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On an Inequality for Nonexpansive Mappings in Uniformly Convex Banach Spaces

Dedicated to the professors of mathematics
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ABSTRACT. Let C be a bounded convex subset of an uniformly convex Banach space X. Then it is shown that there exists an increasing continuous function $h : \mathbb{R}^+ \to \mathbb{R}^+$ depending on the diameter of C so that for any nonexpansive mapping $T : C \to X$ and any convex combination of arbitrarily many elements x_i in C the inequality $h(\|\sum \lambda_i x_i - T(\sum \lambda_i x_i)\|) \leq \sum \lambda_i \|x_i - Tx_i\|$ holds. This inequality has several consequences in the theory of nonexpansive mappings.

KEY WORDS. Inequality, uniformly convex Banach space, nonexpansive mapping.

1 The Inequality

We start with the following result due to Bruck.

Lemma 1.1 ([1]) *Suppose C is a bounded closed convex subset of a uniformly convex Banach space X. Then there exists a strictly increasing continuous function $\gamma : [0,\infty) \to [0,\infty)$ with $\gamma(0) = 0$ depending on the diameter of C such that*

$$\gamma\left(\left\|\sum_{i=1}^{n} \lambda_i T x_i - T\left(\sum_{i=1}^{n} \lambda_i x_i\right)\right\|\right) \leq \max_{1\leq i,j\leq n} \left(\|x_i - x_j\| - \|Tx_i - Tx_j\|\right)$$

holds for any nonexpansive mapping $T : C \to X$, any elements $x_1, \ldots, x_n$ in C and any numbers $\lambda_1, \ldots, \lambda_n \geq 0$ with $\lambda_1 + \ldots + \lambda_n = 1$.

Corollary 1.2 *Under the same suppositions as in Lemma 1.1 there exists a strictly increasing, continuous function α such that*

$$\alpha\left(\left\|\sum_{i=1}^{n} \lambda_i x_i - T\left(\sum_{i=1}^{n} \lambda_i x_i\right)\right\|\right) \leq \max_{1\leq i\leq n} \left(\|x_i - Tx_i\|\right).$$

Proof: We put $t = \max(\|x_i - Tx_i\|)$. Since

$$\|x-y\| - \|Tx - Ty\| \le \|(x-y) - (Tx - Ty)\| \le \|x - Tx\| + \|y - Ty\|$$

it follows by Lemma 1.1 that there is a strictly increasing continuous function γ with

$$\gamma\left(\left\|\sum_{i=1}^n \lambda_i Tx_i - T\left(\sum_{i=1}^n \lambda_i x_i\right)\right\|\right) \le 2t.$$

As γ^{-1} is increasing, we have

$$\begin{aligned}
\left\|\sum_{i=1}^n \lambda_i x_i - T\left(\sum_{i=1}^n \lambda_i x_i\right)\right\| \; &\le \; \left\|\sum_{i=1}^n \lambda_i Tx_i - T\left(\sum_{i=1}^n \lambda_i x_i\right)\right\| + \left\|\sum_{i=1}^n \lambda_i x_i - \sum_{i=1}^n \lambda_i Tx_i\right\| \\
&= \gamma^{-1}(2t) + \sum_{i=1}^n \lambda_i \|x_i - Tx_i\| \\
&\le \gamma^{-1}(2t) + t.
\end{aligned}$$

If α denotes the inverse function $t \to \gamma^{-1}(2t) + t$, we get the assertion. ■

Theorem 1.3 *Suppose C is a bounded convex subset of a uniformly convex Banach space X. Then there exists an increasing continuous function $h : \mathbb{R}^+ \to \mathbb{R}^+$ with $h(0) = 0$ depending on the diameter of C so that for any nonexpansive mapping $T : C \to X$ and any convex combination of elements $x_1, \ldots, x_n$ in C holds*

$$h\left(\left\|\sum_{i=1}^n \lambda_i x_i - T\left(\sum_{i=1}^n \lambda_i x_i\right)\right\|\right) \le \sum_{i=1}^n \lambda_i \|x_i - Tx_i\|.$$

Proof: 1. First we prove that there exists an increasing continuous function $\beta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\beta(0) = 0$ and $\beta(\varepsilon) \le \varepsilon$ such that

$$\beta(\|ax + by - T(ax + by)\|) \le a\|x - Tx\| + b\|y - Ty\| \tag{1}$$

for any $x, y \in C$ and any $a, b \ge 0$ with $a + b = 1$. For this purpose first we show: For any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\|ax + by - T(ax + by)\| \ge \varepsilon \tag{2}$$

implies

$$a\|x - Tx\| + b\|y - Ty\| \ge \delta. \tag{3}$$

Assume this is false. Then there exists a positve real number ε_0 such that for each integer n there are numbers $a_n, b_n \in [0,1]$ with $a_n + b_n = 1$ and elements $x_n, y_n \in C$ satisfying

$$\|a_n x_n + b_n y_n - T(a_n x_n + b_n y_n)\| \geq \varepsilon_0 \tag{4}$$

and

$$a_n\|x_n - Tx_n\| + b_n\|y_n - Ty_n\| < \frac{1}{n}. \tag{5}$$

We may assume that $a_n \to a$ and $b_n \to b = 1 - a$.

1.1. Suppose $\min(a,b) = 0$. Without loss of generality we may assume that $a = 1$ and $b = 0$. In view of (4) and the nonexpansivity of T we have

$$\begin{aligned}\|x_n - Tx_n\| \;&\geq\; \|a_n x_n + b_n y_n + T(a_n x_n + b_n y_n)\| \\ &\quad -\|T(a_n x_n + b_n y_n) - Tx_n\| - \|a_n x_n + b_n y_n - x_n\| \\ &\geq\; \varepsilon_0 - 2b_n\|x_n - y_n\|.\end{aligned}$$

For $b_n \to 0$ this implies $\liminf \|x_n - Tx_n\| \geq \varepsilon_0$. But (5) leads to $\lim \|x_n - Tx_n\| = 0$. This is a contradiction.

1.2. Suppose $\min(a,b) > 0$. Now (5) leads to $\lim \|x_n - Tx_n\| = 0$ and $\lim \|y_n - Ty_n\| = 0$. Since the function α of Corollary 1.2 is increasing, we have from (4) and this corollary

$$\alpha(\varepsilon_0) \leq \max(\|x_n - Tx_n\|,\ \|y_n - Ty_n\|).$$

So we get for $n \to \infty$ again a contradiction.

Thus we have proved that (2) follows from (1). If we put $a = 1$ and $b = 0$, then we see that $\delta \leq \varepsilon$. Obviously the function δ is increasing. Now we put $\beta(0) = 0$ and for $\varepsilon > 0$

$$\beta(\varepsilon) = \frac{1}{\varepsilon}\int_0^{\varepsilon} \delta(t)dt.$$

This function β is increasing and continuous. Because of $\beta(\varepsilon) \leq \delta(\varepsilon) \leq \varepsilon$ inequality (1) holds and it is $\beta(\varepsilon) \leq \varepsilon$, too.

2. Now we shall show that for each given $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that

$$S = \left\|\sum_{i=1}^{n} \lambda_i x_i - T\left(\sum_{i=1}^{n} \lambda_i x_i\right)\right\| \geq \varepsilon \tag{6}$$

implies

$$s = \sum_{i=1}^{n} \lambda_i \|x_i - Tx_i\| \geq \delta, \tag{7}$$

where we may choose

$$\delta = \delta(\varepsilon) = \alpha\left(\frac{\beta(\varepsilon)}{2}\right)\beta(\varepsilon)\min\left\{\frac{1}{2},\ \frac{1}{4d}\right\}, \tag{8}$$

with $d = \operatorname{diam} C$. We put $r = \alpha(\beta(\varepsilon)/2)$ and

$$A = \{i : \|x_i - Tx_i\| \le r\},\ \ B = \{i : \|x_i - Tx_i\| > r\}.$$

Let be

$$a = \sum_{i\in A}\lambda_i,\ \ b = \sum_{i\in B}\lambda_i = 1 - a.$$

Now we consider 3 cases:

2.1. Suppose $a = 0,\ b = 1$. In view of (8) we obtain by definition of s and B

$$s = \sum_{i\in B}\lambda_i\|x_i - Tx_i\| > r = \alpha\left(\frac{\beta(\varepsilon)}{2}\right) \ge \delta.$$

2.2. Suppose $a = 1,\ b = 0$. Since α is increasing, we have by Corollary 1.2

$$\alpha(\varepsilon) \le \alpha(S) \le s = \sum_{i\in A}\lambda_i\|x_i - Tx_i\| \le r = \alpha\left(\frac{\beta(\varepsilon)}{2}\right)$$

and therefore $\beta(\varepsilon) \ge 2\varepsilon$. This is a contradiction to $\beta(\varepsilon) \le \varepsilon$.

2.3. Finally let be $0 < a < 1,\ 0 < b < 1$. We put

$$x = \sum_{i\in A}\frac{\lambda_i}{a}x_i,\ \ y = \sum_{i\in B}\frac{\lambda_i}{b}x_i.$$

Then we get

$$S = \|ax + by - T(ax + by)\|.$$

We obtain by (1)

$$\beta(S) \le a\|x - Tx\| + b\|y - Ty\|. \tag{9}$$

By Corollary 1.2 follows

$$\alpha(\|x - Tx\|) = \alpha\left(\left\|\sum_{i\in A}\frac{\lambda_i}{a}x_i - T\left(\sum_{i\in A}\frac{\lambda_i}{a}x_i\right)\right\|\right) \le \max_{i\in A}\{\|x_i - Tx_i\|\} \le r$$

and therefore $\|x - Tx\| \le \alpha^{-1}(r)$. For $\|y - Ty\|$ we have the estimate

$$\begin{aligned}\|y - Ty\| &\le \|y - x\| + \|x - Tx\| + \|Tx - Ty\| \\ &\le \|x - Tx\| + 2\|x - y\| \\ &\le \alpha^{-1}(r) + 2d.\end{aligned}$$

Thus we obtain in view of (9)

$$\begin{aligned} \beta(S) &\leq a\alpha^{-1}(r) + b[\alpha^{-1}(r) + 2d] \\ &= \alpha^{-1}(r) + 2bd, \end{aligned} \tag{10}$$

since $a + b = 1$. Further it follows by definition of s

$$s \geq \sum_{i \in B} \lambda_i \|x_i - Tx_i\| \geq \sum_{i \in B} \lambda_i r \geq br$$

and therefore $b \leq s/r$. Because of $S \geq \varepsilon$ and the monotony of β we obtain from (10)

$$\beta(\varepsilon) \leq \beta(S) \leq \alpha^{-1}(r) + 2bd \leq \alpha^{-1}(r) + 2\frac{s}{r}d$$

and with $r = \alpha(\beta(\varepsilon)/2)$ and in view of (8) finally

$$s \geq \frac{r}{2d}\left[\beta(\varepsilon) - \alpha^{-1}(r)\right] = \frac{1}{2d}\alpha\left(\frac{\beta(\varepsilon)}{2}\right)\frac{\beta(\varepsilon)}{2} \geq \delta,$$

i.e. (7) follows from (6).

3. Finally we put $h(0) = 0$ and in view of (8) $h(\varepsilon) = \delta(\varepsilon)$ for $\varepsilon > 0$. The theorem is proved. ∎

2 Applications

Let X be a Banach space, $C \subset X$ and $T : C \to C$. For $\mu > 0$ set

$$F_\mu(T) = \{x \in C : \|x - Tx\| \leq \mu\}.$$

A direct consequence of Theorem 1.3 is the following result of Bruck (see also Goebel, Kirk [2], p. 113).

Corollary 2.1 ([1]) *Suppose C is a bounded, closed and convex subset of a uniformly convex Banach space X. Then for each $\varepsilon > 0$ there exists $\sigma > 0$ depending only on X, ε and* diam C *so that if $T : C \to C$ is nonexpansive then*

$$\overline{\operatorname{conv}}\, F_\sigma(T) \subseteq F_\varepsilon(T).$$

Lemma 2.2 *Suppose C is a bounded, closed and convex subset of a uniformly convex Banach space X. If $T : C \to X$ is nonexpansive and $x_n \to x$ (weakly) with $x_n \in C$ then*

$$h(\|x - Tx\|) \leq \liminf_{n \to \infty} \|x_n - Tx_n\|,$$

where h is the function given in Theorem 1.3.

Proof: Without loss of generality we may assume $\|x_n - Tx_n\| \to s$ for $n \to \infty$. Then for each $\varepsilon > 0$ there is an integer N so that

$$\|x_n - Tx_n\| < s + \varepsilon \qquad \text{as } n \geq N.$$

It is $x \in \overline{\text{conv}}\ \{x_n : n \geq N\}$. For this reason there is a finite convex combination of the elements x_n with $n \geq N$ such that

$$\left\| x - \sum_{i=N}^{k} \lambda_i x_i \right\| < \varepsilon.$$

T is nonexpansive, so we have by theorem 1.3

$$\begin{aligned}\|x - Tx\| &\leq \left\| x - \sum_{i=N}^{k} \lambda_i x_i \right\| + \left\| \sum_{i=N}^{k} \lambda_i x_i - T\left(\sum_{i=N}^{k} \lambda_i x_i\right)\right\| + \left\| T\left(\sum_{i=N}^{k} \lambda_i x_i\right) - Tx\right\| \\ &\leq 2\left\| x - \sum_{i=N}^{k} \lambda_i x_i \right\| + h^{-1}\left(\sum_{i=N}^{k} \lambda_i \|x_i - Tx_1\|\right) \\ &\leq 2\varepsilon + h^{-1}(s+\varepsilon).\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary and h^{-1} is continuous, the theorem follows for $\varepsilon \to 0$. ■

Corollary 2.3 (Demiclosedness Principle of Browder, 1968) *Let X be a uniformly convex Banach space, C a closed and convex subset of X and $T : C \to X$ nonexpansive. Then the mapping $I - T$ is demiclosed on C* (cf. Goebel, Kirk [2], p. 109).

In the following we shall prove the demiclosedness principle for asymptotically nonexpansive mappings (cf. also Lin, Tan, Xu [4]).

Corollary 2.4 ([4]) *Let C be a bounded closed convex subset of a uniformly convex Banach space and $T : C \to X$ an asymptotically nonexpansive mapping. Then $I - T$ is demiclosed at 0, i.e. for any sequence (x_n) in C holds: if $x_n \to x$ (weakly) and $(I-T)x_n \to 0$ (strongly) then $x = Tx$.*

Proof: Without loss of generality we may assume $0 \in C$ and diam $C \leq 1$. If q_m denotes the Lipschitz constant of the iterate T^m then $T_m = 1/q_m T^m : C \to X$ is a nonexpansive mapping and we have by Lemma 2.2

$$h(\|x - T_m x\|) \leq \liminf_{n\to\infty} \|x_n - T_m x_n\|$$

with x, x_n in C which means by the way that the norms are bounded by 1. Because of

$$\begin{aligned}\|x_n - T_m x_n\| = \left\| x_n - \frac{1}{q_m} T^m x_n \right\| &\leq \|x_n - T^m x_n\| + \left(1 - \frac{1}{q_m}\right)\|T^m x_n\| \\ &\leq \|x_n - T^m x_n\| + \left(1 - \frac{1}{q_m}\right) q_m \|x_n\| \\ &\leq \|x_n - T^m x_n\| + q_m - 1\end{aligned}$$

and

$$\|x_n - T^m x_n\| \le \sum_{i=0}^{m-1} \|T^i x_n - T^{i+1} x_n\| \le \|x_n - Tx_n\| \sum_{i=0}^{m-1} q_i \to 0$$

it follows

$$\limsup_{n\to\infty} \|x_n - T_m x_n\| \le q_m - 1.$$

Now we obtain in view of $\|T^m x\| \le q_m \|x\| \le q_m$ the estimate

$$\begin{aligned}\|x - T_m x\| &= \left\|x - \frac{1}{q_m} T^m x\right\| \ge \|x - T^m x\| - \left(1 - \frac{1}{q_m}\right) \|T^m x\| \\ &\ge \|x - T^m x\| - (q_m - 1)\end{aligned}$$

and therefore

$$\begin{aligned}\|x - T^m x\| &\le (q_m - 1) + \|x - T_m x\| \\ &\le (q_m - 1) + h^{-1}(q_m - 1).\end{aligned}$$

Hence we get

$$\lim_{m\to\infty} \|x - T^m x\| = 0.$$

Since T is continuous, x is a fixed point of T. The proof is complete. ■

A further application of our inequality is an ergodic theorem for nonexpansive mappings which are moreover asymptotically regular (cf. Hirano, Kido, Takahashi [3] and Tan, Xu [6]). A mapping $T : C \to C$ is said to be *asymptotically regular* if for any $x \in C$

$$\lim_{n\to\infty} \|T^n x - T^{n+1} x\| = 0$$

(cf. Goebel, Kirk, [2], p. 97). It is known that for nonexpansive T the mapping $T_\alpha = (1-\alpha)I + \alpha T$ with $0 < \alpha < 1$ is asymptotically regular.

Corollary 2.5 ([3], [6]) *Let C be a bounded closed convex subset of a uniformly convex Banach space with Fréchet differentiable norm and let T be a nonexpansive, asymptotically regular self-mapping of C. For $x \in C$ let be*

$$A_n x = \sum_{k=0}^{n-1} a_{nk} T^k x,$$

where (a_{nk}) is any regular matrix. Then $\{A_n x\}$ converges weakly to a fixed point of T.

Proof: By Theorem 1.3 we have

$$h(\|A_n x - TA_n x\|) \leq \sum_{k=0}^{n-1} a_{nk} \|T^k x - T^{k+1} x\|.$$

Since the matrix (a_{nk}) is regular and $\|T^k x - T^{k+1} x\| \to 0$ for $k \to \infty$ it follows by a limit theorem, due to Toeplitz, that $\|A_n x - TA_n x\| \to 0$ for $n \to \infty$. By Corollary 2.4 every weak cluster point of $\{A_n x\}$ is a fixed point of T. Now the weak convergence of the sequence $\{A_n x\}$ follows by a theorem due to Reich [5].

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