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Left K-completeness of the Hausdorff quasi-uniformity*

Dedicated to the professors of mathematics
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ABSTRACT. Left K-completeness of the Hausdorff quasi-uniformity is investigated. In particular the restriction of this quasi-uniformity to the (nonempty) compact subsets of a quasi-metric space is studied. Among other things it is shown that for any topological space the Hausdorff quasi-uniformity of the well-monotone quasi-uniformity is left K-complete.

KEY WORDS. Left K-complete, Smyth complete, Hausdorff quasi-uniformity, Bourbaki quasi-uniformity, well-monotone quasi-uniformity.

1 Introduction

For a quasi-uniform space $(X,\mathcal{U})$ we shall denote by $\mathcal{U}_*$ the Hausdorff quasi-uniformity on the set $\mathcal{P}_0(X)$ of nonempty subsets of X. In [13, Proposition 6] Künzi and Ryser characterized those quasi-uniform spaces $(X,\mathcal{U})$ for which $(\mathcal{P}_0(X),\mathcal{U}_*)$ is right K-complete. Similarly Künzi and Romaguera [11, Proposition 5] obtained a characterization of those quasi-uniform spaces $(X,\mathcal{U})$ for which $(\mathcal{P}_0(X),\mathcal{U}_*)$ is compact. On the other hand no useful characterization of those quasi-uniform spaces $(X,\mathcal{U})$ for which $(\mathcal{P}_0(X),\mathcal{U}_*)$ is left K-complete has been found, although it is known that a quasi-uniform space is compact if and only if it is left K-complete and precompact [9, Proposition 13] and that a quasi-uniform space $(X,\mathcal{U})$ is precompact if and only if $(\mathcal{P}_0(X),\mathcal{U}_*)$ is precompact [13, Proposition 1].

By an argument similar to [13, first part of proof of Proposition 6] a quasi-uniform space $(X,\mathcal{U})$ for which $(\mathcal{P}_0(X),\mathcal{U}_*)$ is left K-complete, satisfies the condition that each filter that is stable on $(X,\mathcal{U}^{-1})$ has a cluster point in $(X,\mathcal{U})$. (Let us mention that filters that are stable on $(X,\mathcal{U}^{-1})$ were called Császár filters in [17].) It was noted in [13, p. 169] (compare [18])

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that for quasi-pseudometric spaces the latter condition, which for arbitrary quasi-uniform spaces is stronger than left K-completeness, is indeed equivalent to left K-completeness. But for an arbitrary quasi-uniform space $(X,\mathcal{U})$ it obviously does not imply that $(\mathcal{P}_0(X),\mathcal{U}_*)$ is left K-complete (consider e.g. the compact space in [13, Remark 2]).

In this paper we wish to study this situation more carefully. In particular we investigate left K-completeness of the restriction of the Hausdorff quasi-uniformity to the (nonempty) compact subsets of a quasi-metric space. Among other things we show that for any topological space the Hausdorff quasi-uniformity of the well-monotone quasi-uniformity is left K-complete.

For basic facts about (quasi-)uniform (hyper)spaces we refer the reader to [1, 4, 14].

2 Preliminary results

We begin by recalling some definitions and collecting various basic results.

Let $(X,\mathcal{U})$ be a quasi-uniform space. It is called *precompact* (see e.g. [4]) provided that for each entourage $U \in \mathcal{U}$ there is a finite subset F of X such that $\{U(x) : x \in F\}$ covers X. A filter $\mathcal{F}$ on X is called a *left K-Cauchy filter* [19] provided that for each $U \in \mathcal{U}$ there is $F \in \mathcal{F}$ such that $U(x) \in \mathcal{F}$ whenever $x \in F$. A net $(x_d)_{d\in D}$ in X is called a *left K-Cauchy net* [9] provided that for each $U \in \mathcal{U}$ there is a $d_U \in D$ such that $(x_{d_1}, x_{d_2}) \in U$ whenever $d_1, d_2 \in D$ and $d_2 \geq d_1 \geq d_U$.

A quasi-uniform space is called *left K-complete* [19] provided that each left K-Cauchy filter (equivalently [9], each left K-Cauchy net) converges. It is called *right K-complete* [19] provided that each left K-Cauchy filter (equivalently [12], each left K-Cauchy net) with respect to the conjugate quasi-uniformity $\mathcal{U}^{-1}$ (usually, such filters resp. nets are called *right-K-Cauchy filters* resp. *right K-Cauchy nets* of $(X,\mathcal{U})$) converges in $(X,\mathcal{U})$.

A filter $\mathcal{F}$ on X is called *stable* [3] provided that $\cap_{F\in\mathcal{F}}U(F) \in \mathcal{F}$ whenever $U \in \mathcal{U}$. Each right K-Cauchy filter is stable [13, Lemma 5] and for ultrafilters stability and right K-Cauchyness are equivalent [19, Proposition 1].

A quasi-uniform space $(X,\mathcal{U})$ is called *Smyth completable* (*Smyth complete*), see [9, p. 322] and [21], provided that each left K-Cauchy filter (equivalently, left K-Cauchy net [22]) is Cauchy (converges) with respect to the supremum uniformity $\mathcal{U}^* = \mathcal{U} \vee \mathcal{U}^{-1}$. (Smyth originally requested convergence to a *unique* point in order to obtain T_0-spaces, but this is of no importance here.) It is called *bicomplete* [4] provided that the uniformity $\mathcal{U}^*$ is complete.

For any $U \in \mathcal{U}$ let $U_+ = \{(A,B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : B \subseteq U(A)\}$ and $U_- = \{(A,B) \in \mathcal{P}_0(X) \times \mathcal{P}_0(X) : A \subseteq U^{-1}(B)\}$. Furthermore set $U_* = (U_-) \cap (U_+)$ whenever $U \in \mathcal{U}$. Then $\{U_- : U \in \mathcal{U}\}$ is a base for *the lower quasi-uniformity* on $\mathcal{P}_0(X)$ and $\{U_+ : U \in \mathcal{U}\}$ is a base for *the upper quasi-uniformity* on $\mathcal{P}_0(X)$. Moreover $\mathcal{U}_* = \mathcal{U}_+ \vee \mathcal{U}_-$ is the so-called *Hausdorff* or *Bourbaki quasi-uniformity* of $(X, \mathcal{U})$ (see [1]).

In the following we shall denote the set of the nonempty compact sets of a quasi-uniform space $(X, \mathcal{U})$ by $\mathcal{K}_0(X)$. For simplicity we also denote the restriction of the Hausdorff quasi-uniformity to $\mathcal{K}_0(X)$ by $\mathcal{U}_*$.

Remark 1 *Let $(X, \mathcal{U})$ be a quasi-uniform space. Then $(\mathcal{P}_0(X), \mathcal{U}_*)$ is compact if and only if $(\mathcal{K}_0(X), \mathcal{U}_*)$ is compact.*

Proof: Suppose that $(\mathcal{K}_0(X), \mathcal{U}_*)$ is compact and let $(F_d)_{d\in D}$ be a net in $(\mathcal{P}_0(X), \mathcal{U}_*)$. First note that $(X, \mathcal{U})$ is compact: If $(x_e)_{e\in E}$ is a net in $(X, \mathcal{U})$ and C is a cluster point in $\mathcal{K}_0(X)$ of the net $(\{x_e\})_{e\in E}$, then any $c \in C$ is a cluster point of $(x_e)_{e\in E}$. Consequently $(X, \mathcal{U})$ is compact.

Thus $(\overline{F_d})_{d\in D}$ is a net in $\mathcal{K}_0(X)$. So, it has a cluster point C in $\mathcal{K}_0(X)$. Note now that C is also a cluster point of $(F_d)_{d\in D}$: For each $U \in \mathcal{U}$ and all $d \in D$ there is $d' \in D$ such that $d' \geq d$ and $\overline{F_{d'}} \subseteq U(C)$ and $C \subseteq U^{-1}(\overline{F_{d'}}) \subseteq U^{-2}(F_{d'})$. We conclude that $(\mathcal{P}_0(X), \mathcal{U}_*)$ is compact.

For the converse suppose that $(\mathcal{P}_0(X), \mathcal{U}_*)$ is compact and let $(K_d)_{d\in D}$ be a net in $(\mathcal{K}_0(X), \mathcal{U}_*)$. Then $(K_d)_{d\in D}$ has a cluster point $C \in \mathcal{P}_0(X)$. Since $(X, \mathcal{U})$ is compact [13, Corollary 3], we have that $\overline{C}$ is compact. Then for each $U \in \mathcal{U}$ and all $d \in D$ there is $d' \in D$ with $d' \geq d$ such that $K_{d'} \subseteq U(C) \subseteq U(\overline{C})$ and $C \subseteq U^{-1}(K_{d'})$; thus $\overline{C} \subseteq U^{-2}(K_{d'})$. We conclude that $\overline{C}$ is a cluster point of the net $(K_d)_{d\in D}$ and hence $(\mathcal{K}_0(X), \mathcal{U}_*)$ is compact.

Corollary 1 *Let $(X, \mathcal{U})$ be a quasi-uniform space. Then $(\mathcal{K}_0(X), \mathcal{U}_*)$ is compact if and only if $X \in \mathcal{K}_0(X)$ and $\mathcal{U}^{-1}|M$ is hereditarily precompact where M denotes the set of minimal elements with respect to the specialization quasi-order $\cap\mathcal{U}$ of $(X, \mathcal{U})$.*

Proof: The assertion is an immediate consequence of the preceding result and [11, Proposition 5].

Remark 2 *Let $(X, \mathcal{U})$ be a quasi-uniform space. Then $(\mathcal{K}_0(X), \mathcal{U}_*)$ is precompact if and only if $(X, \mathcal{U})$ is precompact.*

Proof: Only one minor modification in the second part of the proof of the corresponding result for $\mathcal{P}_0(X)$ [13, Proposition 1] is necessary: In the notation given there, if we had $X \setminus \bigcup_{A\in\mathcal{A}} V(x_A) \neq \emptyset$, choose a point y of this set. Then there exists $A \in \mathcal{A}$ such that $A \subseteq V^{-1}(y)$. Thus $y \in V(x_A)$— a contradiction.

The well-monotone quasi-uniformity was introduced by Junnila [6]. It is generated by the transitive neighbornets determined by the well-monotone open covers of a topological space via Fletcher's construction; see e.g. [4, Theorem 2.6]. The following result implies [10, Proposition 1] and [17, Proposition 2].

Proposition 1 *Let $\mathcal{W}$ be the well-monotone quasi-uniformity of a topological space X. Then $(\mathcal{P}_0(X), \mathcal{W}_*)$ is left K-complete.*

Proof: Let $(F_d)_{d\in D}$ be a left K-Cauchy net in $(\mathcal{P}_0(X), \mathcal{W}_*)$, i.e. for each $U \in \mathcal{W}$ there is $d_U \in D$ such that $d_1, d_2 \in D$ and $d_2 \geq d_1 \geq d_U$ imply that $F_{d_2} \subseteq U(F_{d_1})$ and $F_{d_1} \subseteq U^{-1}(F_{d_2})$. Consider the filter $\mathcal{F}$ on X generated by $\{E_d : d \in D\}$ where $E_d = \bigcup_{e\in D, e\geq d} F_e$. Then $\mathcal{F}$ is a Császár filter, i.e. $\cap_{F\in\mathcal{F}} U^{-1}(F) \in \mathcal{F}$ whenever $U \in \mathcal{W}$: Let $x \in E_{d_U}$ and $d \in D$. Therefore $x \in F_{d_0}$ for some $d_0 \in D$ such that $d_0 \geq d_U$. Choose $h \in D$ such that $h \geq d_0, d$. Observe that $x \in F_{d_0} \subseteq U^{-1}(F_h) \subseteq U^{-1}(E_d)$. We conclude that $E_{d_U} \subseteq \bigcap_{d\in D} U^{-1}(E_d)$ whenever $U \in \mathcal{W}$.

Note next that each filter $\mathcal{F}$ that is Császár with respect to the well-monotone quasi-uniformity contains its set C of cluster points (see [17, proof of Proposition 2]).

Hence $C = \bigcap\{\overline{F} : F \in \mathcal{F}\} \in \mathcal{F}$; in particular for each $U \in \mathcal{W}$ there is $d \in D$ such that $\bigcup_{d'\in D, d'\geq d} F_{d'} \subseteq C \subseteq U(C)$. Let $U \in \mathcal{W}$ and let $W \in \mathcal{W}$ be such that $W^4 \subseteq U$. Since $\mathcal{W}^{-1}$ is hereditarily precompact [9, p. 327], there is a nonempty finite set of points $E \subseteq C$ such that $C \subseteq W^{-1}(E)$. Let $e \in E$. Since e is a cluster point of $\mathcal{F}$, there is $d_e \in D$ with $d_e \geq d_W$ such that $W(e) \cap F_{d_e} \neq \emptyset$. Choose $f \in D$ such that $f \geq d_e$ whenever $e \in E$. Then given $c \in C$, we find $e \in E$ such that $c \in W^{-1}(e)$. Thus $W(e) \subseteq W^2(c)$ and $\emptyset \neq W^2(c) \cap F_{d_e} \subseteq W^2(c) \cap W^{-1}(F_f)$. Therefore, for any $l \in D$ such that $l \geq f$ we deduce that $C \subseteq W^{-3}(F_f) \subseteq W^{-3}W^{-1}(F_l) \subseteq U^{-1}(F_l)$. We conclude that $(F_d)_{d\in D}$ converges to C in $(\mathcal{P}_0(X), \mathcal{W}_*)$. Hence $(\mathcal{P}_0(X), \mathcal{W}_*)$ is left K-complete.

Question 1 If $(X, \mathcal{U})$ is left K-complete (or if in $(X, \mathcal{U})$ each Császár filter clusters) and $\mathcal{U}^{-1}$ is hereditarily precompact, is $(\mathcal{P}_0(X), \mathcal{U}_*)$ necessarily left K-complete? Note that the answer to the first version of the question is positive by [11, Proposition 5] provided that $(X, \mathcal{U})$ is precompact.

Remark 3 Let $(X, \mathcal{U})$ be a complete uniform space with a stable filter that has no cluster point (see e.g. [5, p. 31]). Then $(\mathcal{K}_0(X), \mathcal{U}_*)$ is complete by Morita's celebrated result [15], but $(\mathcal{P}_0(X), \mathcal{U}_*)$ is not complete. Hence in this way we obtain a quasi-uniform space $(X, \mathcal{U})$ such that $(\mathcal{K}_0(X), \mathcal{U}_*)$ is right K-complete and left K-complete, but $(\mathcal{P}_0(X), \mathcal{U}_*)$ is neither right-K-complete nor left-K-complete.

Example 1 *There exists a quasi-uniform space $(X, \mathcal{U})$ such that $(\mathcal{P}_0(X), \mathcal{U}_*)$ is right K-complete, but $(\mathcal{K}_0(X), \mathcal{U}_*)$ is not right K-complete.*

Let $X = \mathbf{R}$ be the set of the reals and define $U_n = \{(x, y) : x \in X, y < -2^n\} \cup \{(x, y) : x \in X, x - 2^{-n} < y < x + 2^{-n}\}$ for each $n \in \omega$. Let $\mathcal{U}$ be the quasi-uniformity on X generated by $\{U_n : n \in \omega\}$. In [2, Example 3.9] it is shown that $(\mathcal{K}_0(X), \mathcal{U}_*)$ is not right K-complete. In order to see that $(\mathcal{P}_0(X), \mathcal{U}_*)$ is right K-complete, it suffices to show that each stable filter in $(X, \mathcal{U})$ has a cluster point [13, Proposition 6]. Suppose that $\mathcal{F}$ is a stable filter on

$(X,\mathcal{U})$ and assume that for each $n \in \omega$ there is $F_n \in \mathcal{F}$ such that $\emptyset =]\leftarrow, n] \cap F_n$. Then $\cap_{F\in\mathcal{F}} U_1(F) \subseteq]\leftarrow, 0]$, contradicting the stability of $\mathcal{F}$, since $]\leftarrow, 0] \cap F_1 = \emptyset$. Thus there exists $n \in \omega$ such that for all $F \in \mathcal{F}$, $]\leftarrow, n] \cap F \neq \emptyset$. Since $]\leftarrow, n]$ is compact in $(X,\mathcal{U})$, we conclude that $\mathcal{F}$ has a cluster point in $(X,\mathcal{U})$.

Question 2 Suppose that $(\mathcal{P}_0(X), \mathcal{U}_*)$ is left K-complete. Is $(\mathcal{K}_0(X), \mathcal{U}_*)$ left K-complete? Is $(\mathcal{K}_0(X), \mathcal{W}_*)$ left K-complete? Here $\mathcal{W}$ denotes the well-monotone quasi-uniformity of a topological space X.

We next give an example of a quasi-uniform T_1-space $(X,\mathcal{U})$ that is not Smyth completable, although $(\mathcal{K}_0(X), \mathcal{U}_*)$ is left K-complete. The set of nonempty finite subsets of the set X will be denoted by $\mathcal{F}_0(X)$. This example should be compared with Corollary 2 in the next section.

Example 2 Let $X = \omega_1 + 1$. For each $\alpha \in \omega_1$ define $T_\alpha(\beta) = \{\beta\}$ if $\beta < \alpha$, $T_\alpha(\beta) = [\beta, \omega_1)$ if $\alpha \leq \beta < \omega_1$, and $T_\alpha(\beta) = [\alpha, \omega_1]$ if $\beta = \omega_1$. Note that $\{T_\alpha : \alpha \in \omega_1\}$ yields a base for a transitive quasi-uniformity $\mathcal{U}$ on X such that the uniformity $\mathcal{U}^*$ is discrete. Evidently $\mathcal{K}_0(X) = \mathcal{F}_0(X)$, since $\mathcal{U}$ induces the topology of the one-point-Lindelöfication of $\aleph_1$ on X. Note that $(\alpha)_{\alpha\in(\omega_1,\leq)}$ is a left K-Cauchy net that is not Cauchy with respect to $\mathcal{U}^*$. Thus $\mathcal{U}$ is not Smyth completable. We wish to show that $(\mathcal{K}_0(X), \mathcal{U}_*)$ is left K-complete. To this end suppose that $(F_d)_{d\in D}$ is a left K-Cauchy net in $\mathcal{K}_0(X)$. Then for each $U \in \mathcal{U}$ there is $d_U \in D$ such that $d_1, d_2 \in D$ and $d_2 \geq d_1 \geq d_U$ imply that $F_{d_2} \subseteq U(F_{d_1})$ and $F_{d_1} \subseteq U^{-1}(F_{d_2})$. We begin the proof with some general observations.

Let $\alpha \in \omega_1$. Note that $\bigcup_{d\geq d_{T_\alpha}} F_d$ contains at most finitely many ordinals smaller than α, because $\bigcup_{d\geq d_{T_\alpha}} F_d \subseteq T_\alpha(F_{d_{T_\alpha}})$; indeed, since for any $x < \alpha$ we have $T_\alpha^{-1}(x) = \{x\}$, we conclude that $x < \alpha$ and $x \in \bigcup_{d\geq d_{T_\alpha}} F_d$ imply that $x \in F_{d_{T_\alpha}}$.

Suppose that $M = \{x \in \omega_1 :$ There is a cofinal subset C_x of D such that $x \in F_d$ whenever $d \in C_x\}$ contains a countably infinite subset C. Choose $\alpha \in \omega_1$ such that $c < \alpha$ whenever $c \in C$. For any $c \in C$ there exists $d_c \in D$ such that $d_c \geq d_{T_\alpha}$ and $c \in F_{d_c}$. Thus $C \subseteq \bigcup_{d\geq d_{T_\alpha}} F_d$, a contradiction, since $\bigcup_{d\geq d_{T_\alpha}} F_d$ contains only finitely many ordinals smaller than α. Thus M is finite.

Let $\beta \in \omega_1$ be larger than any element of M. Since $F_{d_2} \subseteq T_\beta(F_{d_1})$ whenever $d_1, d_2 \in D$ and $d_2 \geq d_1 \geq d_{T_\beta}$, we conclude by a similar argument as above that $M \subseteq F_{d'}$ whenever $d' \in D$ and $d' \geq d_{T_\beta}$. Now we are ready for the main idea of the proof.

Case 1: Suppose that there exist $d \in D$ and $\alpha \in \omega_1$ such that $F_{d'} \cap [\alpha, \omega_1] = \emptyset$ whenever $d' \in D$ and $d' \geq d$. Choose $d'' \in D$ such that $d'' \geq d, d_{T_\alpha}$. Then $d_2 \in D$ and $d_2 \geq d''$ imply that $F_{d_2} \subseteq T_\alpha(F_{d''}) = F_{d''}$ by our assumption just made. Since $F_{d''}$ is finite and D is directed, we conclude that there is a cofinal subset C of D such that $F_c = F_{c'}$ whenever $c, c' \in C$. Thus $(F_d)_{d\in D}$ has a cluster point and therefore converges as a left K-Cauchy net [19, Lemma 1].

Case 2: Suppose now that for all $\alpha \in \omega_1$ and all $d \in D$ there exists $d' \in D$ such that $d' \geq d$ and $F_{d'} \cap [\alpha, \omega_1] \neq \emptyset$.

We show that $M \cup \{\omega_1\}$ is a cluster point of $(F_d)_{d\in D}$: Let $\alpha \in \omega_1$ and choose $\beta > \sup M$ such that $\beta > \alpha$. Since $\bigcup_{d \geq d_{T_\beta}} F_d$ contains only finitely many elements smaller than β, we find $e \in D$ such that $e \geq d_{T_\beta}$ and $(\bigcup_{d\geq e} F_d) \setminus [\beta, \omega_1] \subseteq M$. Therefore $\bigcup_{d\geq e} F_d \subseteq T_\beta(M \cup \{\omega_1\}) \subseteq T_\alpha(M \cup \{\omega_1\})$.

Let $d \in D$ be arbitrary. Since $M \subseteq F_{d^*}$ whenever $d^* \in D$ and $d^* \geq d_{T_\beta}$ by the observation made in the beginning of the proof and since by our assumption there is $d'' \in D$ such that $d'' \geq d, e$ and $F_{d''} \cap [\beta, \omega_1] \neq \emptyset$, we deduce that $M \cup \{\omega_1\} \subseteq T_\beta^{-1}(F_{d''}) \subseteq T_\alpha^{-1}(F_{d''})$. We have shown that the left K-Cauchy net $(F_d)_{d\in D}$ has a cluster point and thus converges in $(\mathcal{K}_0(X), \mathcal{U}_*)$.

Therefore $(\mathcal{K}_0(X), \mathcal{U}_*)$ is left K-complete.

3 Main results

As usual, a *quasi-(pseudo)metric* d on a set X is a (pseudo)metric except that it does not necessarily satisfy the symmetry condition; furthermore d^{-1} will denote the conjugate of d. For each $n \in \omega$ we set $Z_n := \{(x, y) \in X \times X : d(x, y) < 2^{-n}\}$.

Lemma 1 *Let A be a precompact subspace of a quasi-uniform T_1-space $(X, \mathcal{U})$ where $(\mathcal{K}_0(X), \mathcal{U}_*)$ is left K-complete. Then $(A, \mathcal{U}^{-1}|A)$ is precompact.*

Proof: Suppose that A is nonempty. Let $[A]^{<\omega}$ be the set of nonempty finite subsets of A directed by set-theoretic inclusion. Then $[A]^{<\omega}$ can be considered a left K-Cauchy net in $(\mathcal{K}_0(X), \mathcal{U}_*)$: Indeed by precompactness of A for any $U \in \mathcal{U}$ there is $A_U \in [A]^{<\omega}$ such that $A \subseteq U(A_U)$. Thus for any $B, C \in [A]^{<\omega}$ such that $A_U \subseteq B \subseteq C$ we have that $C \subseteq A \subseteq U(A_U) \subseteq U(B)$ and $B \subseteq U^{-1}(C)$.

Then $[A]^{<\omega}$ converges to some $C \in \mathcal{K}_0(X)$ by left K-completeness. Suppose that there is $a \in A$ such that $a \notin C$. Consequently in the light of [4, Proposition 1.43] there exists $U \in \mathcal{U}$ such that $a \notin U(C)$ by compactness of C and the T_1-property of $(X, \mathcal{U})$. Since $[A]^{<\omega}$ converges to C, there is $B \in [A]^{<\omega}$ such that $\{a\} \subseteq B$ and $B \subseteq U(C)$— a contradiction. Thus $A \subseteq C$. Furthermore for any $U \in \mathcal{U}$, there is $B \in [A]^{<\omega}$ such that $C \subseteq U^{-1}(B)$. Since $A \subseteq C$, we conclude that A is precompact in $(X, \mathcal{U}^{-1})$.

Corollary 2 *Let (X, d) be a quasi-metric space such that $(\mathcal{K}_0(X), (\mathcal{U}_d)_*)$ is left K-complete. Then $(X, \mathcal{U}_d)$ is Smyth completable.*

Proof: By a result of Schellekens [20, Proposition 4] it suffices to see that each left K-Cauchy sequence $(x_n)_{n\in\omega}$ in (X,d) is Cauchy with respect to the supremum metric $d^* = d \vee d^{-1}$. But this follows from the preceding result, since as a left K-Cauchy sequence, $P = \{x_n : n \in \omega\}$ is (hereditarily) precompact in (X,d). Let $\mathcal{F}$ be the filter generated by $\{\{x_n : n \in \omega, n \geq k\} : k \in \omega\}$ on P. Thus it is a left K-Cauchy filter and stable by [9, p. 315], since $d^{-1}|P$ is hereditarily precompact by the preceding result; therefore it is a Cauchy filterbase with respect to $(\mathcal{U}_d)^*$ (see [9, p. 320]). It follows that the sequence $(x_n)_{n\in\omega}$ is Cauchy in (X,d^*).

The following example shows that for a quasi-metric space (X,d) the condition that $(\mathcal{K}_0(X),(\mathcal{U}_d)_*)$ is left K-complete does not imply that $(X,\mathcal{U}_d)$ is Smyth complete.

Example 3 Let $X = \mathbf{N}$ be the set of positive integers equipped with the cofinite topology. Consider a countable base $\mathcal{B}$ for the topology of X where we suppose that for each $n \in \mathbf{N}$, $\mathbf{N} \setminus \{n\}$ belongs to $\mathcal{B}$. Let $\mathcal{U}$ be the quasi-metrizable quasi-uniformity generated by $\{[(X \setminus G) \times X] \cup [X \times G] : G \in \mathcal{B}\}$. Then $(\mathcal{P}_0(X),\mathcal{U}_*)$ is compact by [11, Proposition 5], because $\mathcal{U}$ is totally bounded and $(X,\mathcal{U})$ is compact; hence $(\mathcal{P}_0(X),\mathcal{U}_*)$ is left K-complete. Since X is hereditarily compact, $\mathcal{P}_0(X) = \mathcal{K}_0(X)$. Let $\mathcal{G}$ be a free ultrafilter on $(X,\mathcal{U})$. Since $\mathcal{U}$ is totally bounded, $\mathcal{G}$ is $\mathcal{U}^*$-Cauchy [4, Proposition 3.14], but because the topology induced by $\mathcal{U}^{-1}$ is discrete, it does not converge with respect to this topology. In particular, $(X,\mathcal{U})$ is not bicomplete and thus not Smyth complete. (Readers who prefer to work with an explicit quasi-metric d on $\mathbf{N}$ may wish to consider the classical example: $d(n,m) = 1/m$ if n and m are distinct, and $d(n,m) = 0$ otherwise.)

Proposition 2 *Let (X,d) be a quasi-metric space such that $\mathcal{U}_d$ is bicomplete. Then $(X,\mathcal{U}_d)$ is Smyth complete provided that $(\mathcal{K}_0(X),(\mathcal{U}_d)_*)$ is left K-complete.*

Proof: The result follows immediately from Corollary 2 above.

Finally we give a sufficient condition that for a quasi-metric space (X,d) the Hausdorff quasi-uniformity on the nonempty compact subsets of (X,d) is left K-complete.

Proposition 3 *Let (X,d) be a quasi-metric space such that $\mathcal{U}_d$ is Smyth complete and such that on each compact subset K of (X,d), $d^{-1}|K$ is precompact. Then $(\mathcal{K}_0(X),(\mathcal{U}_d)_*)$ is left K-complete.*

Proof: By [18] it will be sufficient to show that each left K-Cauchy sequence in $(\mathcal{K}_0(X),(\mathcal{U}_d)_*)$ converges. Let $(K_n)_{n\in\omega}$ be a left K-Cauchy sequence in $(\mathcal{K}_0(X),(\mathcal{U}_d)_*)$.

Set $C = \bigcap\{\mathrm{cl}_{\tau(d^{-1})}(\cup_{s\geq p} K_s) : p \in \omega\}$. We choose a base $\{U_n : n \in \omega\}$ of entourages of the quasi-uniformity $\mathcal{U}_d$ such that $U_{n+1}^3 \subseteq U_n \subseteq Z_n$ whenever $n \in \omega$.

For each $n \in \omega$ there is $k_n \in \omega$ such that $a, b \in \omega$ and $k_n \leq a \leq b$ imply that $K_b \subseteq U_n(K_a)$ and $K_a \subseteq U_n^{-1}(K_b)$. Without loss of generality we can suppose that the sequence $(k_n)_{n\in\omega}$ is strictly increasing. Thus $K_{k_{n+1}} \subseteq U_n(K_{k_n})$ whenever $n \in \omega$.

We first show that $C \neq \emptyset$: For each $n \in \omega$ choose a finite subset F_{k_n} of K_{k_n} such that $K_{k_n} \subseteq U_n(F_{k_n})$. Thus $F_{k_{n+1}} \subseteq K_{k_{n+1}} \subseteq U_n(K_{k_n}) \subseteq (U_n)^2(F_{k_n})$. Define a binary relation R on $\bigcup_{n\in\omega} F_{k_n}$ as follows. For any $n \in \omega$, $f \in F_{k_{n+1}}$, $g \in F_{k_n}$ set gRf iff $f \in (U_n)^2(g)$. By König's Lemma [7] there is a sequence $(a_n)_{n\in\omega}$ such that $a_n R a_{n+1}$ and $a_n \in F_{k_n}$ whenever $n \in \omega$. Since (X, d) is a quasi-metric space, $(a_n)_{n\in\omega}$ is a left K-Cauchy sequence in (X, d). By Smyth completeness, $(a_n)_{n\in\omega}$ converges to some x in (X, d^*). Thus $x \in C$. We have shown that $C \neq \emptyset$.

We are going to show next that $C \in \mathcal{K}_0(X)$: Since (X, d) is quasi-metrizable, it suffices to show that C is a countably compact subspace of (X, d) [16]. Suppose that $(c_n)_{n\in\omega}$ is an arbitrary sequence in C. By the definition of C for each $n \in \omega$ we can choose $b_n \in \omega$ and $x_n \in K_{b_n}$ such that $b_n \geq k_n$ and $d(x_n, c_n) < 1/n$. Apply König's Lemma now to the sequence $(E_n)_{n\in\omega}$ of (finite) level sets, where for each $n \in \omega$, $E_n = \{f \in F_{k_n} : U_{n-1}(f)$ contains x_m for infinitely many $m\}$: First note that each $E_n \neq \emptyset$, because $K_{b_m} \subseteq U_n(K_{k_n})$ and thus $K_{b_m} \subseteq (U_n)^2(F_{k_n})$ whenever $n, m \in \omega$ and $m > n$. One checks as in [8, Démonstration du Théorème 2] that the assumption of König's Lemma is satisfied for the relation Q defined on $\bigcup_{n\in\omega} E_n$ by gQf if $n \in \omega$, $f \in E_{k_{n+1}}, g \in E_{k_n}$ and $f \in (U_n)^2(g)$. Hence there are a left K-Cauchy sequence $(a_n)_{n\in\omega}$ with $a_n \in E_n$ and a strictly increasing sequence $(l_n)_{n\in\omega}$ of nonnegative integers such that $d(a_n, x_{l_n})$ converges to 0. By Smyth completeness of $(X, \mathcal{U}_d)$ we conclude that there is $x \in X$ such that $d^*(x, a_n)$ converges to 0. Thus $x \in C$ by definition of C. Furthermore x is a cluster point of $(c_n)_{n\in\omega}$ in (X, d). Consequently C is (countably) compact in (X, d).

Note that if in the proof above we suppose that c_n is equal to some fixed $c \in C$ whenever $n \in \omega$, then $d(x, c) = 0$, thus $x = c$. In particular, since then $d(c, a_n)$ converges to 0, in this way we see that $C \subseteq \bigcap_{p\in\omega}(\mathrm{cl}_{\tau(d)} \cup_{s\geq p} K_s)$.

Finally we want to show that $(K_n)_{n\in\omega}$ converges to C in $(\mathcal{K}_0(X), (\mathcal{U}_d)_*)$. Because by our assumption C is precompact in (X, d^{-1}), the inequality obtained in the last paragraph together with the argument given at the end of the proof of Proposition 1 yields that for each $U \in \mathcal{U}_d$ there is $c \in \omega$ such that for any $b \in \omega$ satisfying $b \geq c$ we have that $C \subseteq U^{-1}(K_b)$.

Suppose now that there are $p \in \omega$ and a strictly increasing sequence $(a_n)_{n\in\omega}$ such that $K_{a_n} \setminus Z_p(C) \neq \emptyset$ whenever $n \in \omega$.

We can assume that $a_n \geq k_n$ whenever $n \in \omega$. Thus $K_{a_{n+1}} \subseteq Z_n(K_{a_n})$. For each $n \in \omega$ set $B_n = K_{a_{p+n+3}} \setminus Z_{p+n+2}(\ldots(Z_{p+1}(C)))$. Then for each $n \in \omega$, $B_n \in \mathcal{K}_0(X)$ and one readily checks that $B_{n+1} \subseteq Z_{p+n+3}(B_n)$.

By the same argument as in the first part of the proof, by compactness of each B_n and König's Lemma for each $n \in \omega$ there is $b_n \in B_n$ such that $(b_n)_{n\in\omega}$ is a left K-Cauchy sequence in (X, d). Then by Smyth completeness there is $x \in X$ such that $d^*(x, b_n)$ converges to 0. Hence $x \in C$, by definition of C. Furthermore, since $d(x, b_n)$ converges to 0, we see that $x \notin Z_{p+1}(C)$, a contradiction. We conclude that $(K_n)_{n\in\omega}$ converges to C and, hence, $(\mathcal{K}_0(X), (\mathcal{U}_d)_*)$ is left K-complete.

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