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Fell topology and uniform topology on compacta on spaces of multifunctions

Dedicated to the professors of mathematics
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ABSTRACT. The set $2^{X\times Y}$ of closed subsets of $X \times Y$ may be identified with the set of upper semicontinuous multifunctions from X into 2^Y. This set contains the set $C(X,Y)$ of continuous functions and the larger set $D(X,Y)$ of densely continuous forms. In this paper, the Fell topology (a hyperspace topology) and the uniform topology on compacta (a function space topology) are both imposed on $2^{X\times Y}$ and compared. Conditions are determined for the subspace $D(X,Y)$ to be dense in $2^{X\times Y}$ under the Fell topology and to be closed in $2^{X\times Y}$ under the uniform topology on compacta.

KEY WORDS. Hyperspace, multifunction, Fell topology, uniform topology on compacta, densely continuous form.

The set $C(X,Y)$ of continuous functions from X into Y is a subset of the set $2^{X\times Y}$ of all closed subsets of $X \times Y$ under the identification of each function with its graph. So any hyperspace topology on $2^{X\times Y}$ induces a topology on $C(X,Y)$.

The hyperspace topology that we use is the Fell topology. For a space Z, the Fell topology on the set 2^Z of closed subsets of Z is the topology with the following two kinds of subbasic open sets. The "hit sets" are the sets

$$W^- = \{A \in 2^Z : A \cap W \neq \emptyset\}$$

where W is an open subset of Z, and the "miss sets" are the sets

$$K^+ = \{A \in 2^Z : A \cap K = \emptyset\}$$

where K is a compact subset of Z. Whenever Z is a locally compact Hausdorff space, 2^Z is a compact Hausdorff space. If Z is compact, then $\emptyset$ is an isolated point in 2^Z; so in this case we use $2^Z_* \equiv 2^Z \setminus \{\emptyset\}$. See [2], [3], or [5] for properties of the Fell topology.

Let $C_F(X,Y)$ be the set $C(X,Y)$ with the topology inherited from the Fell topology on $2^{X\times Y}$. Then if X and Y are locally compact Hausdorff spaces, the closure of $C(X,Y)$ in $2^{X\times Y}$ is a compactification of $C_F(X,Y)$.

If Y is a metric space, $C_k(X,Y)$ denotes the set $C(X,Y)$ with the uniform topology on compacta (compact-open topology). This topology on $C(X,Y)$ is finer than the Fell topology, and is equal to it whenever X is locally connected [7]. Topological properties of $C_k(X,Y)$ can be found in, for example, in [1], [4], or [8].

A larger function space that is also contained in $2^{X\times Y}$ is the space $D(X,Y)$ of densely continuous forms from X into Y. This is defined as follows. First let $DC(X,Y)$ be the set of all functions f from X into Y whose set of points of continuity, $C(f)$, is a dense subset of X. Then for such f, let $\overline{f}$ be the closure of $f|_{C(f)}$ in $X\times Y$. The set $D(X,Y)$ is the set of such forms $\overline{f}$ for all $f\in DC(X,Y)$. Then $D(X,Y)$ is a subset of $2^{X\times Y}$ that contains $C(X,Y)$. We denote this space having the inherited Fell topology from $2^{X\times Y}$ by $D_F(X,Y)$.

Each element Φ of $2^{X\times Y}$ can be thought of as a function from X into 2^Y by letting

$$\Phi(x)=\{y\in Y:(x,y)\in\Phi\}$$

for all $x\in X$. This is also referred to as a multifunction from X to Y. We define a function Φ from X into 2^Y to be *upper (Fell) semicontinuous* provided that for each compact set K in Y, $\Phi^{-1}(K^+)$ is open in X. Let $U(X,2^Y)$ be the set of all such upper semicontinuous functions from X into 2^Y. Then $2^{X\times Y}$ may be identified with $U(X,2^Y)$, as given by Proposition 1 below. The following well-known fact (see tube lemma in [9]) is used in Proposition 1 and Theorems 3, 4 and 5.

Lemma 1 *If A and B are compact subsets of X and Y, respectively, and if W is an open subset of $X\times Y$ containing $A\times B$, then there exist open sets U and V in X and Y, respectively, such that $A\times B\subseteq U\times V\subseteq W$.*

Proposition 2 *If Y is a locally compact Hausdorff space, then $2^{X\times Y}=U(X,2^Y)$.*

Proof: Let $\Phi\in 2^{X\times Y}$. To show that Φ is upper semicontinuous, let $x\in X$ and let K be a compact subset of Y such that $\Phi(x)\in K^+$. Then Φ and $\{x\}\times K$ are disjoint, so that by Lemma 1, there exists a basic open set $U\times V$ in $X\times Y$ containing $\{x\}\times K$ and disjoint from Φ. Then U is a neighborhood of x such that $\Phi(U)\subseteq K^+$.

Now for the other direction, let $\Phi\in U(X,2^Y)$. To show that Φ is closed in $X\times Y$, let $(x,y)\in X\times Y\setminus\Phi$. Then y has a compact neighborhood K that is disjoint from $\Phi(x)$. So $\Phi(x)\in K^+$, and hence x has a neighborhood U such that $\Phi(U)\subseteq K^+$. Therefore $U\times K$ is a neighborhood of (x,y) in $X\times Y$ that is disjoint from Φ. $\square$

Since $2^{X\times Y}$ has the Fell topology and is equal to $U(x, 2^Y)$ when X is locally compact, we now denote this hyperspace space by $U_F(X, 2^Y)$, and we think of it as a function space. In the case that Y is compact, we use the subspace $U_F(X, 2^Y_*)$. The following lemma gives a convenient base for the topology on this space.

Lemma 3 *If X and Y are locally compact Hausdorff spaces, then the topology on $U_F(X, 2^Y)$ is generated by the subbasic open sets of the form $(U\times V)^-$ and $(L\times M)^+$, where U is open in X, V is open in Y, L is compact in X, and M is compact in Y.*

Proof: Let W be an open subset of $X\times Y$, and let $\Phi\in W^-$. Then there exists an $(x,y)\in W\cap\Phi$. If $U\times V$ is a basic open neighborhood of (x,y) contained in W, then $\Phi\in(U\times V)^-\subseteq W^-$.

Now let K be a compact subset of $X\times Y$, and let $\Phi\in K^+$. For every $z\in K$, there is a basic open neighborhood $U_z\times V_z$ of z in $X\times Y$ such that $L_z\times M_z\subseteq X\times Y\setminus\Phi$ where L_z and M_z are the closures of U_z and V_z and are compact. Because K is compact, there are $z_1,\dots,z_n\in K$ with

$$K\subseteq(U_{z_1}\times V_{z_1})\cup\dots\cup(U_{z_n}\times V_{z_n}).$$

Then

$$\Phi\in(L_{z_1}\times M_{z_1})^+\cap\dots\cap(L_{z_n}\times M_{z_n})^+\subseteq K^+,$$

which completes the proof. □

As a function space, $U(X, 2^Y)$ can also have the uniform topology on compacta whenever Y is a metric space. This space, denoted by $U_k(X, 2^Y)$, is defined as follows. Let H be the Hausdorff metric on 2^Y induced by the metric on Y (take the distance between the empty set and a nonempty set to be ∞). Then the basic open sets for the topology on $U_k(X, 2^Y)$ are the sets

$$\langle\Phi, A, \varepsilon\rangle=\{\Psi\in U(X, 2^Y): H(\Phi(x),\Psi(x))<\varepsilon \text{ for all } x\in A\}\ ,$$

where $\Phi\in U(X, 2^Y)$, A is a compact subset of X, and $\varepsilon>0$. Also for compact Y, we use the subspace $U_k(X, 2^Y_*)$.

The subset $D(X,Y)$ of $U_k(X, 2^Y)$ with the subspace topology is denoted by $D_k(X,Y)$. This space has been studied in [6], where an Ascoli theorem is established for it, and its properties such as metrizability are characterized. Note that $D_k(X,Y)$ contains $C_k(X,Y)$ as a subspace, and when Y is compact $D_k(X,Y)$ is a subspace of $U_k(X, 2^Y_*)$.

For the space of continuous functions $C(X,Y)$, the comparison of the Fell topology and the uniform topology on compacta is given in [10] and [7]. The following theorem extends this comparison to the multifunction space $U(X,Y)$.

Theorem 4 *Let X be a locally compact Hausdorff space, and let Y be a nontrivial locally compact metric space. Then $U_k(X,2^Y)$ has finer topology than $U_F(X,2^Y)$. Furthermore, the topologies of these two spaces are equal if and only if X is discrete and Y is compact.*

Proof: Denote the metric on Y by d and the induced Hausdorff metric on 2^Y by H. First let $\Phi \in W^-$, where $W = U \times V$. Then there exists an $(x,y) \in W \cap \Phi$; and so for some ε, the ε-ball centered at y, $B(y,\varepsilon)$, is contained in V. Let A be a compact neighborhood of x contained in U. If $\Psi \in \langle \Phi, A, \varepsilon\rangle$, then $H(\Phi(x),\Psi(x)) < \varepsilon$. But then there is a $z \in \Psi(x)$ with $d(z,y) < \varepsilon$, so that $z \in V$. This means that $\Psi \in W^-$, and hence $\langle \Psi, A, \varepsilon\rangle \subseteq W^-$.

Next let $\Phi \in K^+$, where $K = L \times M$. Then $\Phi \cap (L \times M) = \emptyset$. By Lemma 1, there exists an $\varepsilon > 0$ such that $\Phi \cap (L \times V) = \emptyset$, where $V = B(M,\varepsilon)$ is the ε-neighborhood about M. Let $\Psi \in \langle \Phi, L, \varepsilon\rangle$, and let $x \in L$. Then $H(\Phi(x),\Psi(x)) < \varepsilon$. If $y \in \Psi(x)$, then there is a $z \in \Phi(x)$ with $d(y,z) < \varepsilon$, and so $z \notin B(M,\varepsilon)$; showing that $y \notin M$. Therefore $\Psi \cap (L \times M) = \emptyset$, and thus $\Psi \in K^+$. This finishes the argument that the uniform topology on compacta is finer than the Fell topology.

Now suppose that Y is not compact. Let $A = \{x_0\}$, where x_0 is any point in X. Let Φ be any member of $C(X,Y)$ and choose $\varepsilon > 0$ such that the closure of the ε-ball, $B(\Phi(x_0),\varepsilon)$, about $\Phi(x_0)$ is compact. To show that $\langle \Phi, A, \varepsilon\rangle$ has no interior in $U_F(X,2^Y)$, let

$$B = W_1^- \cap \cdots \cap W_m^- \cap K_1^+ \cap \cdots \cap K_n^+$$

be a basic open set in $U_F(X,2^Y)$ containing some Ψ. For each $i = 1,\ldots,m$, there exists $(x_i,y_i) \in W_i \cap \Psi$. Define $\Omega \in B \setminus \langle \Phi, A, \varepsilon\rangle$ as follows. Choose

$$y_0 \in Y \setminus (\pi_Y(K_1 \cup \cdots \cup K_n) \cup B(\Phi(x_0),\varepsilon)),$$

and then

$$\Omega = \{(x_i,y_i) : i = 1,\ldots,m\} \cup \{(x_0,y_0)\}$$

is the desired element of $U(X,2^Y)$.

Next suppose that X is not discrete. Let A be an infinite compact subset of X, let z_1 and z_2 be distinct points in Y, and let $\varepsilon = \frac{1}{2}d(z_1,z_2)$. Let Φ be the constatnt function in $U(X,2^Y)$ mapping each x in X to the closure of $B(z_1,2\varepsilon)$. To show that $\langle \Phi, A, \varepsilon\rangle$ contains no neighborhood of Φ in $U_F(X,2^Y)$, let

$$B = W_1^- \cap \cdots \cap W_m^- \cap K_1^+ \cap \cdots \cap K_n^+$$

be a basic open set in $U_F(X,2^Y)$ containing Φ. Then for each $i = 1,\ldots,m$, there exists $(x_i,y_i) \in W_i \cap \Phi$. Define $\Omega \in B \setminus \langle \Phi, A, \varepsilon\rangle$ as follows. Choose $x_0 \in A \setminus \{x_1,\ldots,x_m\}$, and then

$$\Omega \in \{(x_i,y_i) : i = 1,\ldots,m\} \cup \{(x_0,z_2)\}$$

is the desired element of $U(X, 2^Y)$.

It remains to show that if X is discrete and Y is compact then the topology on $U_F(X, 2^Y)$ is finer than that on $U_k(X, 2^Y)$. So let $\langle \Phi, A, \varepsilon \rangle$ be a basic open set in $U_k(X, 2^Y)$, where $A = \{a_1, \ldots, a_n\}$. Define

$$K = \bigcup_{i=1}^{n} (\{a_i\} \times (Y \setminus B(\Phi(a_i), \varepsilon))),$$

which is compact in $X \times Y$. For each $i = 1, \ldots, n$, there exists a finite subset $A_i \subseteq \Phi(a_i)$ such that $\Phi(a_i) \subseteq \cup\{B(a, \varepsilon) : a \in A_i\}$. For each $i = 1, \ldots, n$ and $a \in A_i$, let $W_{i,a} = \{a\} \times B(a, \varepsilon)$; and define

$$B = \left(\bigcap_{i=1}^{n} \bigcap_{a \in A_i} W_{i,a}^- \right) \bigcap K^+.$$

Then $\Phi \in B \subseteq \langle \Phi, A, \varepsilon \rangle$, showing that the topologies are equal in this case. $\square$

We now determine the closure of $D(X, Y)$ in $U(X, 2^Y)$ with respect to both the Fell topology and the uniform topology on compacta. As can be seen from the next two theorems, these two topologies are rather antithetical in this aspect.

The following notation is used in our next result. Let $U^i(X, 2^Y)$ be the set of all Φ in $U(X, 2^Y)$ such that the cardinality of $\Phi(x)$ is 1 or 0 for each isolated point x in X, and let $U^i(X, 2^Y_*) = U^i(X, 2^Y) \cap U(X, 2^Y_*)$.

Theorem 5 *Let X and Y be locally compact Hausdorff spaces.*

(a) *If Y is not compact, then the closure of $D(X, Y)$ in $U_F(X, 2^Y)$ is $U^i(X, 2^Y)$. In particular, if X has no isolated point, then $D(X, Y)$ is dense in $U_F(X, 2^Y)$.*

(b) *If Y is compact, then the closure of $D(X, Y)$ in $U_F(X, 2^Y_*)$ is $U^i(X, 2^Y_*)$. In particular, if X has no isolated point, then $D(X, Y)$ is dense in $U_F(X, 2^Y_*)$.*

Proof: First let Φ be in the closure of $D(X, Y)$ in $U_F(X, 2^Y)$. Assume, by way of contradiction, that there is an isolated point x_0 in X such that $\phi(x_0)$ has more than one element. Then there are disjoint open sets V_1 and V_2 in Y which intersect $\phi(x_0)$; let $W_1 = \{x_0\} \times V_1$ and $W_2 = \{x_0\} \times V_2$. Then $\phi \in W_1^- \cap W_2^-$, so that there is an $\overline{f} \in D(X, Y) \cap W_1^- \cap W_2^-$. But this is a contradiction because f is continuous at x_0 and single-valued there. Therefore Φ is in $U^i(X, 2^Y)$, and in $U^i(X, 2^Y_*)$ whenever X is compact.

For the remainder of the proof, let $\Phi \in U^i(X, 2^Y)$ and let

$$B = W_1^- \cap \cdots \cap W_m^- \cap K_1^+ \cap \cdots \cap K_n^+$$

be a baisc neighborhood of Φ in $U_F(X,2^Y)$, where each $K_j = L_j \times M_j$. Also for each $i = 1,\ldots,m$, there is a $(x_i,y_i) \in W_i$ with $y_i \in \Phi(x_i)$. Since $\Phi \in U^i(X,2^Y)$, we may choose the x_i to all be distinct.

In the case that Y is not compact, there exists a $y_0 \in Y \setminus (M_1 \cup \cdots \cup M_n)$. Then define $f : X \to Y$ by

$$f(x) = \begin{cases} y_i, & \text{if } x = x_i \text{ for some } i, \\ y_0, & \text{otherwise.} \end{cases}$$

Clearly $f \in DC(X,Y)$, so that $\overline{f} \in D(X,Y)$. Also by construction, $\overline{f} \in B$; which shows that in this case Φ is in the closure of $D(X,Y)$ in $U_F(X,2^Y)$.

We now consider the harder case that Y is compact, and that Φ is in $U^i(X,2^Y_*)$. In this case we use Lemma 1 to choose compact sets $L'_1,\ldots,L'_n$ such that each L'_j contains L_j in its interior and

$$\Phi \in W_1^- \cap \cdots \cap W_m^- \cap K_1'^+ \cap \cdots \cap K_n'^+$$

where each $K'_j = L'_j \times M_j$. Let $X_0 = \{x_1,\ldots,x_m\}$, and let P be the power set of $\{1,\ldots,n\}$ partially ordered by inclusion: if $p,q \in P$ then $p \le q$ if and only if $p \subseteq q$. For every $p \in P$, define $L^p = \cup\{L'_j : j \in p\}$ and $M^p = \cup\{M_j : j \in p\}$.

We define by induction families $\{P_1,\ldots,P_s\}$ of subsets of P and $\{X_1,\ldots,X_s\}$ of subsets of X such that for all $1 \le k \le s$,

1) P_k is the set of maximal elements in $\{p \in P : L^p \setminus (X_0 \cup \cdots \cup X_{k-1}) \ne \emptyset\}$, and

2) $X_k = \cup\{L^p \setminus (X_0 \cup \cdots \cup X_{k-1}) : p \in P_k\}$.

The induction must stop after s steps when $\{p \in P : L^p \setminus (X_0 \cup \cdots \cup X_s) \ne \emptyset\} = \emptyset$. Note that because of the maximality in 1) and because $\emptyset \in P$, the family

$$\{L^p \setminus (X_0 \cup \cdots \cup X_{k-1}) : 1 \le k \le s \text{ and } p \in P_k\} \cup \{X_0\}$$

partitions X. Also for each $p \in P_1$, $M^p \ne Y$. This is because if $x \in L_p$, then $\Phi(x) \cap M^p = \emptyset$; and $\Phi(x) \ne \emptyset$ since $\Phi \in U(X,2^Y_*)$. Therefore, for every $p \in P_1$, we choose a $y^p \in Y \setminus M^p$.

If $s > 1$, then also choose by induction, for every $1 < k \le s$ and every $p \in P_k$, an element

$$y^p \in \{y^q : q \in P_{k-1} \text{ and } p \le q\}.$$

This can be done because an element in P_k is in $\{p \in P : L^p \setminus (X_0 \cup \cdots \cup X_{k-1}) \ne \emptyset\}$ which is contained in $\{p \in P : L^p \setminus (X_0 \cup \cdots \cup X_{k-2}) \ne \emptyset\}$; and such an element is therefore contained in a maximal element of this latter set, and this maximal element is a member of P_{k-1}.

Finally, define $f : X \to Y$ by

$$f(x) = \begin{cases} y_i, & \text{if } x = x_i \text{ for some } x_i \in X_0, \\ y^p, & \text{if } x \in L^p \setminus (X_0 \cup \cdots \cup X_{k-1}) \text{ for some } 1 \le k \le s \text{ and some } p \in P_k. \end{cases}$$

Since f is a finite step function on X with each value taken on a set that can be written as a closed set minus a closed set, the discontinuities of f occur on the boundaries of a finite number of closed sets, which are nowhere dense in X. Therefore $f \in DC(X, Y)$, and hence $\overline{f} \in D(X, Y)$.

It remains to show that $\overline{f} \in B$, thus finishing the argument that Φ is in the closure of $D(X, Y)$ in $U_F(X, 2^Y_*)$. Let $1 \le k \le s$ and let $p \in P_k$. Now there is a $q \in P_1$ with $p \le q$ and $y^p = y^q$. Let $x \in L^p \setminus (X_0 \cup \cdots \cup X_{k-1})$. For each $j \in p$, since $j \in q$, $y^p \notin M_j$. So for each $j \in p$, $(x, y^p) \notin L'_j \times M_j$. Also for each $j \in \{1, \ldots, n\} \setminus p$, $x \notin L'_j$; and hence $(x, y^p) \notin L'_j \times M_j$. So the graph of f misses each K'_j. Because every L_j is contained in the interior of L'_j, the closure of the graph of f misses each K_j. This shows that $\overline{f} \in B$, and finishes the proof. $\square$

We need the following property in our next theorem. A metric d on Y has the *Heine-Borel property* provided that each closed bounded subset is compact. Such a metric is complete; and a space has such a metric inducing its topology if and only if it is a locally compact separable metrizable space [11].

Theorem 6 *Let X be a locally compact Hausdorff space, and let Y be a locally compact metric space. Then $C(X, Y)$ is closed in $D_k(X, Y)$; and, if the metric d on Y has the Heine-Borel property, $D(X, Y)$ is closed in $U_k(X, 2^Y)$.*

Proof: First we show that $C(X, Y)$ is closed in $D_k(X, Y)$. Let $\overline{f} \in D_k(X, Y) \setminus C(X, Y)$. Then there exists an $x_0 \in X$ such that the oscillation of f at x_0 is not 0; that is, $\operatorname{osc}(f, x_0) = \delta > 0$. Define $\varepsilon = \frac{1}{9}\delta$, and let W be a neighborhood of x_0 with compact closure A.

To show that $\langle \overline{f}, A, \varepsilon \rangle$ is a neighborhood of $\overline{f}$ in $D_k(X, Y)$ that is disjoint from $C(X, Y)$, let $\overline{g} \in \langle \overline{f}, A, \varepsilon \rangle$. It suffices to show that $\operatorname{osc}(g, x_0) > 0$. Suppose not; then $\overline{g}(x_0) = \{g(x_0)\}$. Now let U be any neighborhood of x_0. Then there exists an $x \in U \cap W \cap C(f) \cap C(g)$ such that $d(f(x), f(x_0)) > \frac{1}{3}\delta = 3\varepsilon$. Also $d(g(x), f(x)) < \varepsilon$, so that $d(g(x), f(x_0)) > 2\varepsilon$. Because $\operatorname{osc}(g, x_0) = 0$, $d(g(x_0), f(x_0)) < \varepsilon$; so $d(g(x), g(x_0)) > \varepsilon$. This is true for neighborhoods U of x_0, so that $\operatorname{osc}(g, x_0) \ge \varepsilon$. This completes the proof that $C(X, Y)$ is closed in $D_k(X, Y)$.

To show that $D(X, Y)$ is closed in $U_k(X, 2^Y)$, let $\Phi \in U_k(X, 2^Y) \setminus D(X, Y)$. Define the subset

$$Z = \{x \in X : \text{the cardinality of } \Phi(x) \text{ is not } 1\}$$

of the space X.

First suppose that Z is of second category in X. For each positive integer n, let

$$Z_n = \{x \in Z : \mathrm{diam}(\Phi(x)) \geq \frac{1}{n}\}.$$

Then $\cup_{n=1}^{\infty} Z_n = Z$, so that there is an n such that $\overline{Z_n}$ has nonempty interior. Let W be a nonempty open set such that its closure, A, is compact and contained in the interior of $\overline{Z_n}$. Define $\varepsilon = \frac{1}{3n}$, and let $\Psi \in \langle \Phi, A, \varepsilon \rangle$.

We need to show that $\Psi \notin D(X,Y)$. Suppose, to the contrary, that $\Psi = \overline{g} \in D(X,Y)$. Then let $x_0 \in W \cap C(g)$. For any neighborhood U of x_0, there is an $x \in U \cap Z_n$; so that $\mathrm{diam}(\Phi(x)) \geq \frac{1}{n}$. For such an x, $\mathrm{diam}(\overline{g}(x)) > \frac{1}{3n} = \varepsilon$. Therefore, for all neighborhoods U of x_0, $\mathrm{diam}(g(U)) > \varepsilon$. This implies that $\mathrm{osc}(g, x_0) \geq \varepsilon$, and contradicts the continuity of g at x_0.

We now consider the harder case that Z is of first category in X. Then $Z = \cup_{n=1}^{\infty} Z_n$ where each Z_n is nowhere dense in X. Then for each n, the set $G_n = X \setminus \overline{Z_n}$ is an open dense subset of X. Define $G = \cap_{n=1}^{\infty} G_n$, which is a dense G_δ-subset of X because X is locally compact.

Let $f : X \to Y$ be a selection for Φ; that is, $f(x) \in \Phi(x)$ for all $x \in X$ with $\Phi(x) \neq \emptyset$. Let $y_0 \in Y$ be arbitary. Because d has the Heine-Borel property, for every n, the closure, $\overline{B(y_0, n)}$, of the ball centered at y_0 with radius n is compact. For each n, let $F_n = G \cap f^{-1}(\overline{B(y_0, n)})$.

Next we show that each F_n is closed in G. Note that $f \cap (G \times Y) = \Phi \cap (G \times Y)$, which is a closed subset of $G \times Y$. So for every $x \in G \setminus F_n$, $\{x\} \times \overline{B(y_0, n)}$ is a compact subset of $G \times Y$ disjoint from $f \cap (G \times Y)$. By Lemma 1, there exist an open neighborhood U of x in G and an open set V in Y with $\overline{B(y_0, n)} \subseteq V$ such that $(U \times V) \cap F = \emptyset$. Hence $U \subseteq G \setminus F_n$, showing that F_n is closed in G.

For each n, let F_n° be the interior of F_n in G. Now $F_n \setminus F_n^\circ$ is nowhere dense in G because F_n is closed in G. Since G is a Baire space, the set $W = \cup_{n=1}^{\infty} F_n^\circ$ is a dense subset of G, and is thus dense in X.

To show that f is continuous at each point of W, let $x \in W$. Then $x \in F_n^\circ$ for some n. Suppose, by way of contradiction, that $\mathrm{osc}(f, x) = \delta > 0$. Then for any neighborhood U of x, there exists an $x_U \in U \cap F_n^\circ$ such that $d(f(x_U), f(x)) > \frac{1}{3}\delta$. Now the net $(f(x_U))$ is in the compact set $\overline{B(y_0, n)}$, and hence has a subnet that converges to some y in Y. We see that $d(y, f(x)) \geq \frac{1}{3}\delta$. Also Φ is closed in $X \times Y$, so that $y \in \Phi(x)$. But $\Phi(x) = \{f(x)\}$ and $y \neq f(x)$, which is a contradiction.

This means that $f \in DC(X,Y)$, and so we have $\overline{f} \in D(X,Y)$. Also note that because Φ is closed in $X \times Y$, $\overline{f} \subseteq \Phi$. Since $\Phi \notin D(X,Y)$, there exists an $x_0 \in X$ with $\Phi(x_0) \setminus \overline{f}(x_0) \neq \emptyset$; let $y_0 \in \Phi(x_0) \setminus \overline{f}(x_0)$. Then there is an $\epsilon > 0$ and an open neighborhood U of x_0 with compact closure, A, such that $(U \times B(y_0, 2\varepsilon)) \cap \overline{f} = \emptyset$.

To show that $\langle \Phi, A, \varepsilon \rangle$ is disjoint from $D(X,Y)$, let $\Psi \in \langle \Phi, A, \varepsilon \rangle$. for every $x \in U \cap W$, $\Phi(x) = \{f(x)\}$ and $f(x) \notin B(y_0, 2\varepsilon)$. Also for all such x, $\Psi(x) \subseteq B(f(x), \varepsilon)$, so that $\Psi(x) \cap B(y_0, \varepsilon) = \emptyset$. If $\Psi = \overline{g} \in D(X,Y)$, then there is a $y \in \overline{g}(x_0) \cap B(y_0, \varepsilon)$; and hence there is an $x \in C(g) \cap U \cap W$ with $g(x) \in B(y_0, \varepsilon)$, which contradicts the fact that $\Psi(x) \cap B(y_0, \varepsilon) = \emptyset$. Therefore $\Psi \notin D(X,Y)$, showing that $\langle \Phi, A, \varepsilon \rangle$ is contained in $U_k(X, 2^Y) \setminus D(X,Y)$, and hence showing that $D(X,Y)$ is closed. □

Question 7 *What is the closure of $C(X,Y)$ in $U_F(X, 2^Y)$?*

To obtain a satisfactory answer to this question, it may be necessary to assume that $Y = \mathbf{R}^n$, so that an extension theorem can be used. We do have the following partial result when Y is the space of real numbers, $\mathbf{R}$, that points out that connectedness is involved with the answer to this question.

Proposition 8 *Let X be a locally compact Hausdorff space. If Φ is in the closure of $C(X, \mathbf{R})$ in $U_F(X, 2^{\mathbf{R}})$, then $\Phi(x)$ is connected for every point x at which X is locally connected.*

Proof: Suppose X is locally connected at x. We assume, by way of contradiction, that $\Phi(x)$ is not connected. So let $r < s < t$ be in $\mathbf{R}$ such that $r, t \in \Phi(x)$ while $s \notin \Phi(x)$. Define $V_1 = (r-1, s)$ and $V_2 = (s, t+1)$, and let U be a connected neighborhood of x such that its closure, A, is compact and $(A \times \{s\}) \cap \Phi = \emptyset$. Define

$$B = (U \times V_1)^- \cap (U \times V_2)^- \cap (A \times \{s\})^+,$$

so that $\Phi \in B$. Because Φ is in the closure of $C(X,Y)$ in $U_F(X, 2^{\mathbf{R}})$, there is an $f \in B \cap C(X, \mathbf{R})$. But then $f(U) \cap (r-1, s) \neq \emptyset$ and $f(U) \cap (s, t+1) \neq \emptyset$, while $s \notin f(U)$. This contradicts the fact that $f(U)$ is connected because f is continuous, thus implying that $\Phi(x)$ must be connected. □

If the converse of Proposition 8 were true, this would answer Question 7 for $Y = \mathbf{R}$. In any case, Theorem 5 and Proposition 8 show that $C(X, \mathbf{R})$ is not dense in $D_F(X, \mathbf{R})$.

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