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The Ascoli Theorem is equivalent to the Boolean Prime Ideal Theorem

Dedicated to the professors of mathematics
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It is well-known that in **ZF** (i.e., Zermelo-Fraenkel set theory without the Axiom of Choice) the following hold:

Theorem [3] *The Tychonoff Product Theorem is equivalent to the Axiom of Choice.*

Theorem [5] *The Čech-Stone Theorem is equivalent to the Boolean Prime Ideal Theorem.*

What is the corresponding status of the Ascoli Theorem? It is the purpose of this note to settle this question. Since the Ascoli Theorem occurs in a variety of forms (see the comprehensive study in [4]), the form used here needs to be specified (although the title-result is rather stable). For the purpose of this paper the following version is used:

Ascoli Theorem *If $\mathbf{X}$ is a locally compact Hausdorff space, $\mathbf{Y}$ is a metric space, $C_{co}(\mathbf{X},\mathbf{Y})$ is the space of all continuous maps from $\mathbf{X}$ to $\mathbf{Y}$ with the compact-open topology, and F is a subspace of $C_{co}(\mathbf{X},\mathbf{Y})$, then the following conditions are equivalent:*

(1) *F is compact,*

(2) (a) *for each $x \in \mathbf{X}$ the set $F(x) = \{f(x) | f \in F\}$ is compact in $\mathbf{Y}$,*

(b) *F is closed in the product space $\mathbf{Y}^{\mathbf{X}}$,*

(c) *F is equicontinuous, i.e.,*

$$\forall x \in X \quad \forall \varepsilon > 0 \quad \exists U \in \mathfrak{U}(x) \quad \forall f \in F \quad \forall y \in U \quad d\big(f(x), f(y)\big) < \varepsilon$$

(where $\mathfrak{U}(x)$ is the neighbourhood-filter of x in $\mathbf{X}$).

Theorem *In* **ZF** *the following statements are equivalent:*

(α) *the Ascoli Theorem,*

(β) *the Boolean Prime Ideal Theorem.*

Proof: $(\alpha) \Rightarrow (\beta)$ Let X be an arbitrary set, let $\mathbf{X}$ be the discrete topological space with underlying set X, let $\mathbf{Y}$ be the metric space with underlying set $\{0,1\}$ and $d(0,1) = 1$, and let F be $C(\mathbf{X}, \mathbf{Y}) = \mathbf{Y}^X$. Then condition (2) of the Ascoli Theorem is satisfied. Thus (α) implies that (1) holds, i.e., that $\mathbf{Y}^X$ is compact. By [5], this implies (β).

$(\beta) \Rightarrow (\alpha)$ Let $\mathbf{X}$, $\mathbf{Y}$, and F be specified as in the Ascoli Theorem.

$(1) \Rightarrow (2a)$ holds, since continuous images of compact spaces are compact.

$(1) \Rightarrow (2b)$ holds, since compactness of F in the compact-open topology implies compactness of F in the (weaker) product topology, and compact subspaces of Hausdorff spaces are closed (see [1]).

$(1) \Rightarrow (2c)$ Choose $x \in X$ and $\varepsilon > 0$. Then for each $f \in F$ the set $B_f = \{y \in Y \mid d\big(f(x), y\big) < \frac{\varepsilon}{2}\}$ is open in $\mathbf{Y}$. Thus, by continuity of f and local compactness of $\mathbf{X}$, there exists a compact neighbourhood of K_f of x with $f[K_f] \subset B_f$. Thus $U_f = \{g \in F \mid g[K_f] \subset B_f\}$ is an open neighbourhood of f in F. Let $\omega : X \times F \to Y$ be defined by $\omega(y, g) = g(y)$. Then the above implies $\omega[K_f \times U_f] \subset B_f$. Consider the collection $\mathcal{C}$ of all tripels (f, K, U) with $f \in F$, K a neighbourhood of x in $\mathbf{X}$, and U an open neighbourhood of f in F with $\omega[K \times U] \subset B_f$. Then, by the above, $\mathfrak{U} = \{U \subset F \mid \exists f \in F\ \exists K \subset \mathbf{X}\ (f, K, U) \in \mathcal{C}\}$ is an open cover of F. Thus, by (1), there exist finitely many members $U_1, \ldots, U_n$ of $\mathfrak{U}$ wich cover F. For each $i \in \{1, \ldots, n\}$ select $f_i \in F$ and $K_i \subset \mathbf{X}$ with $(f_i, K_i, U_i) \in \mathcal{C}$. Then $U = \bigcap\limits_{i=1}^{n} K_i$ is a neighbourhood of x in $\mathbf{X}$.

Claim: $\forall f \in F \quad \forall z \in U \quad d\big(f(x), f(y)\big) < \varepsilon$.

Proof: For $f \in F$ there exists $i \in \{1, \ldots, n\}$ with $f \in U_i$. Thus $z \in U$ implies

$$f(z) = \omega(y, f) \subset \omega[U \times U_i] \subset \omega[K_i, U_i] \subset B_{f_i}\,,$$

i.e. $d\big(f_i(x), f(z)\big) < \frac{\varepsilon}{2}$. In particular, $x \in U$ implies $d\big(f_i(x), f(x)\big) < \frac{\varepsilon}{2}$. Thus $d\big(f(x), f(z)\big) \leq d\big(f(x), f_i(x)\big) + d\big(f_i(x), f(z)\big) < \varepsilon$.

$(2) \Rightarrow (1)$ If (β) holds, then (2a) implies that $\mathbf{Y}^{\mathbf{X}}$ is compact ([5]). Thus (2b) implies that F is compact in the product topology τ on F. It remains to be shown that (2c) implies that τ equals the (generally finer) compact-open topology τ' on F. Consider $f \in F$ and $V \in \tau'$ with $f \in V$. Then there exists a compact subset K of $\mathbf{X}$ and an open subset U of $\mathbf{Y}$ with

$$f \in \{g \in F \mid g[K] \subset U\} \subset V\,.$$

In particular $f[K] \subset U$ implies that, for each $x \in K$, $r_x = dist\big(f(x), Y \backslash U\big) > 0$. Thus $U_x = \big\{z \in X \mid d\big(f(x), f(z)\big) < \frac{r_x}{2}\big\}$ is an open neighbourhood of x. Consequently $\mathfrak{U} = \{U_x \mid x \in K\}$ is an open cover of K. By compactness of K there exist finitely many members $x_1, \ldots, x_n$ of K such that $K \subset \bigcup_{i=1}^{n} U_{x_i}$. Thus $r = \min\{r_{x_1}, \ldots, r_{x_n}\} > 0$ and for each $x \in K$ and each $y \in Y \backslash U$ the inequality $d\big(f(x), y\big) \geq \frac{r}{2}$ follows; in other words: $x \in K$ and $d\big(f(x), y\big) < \frac{r}{2}$ imply $y \in U$. By equicontinuity of F there exists for each $x \in \mathbf{X}$ some neighbourhood W of x such that

$$\forall g \in F \quad \forall z \in W \quad d\big(g(x), g(z)\big) < \frac{r}{4}. \tag{$*$}$$

Consider the set $\mathcal{C}$ of all pairs (x, W) with $x \in \mathbf{X}$ and W an open neighbourhood of x such that $(*)$ holds. Then $\mathfrak{U} = \{W \mid \exists x \in K\ (x, W) \in \mathcal{C}\}$ is an open cover of K. By compactness there exist finitely many members $W_1, \ldots, W_m$ in $\mathfrak{U}$ which cover K. For each $i \in \{1, \ldots, m\}$ select some x_i with $(x_i, W_i) \in \mathcal{C}$. Thus $B = \{g \in F \mid d\big(f(x_i), g(x_i)\big) < \frac{r}{4}$ for $i = 1, \ldots, m\}$ is a neighbourhood of f in product topology τ on F.

Claim: $B \subset V$.

Proof: Consider $g \in B$. For each $x \in K$ There exists $i \in \{1, \ldots, m\}$ with $x \in W_i$. This implies $d\big(g(x_i), g(x)\big) < \frac{r}{4}$ by $(*)$. Since $g \in B$, the inequality $d\big(f(x_i), g(x_i)\big) < \frac{r}{4}$ holds. Thus

$$d\big(f(x_i), g(x)\big) \leq d\big(f(x_i), g(x_i)\big) + d\big(g(x_i), d(x)\big) < \frac{r}{2}.$$

Consequently $g(x) \in U$; hence $g[K] \subset U$; hence $g \in V$. This completes the proof. □

Remarks

(1) Observe that the implication (1) $\Rightarrow$ (2) in Ascoli's Theorem holds in **ZF**. Thus the Boolean Prime Ideal Theorem is equivalent to the implication (2) $\Rightarrow$ (1) in Ascoli's Theorem.

(2) Observe further that in **ZF** the familiar descriptions of compactness fail to remain equivalent (see [1]). The concept used here (as well as in the two theorems mentioned at the beginning of this note) is *Heine-Borel-compactness*, (i.e., every open cover contains a finite one). For the set theoretical status of Ascoli's Theorem with respect to other forms of compactness see [2].

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