions may be non-differentiable and therefore we should not expect to find stable numerical methods of high order.

In the case of bounded nonlinearity, the results of Section 3 may give further insight.

5 Numerical Results

In this section, we give numerical evidence in support of the provisional conclusions of the preceding section. We begin by considering the performance of professional packages on the ODE for the special kernel case. In fact we solve the Bernoulli equation:

$$y'(t) = -\lambda y(t) - \kappa y^{(1/3)}(t), \quad y(0) = \epsilon. \tag{26}$$

The numerical solution given by Mathematica (for $\lambda = 1$, $\kappa = 1$, $\epsilon = 1$), starts chaotic oscillations after approaching the equilibrium. Here are the first 180 steps, we stop just before breakdown.

Figures 3 and 4 show the transient branch and the zoomed "equilibrium" branch.

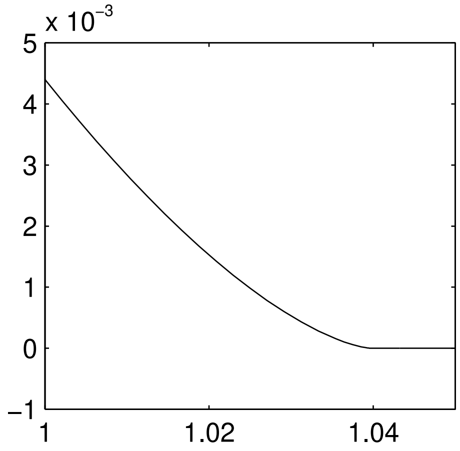

Fig. 3: Bernoulli case

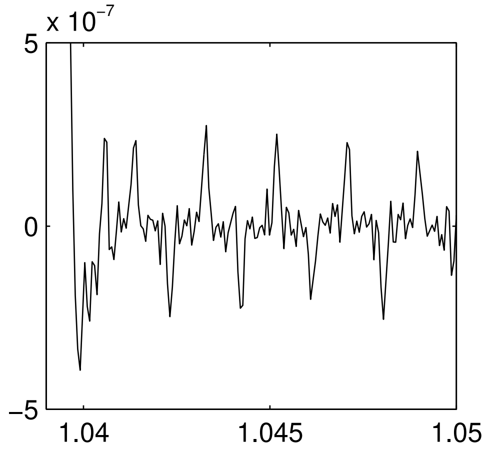

Fig. 4: Zoom

Matlab's standard ODE solvers exhibit similar problems. For example, applied to (18) with $g(t) = e^{-0.5(t-5)^2}, y(0) = 0$, Matlab's ode23 procedure yields the following Figure 5 (the accuracy parameter was chosen as 0.0005). The solver took 2297 steps on the interval [0,25].

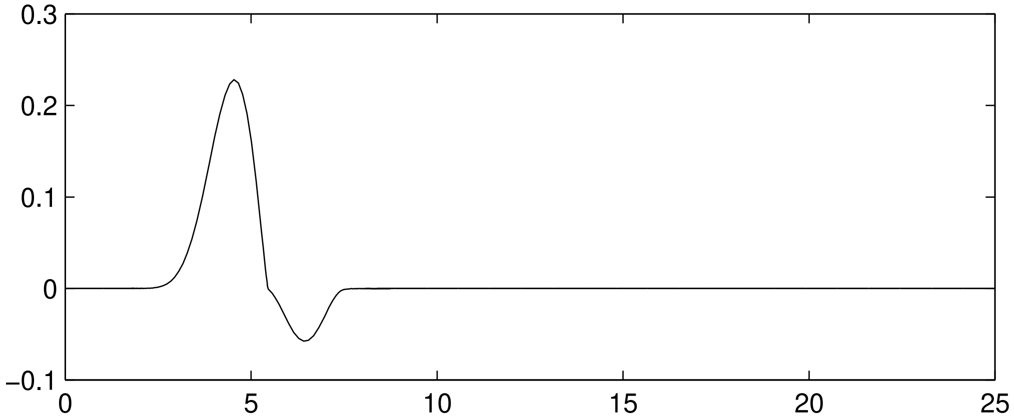

Fig. 5: Matlab's solver

Surprisingly (but not unexpectedly given our previous observations), it needed just 35 steps to calculate all function values outside the interval $[-0.01, 0.01]$. After the solution dropped from it's maximum below 0.01, i.e. from abscissae 7.1963 to 25, it needed as many as 2006 steps, which makes the average step size 0.0089. The average value of h decreases by a factor of about 20 in a region where nothing happens – and moreover, we know that the solution is zero over this interval!

However, implicit quadrature methods generally seem to perform rather better. The repeated trapezium rule provides an implicit method that gives very acceptable results. However, it is not naturally in convolution form, except if we fix the weights by choosing $w_0 = \frac{1}{2}$ and $w_i = 1$ for $i \geq 0$. Some authors (see for example [3]) suggest the use of quadratures derived from BDF formulae for stiff problems. The implicit Euler rule provides an implicit method with $w_i = 1$ for each i.

Remark: For many kernels, which vanish either exactly or approximately for larger values of the argument, the values of w_i for larger values of i play no part in the calculations after finitely many applications of the quadrature.

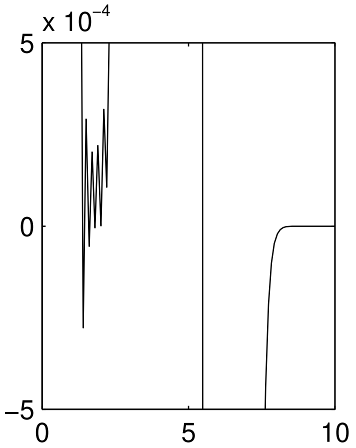

Fig. 6: Trapezium rule

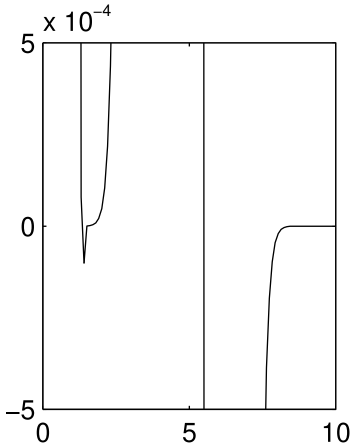

Fig. 7: BDF

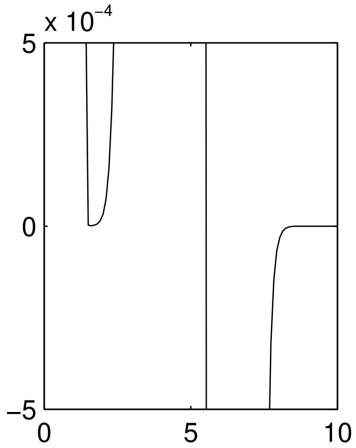

Fig. 8: Implicit Euler

Figures 6, 7 and 8 show the performance of these three methods in solving (18) with

$$g(t) = e^{-0.5(t-5)^2}, \quad y(0) = 1\,.$$

All methods are tested with $h = 0.1$. Note that the repeated trapezium rule (which is not in convolution form) produces several transient oscillations close to the equilibrium solution. The BDF formula is better. We do not see oscillations in this case, but the Implicit Euler rule gives an even better reproduction of the expected behaviour of the analytical solution. We surmise that the BDF formula fails to give any improvement on the Implicit Euler rule because it attempts to smooth the solution, and the solution we seek is not smooth.

We conclude this section with two examples that we will consider further in a subsequent paper. For the first one we take a periodic forcing function g with the same φ, k in (18).

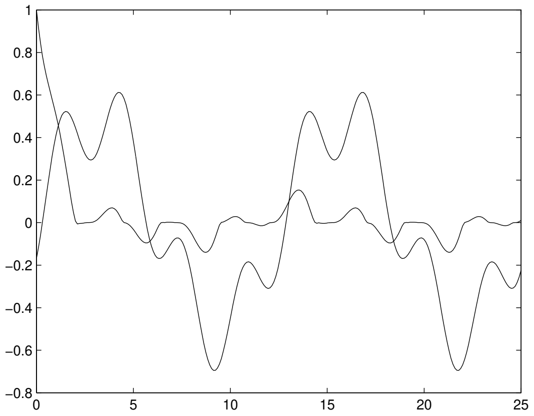

Fig. 9: Periodic case, forcing and solution

From Figure 9 we can see that the solution tends to a periodic function. It is an open question whether the solution becomes exactly periodic after finite time.

The last example relates to work undertaken by two of the authors with Jason Roberts (see [21]). We consider equations of the form:

$$y'(t) = g(t) + \int_0^t k(t-s)\varphi(y(s))ds, \quad t \geq 0, \quad y(0) = y_0. \tag{27}$$

One possible approach to problems of this type is to integrate both sides using Fubini's Theorem, yielding a Volterra integral equation of convolution type. Without going into details, it can be shown that the resulting kernel may fulfil the assumptions of the Corduneanu Theorem and can then be handled by the BDF method or the Implicit Euler rule. This example cannot be reduced to an ordinary differential equation.

Figure 10 shows the solution we derived for a forcing function that exhibits decaying oscillations.

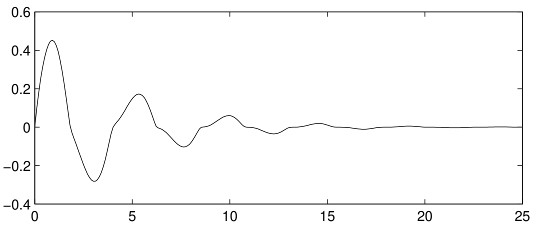

Fig. 10: Integrodifferential case

References

[1] **Bellman, R. K.**, and **Danskin, J. M.** : *A survey of the mathematical theory of time-lag, retarded control, and hereditary processes.* The Rand Corporation, California 1954

[2] **Bochner, S.** : *Lectures on Fourier Integrals.* Princeton University Press 1959

[3] **Brunner, H.**, and **Van der Houwen, P. J.** : *The numerical solution of Volterra equations.* North Holland 1986

[4] **Burton, T. A.** : *Volterra integral and differential equations.* New York 1983

[5] **Corduneanu, C.** : *Integral Equations and Stability of Feedback Systems.* New York 1973

[6] **Cotlar, M.**, and **Cignoli, R.** : *An Introduction to Functional Analysis.* North Holland 1974

[7] **Doetsch, G.** : *Guide to the applications of the Laplace and z-transforms.* Van Nostrand 1971

[8] **Ford, N. J.** : *Convolution Equations and their Discretised Versions.* MSc Thesis, University of Manchester, Manchester 1986

[9] **Ford, N. J.**, and **Baker, C. T. H.** : *Qualitative behaviour and stability of solutions of discretised non-linear Volterra integral equations of convolution type.* Numerical Analysis Report **261**, Manchester Centre for Computational Mathematics (1994)

[10] **Ford, N. J.**, and **Baker, C. T. H.** : *Preserving transience in numerical solutions of Volterra integral equations of convolution type.* Numerical Analysis Report **284**, Manchester Centre for Computational Mathematics (1996)

[11] **Gripenberg, G., Londen, S.-O.**, and **Staffans, O.** : *Volterra integral and functional equations.* Cambridge University Press 1990

[12] **Linz, P.** : *Analytical and Numerical Methods for Volterra Equations.* Philadelphia 1985

[13] **Lubich, Ch.** : *On the stability of linear multistep methods for Volterra Convolution Equations.* IMA J. Numer. Anal. **3**, 439–465 (1983)

[14] **MacCamy, R. C.**, and **Smith, R. L.** : *Limits of Solutions of Nonlinear Volterra Equations.* Appl. Anal. **7**, 19–27 (1977)

[15] **MacCamy, R. C.**, and **Weiss, P.** : *Numerical Solution of Volterra Integral Equations.* Nonlinear Anal. **3**, 677–695 (1979)

[16] **Maeß, G.** : *Vorlesungen über numerische Mathematik II.* Berlin 1988

[17] **Miller, R. K. :** *Nonlinear Volterra Integral Equations.* New York 1971

[18] **Nevanlinna, O. :** *Positive quadratures for Volterra equations.* Computing **16**, 349–357 (1976)

[19] **Paley, R. E. A. C.**, and **Wiener, N. :** *Fourier transforms in the complex domain.* Providence 1934

[20] **Ramachandran, B.**, and **Lau, Ka-Sing :** *Functional Equations in Probability Theory.* San Diego 1991

[21] **Roberts, J. A. :** *Novel methods for the approximate solution of delay, integral, and integrodifferential equations.* PhD Thesis, to be submitted

[22] **Rudin, W. :** *Real and Complex Analysis.* McGraw Hill 1970

[23] **Saaty, T. L. :** *Modern Nonlinear Equations, 2Ed.* Dover 1981

[24] **Smith, D.**, and **Keyfitz, N. :** *Mathematical Demography.* Berlin 1977

[25] **Thieme, H. R. :** *Density-dependent regulation of spatially distributed populations and their asymptotic speed of spread.* J. Math. Biol. **8**, 173–187 (1979)

[26] **Tsalyuk, Z. B. :** *Volterra Integral Equations.* J. Soviet Math. **12**, 715–758 (1979)

[27] **Volterra, V. :** *Sur la théorie mathématique des phénomènes hereditaires.* J. Math. Pures Appl. **7**, 251–268 (1928)

[28] **Wolkenfelt, P. H. M. :** *The construction of reducible quadrature rules for Volterra integral and integro-differential equations.* IMA J. Numer. Anal. **2**, 131–152 (1982)

received: April 11, 1997

Authors:

Kurt Frischmuth
Universität Rostock
Fachbereich Mathematik
Universitätsplatz 1
18051 Rostock
Germany

kurt@sun2.math.uni-rostock.de

Neville J. Ford; John. T. Edwards
University College Chester
Department of Mathematics
Chester CH1 4BJ
UK

njford@chester.ac.uk