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# Approximation in Hölder norms with higher order differences

*Dedicated to the professors of mathematics*
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ABSTRACT. In this paper we obtain Markov-, Markov-Bernstein- and Jackson-type estimates in Hölder-norms, where the Hölder-terms are constructed from first and second order differences. An application to approximation results for interpolatory processes is outlined.

KEY WORDS. Hölder-Zygmund-norms, Markov-Bernstein-type inequalities, Jackson-type estimates.

## 1 Introduction

The asymptotic convergence order in periodic Hölder or Hölder-Zygmund norms for the Fourier sum and interpolatory polynomials on equidistant nodes has been studied in a series of papers in the last 20 years. In this connection, we mention L. Leindler [11], S. Prößdorf [16], S. Prößdorf/B. Silbermann [17] and the literature cited there. In particular, the case of periodic Hölder spaces with arbitrary order of differences is studied in more detail in [14]. Such approximation results have been extensively used to prove convergence theorems for quadrature rules for Cauchy-type principal value integrals and for the numerical solution of singular integral equations (see e.g. [17]).
Here we are interested in the algebraic approximation and interpolation. First approximation results for continuous functions in Hölder norms were obtained by A. I. Kalandiya [9] and N. I. Ioakimidis [7], [8], see also [13] and D. Elliott [4]. The aim of this paper is to discuss estimates for Hölder norms with varying order of difference. However, for the sake of simplicity of the representation we restrict ourselves to the case of first and second order differences. The general situation can be handled by the same methods of proof.

Analogous results can be obtained for general modulus type functions $\omega(h)$ (see [13]). Only to simplify the notation we focus on the classical growth condition $\omega(h) = h^\alpha$.

After studying the Jackson-type inequality and their so-called inverse Markov- and Markov-Bernstein-type inequalities, we apply them to estimate linear approximation processes. In particular, we discuss Lagrange interpolation, where it turns out that the error estimates are not as good as "Lebesgue constant times order of best approximation". Therefore we use also Lagrange interpolation with additional nodes near the endpoints $\pm 1$ to improve the order of convergence (compare e.g. [19], Chap. 8 and [1]).

## 2 Preliminaries

Let us start with the definition of the norms and spaces. As usual we denote by $C[-1,1]$ the Banach space of continuous functions on $[-1,1]$ equipped with the maximum norm $\|f\| = \max\{|f(x)| \,:\, x \in [-1,1]\}$. For $f \in C[-1,1]$ and $x \in J_{sh} = [-1, 1-sh]$ we write

$$\Delta_h^s f(x) = \sum_{i=0}^{s} (-1)^{i+s} \binom{s}{i} f(x+ih)\ .$$

Then we say $f \in C[-1,1]$ belongs to $C^{m,\alpha,s}$ $(m \in \mathbb{N}_0\ ,\ s \in \mathbb{N}$ and $0 \le \alpha \le s)$ , iff the so-called Hölder term

$$\|f^{(m)}\|_{\alpha,s} = \sup_{h>0} h^{-\alpha} \max_{x \in J_{sh}} |\Delta_h^s f^{(m)}(x)|$$

is finite. Especially we write $C^m$ instead of $C^{m,0,s}$. A norm in the Hölder space $C^{m,\alpha,s}$ is given by

$$\|f\|_{m,\alpha,s} = \sum_{k=0}^{m} \|f^{(k)}\| + \|f^{(m)}\|_{\alpha,s}\ .$$

For $0 \le \alpha < s$ we define separable subspaces $\tilde{C}^{m,\alpha,s}$ of $C^{m,\alpha,s}$ by the condition

$$\lim_{h \to 0+} h^{-\alpha} \max_{x \in J_{sh}} |\Delta_h^s f^{(m)}(x)| = 0\ .$$

Sometimes we will also use the modulus of continuity

$$\omega_s(f,\delta) = \sup_{h \le \delta} \max_{x \in J_{sh}} |\Delta_h^s f(x)|$$

with $\delta < 2/s$. Throughout the paper we will apply the following simple observations

$$\begin{aligned}
\omega_{s+r}(f,\delta) &\le \delta^r \omega_s(f^{(r)},\delta) \le \delta^{s+r} \|f^{(s+r)}\|\ ,\\
\omega_{s+r}(f,\delta) &\le 2^r \omega_s(f,\delta) \le 2^{r+s} \|f\|\ ,\\
\omega_s(f,\delta) &\le \delta^\alpha \|f\|_{\alpha,s}\ ,
\end{aligned}$$

(see e.g. [12], Chap. 3). Furthermore we abbreviate

$$\delta_n(x) = \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2} .$$

Then we have

$$\frac{1}{n^2} \le \delta_n(x) \le \frac{2}{n} .$$

In what follows $C$ denotes positive constants depending only on parameters of the Hölder spaces and other fixed parameters involved, but their values may be different at each occurence. In particular, $C$ will be independent of $n, x$, the functions $f$ and polynomials $p_n$ and $q_n$.

Now let us summarize some basic results for algebraic polynomials and best approximation which we will need in the sequel. The set of algebraic polynomials of degree at most $n$ will be denoted by $\Pi_n$.

**Proposition 2.1** **(A.F. Timan [20], V.K. Dzjadyk [3])** *Let $q_n \in \Pi_n$, $k \in I\!N$ and $x \in [-1,1]$. Then*

$$|q_n^{(k)}(x)| \le C\delta_n(x)^{-k}\|q_n\| .$$

*Moreover, if*

$$|q_n(x)| \le C\delta_n(x)^p$$

*for some $p \in I\!R$ and all $x \in [-1,1]$, then*

$$|q_n^{(k)}(x)| \le C\delta_n(x)^{p-k} \tag{1}$$

*for all $k \in I\!N$ and all $x \in [-1,1]$ .*

**Proposition 2.2** **(I.E. Gopengauz [6], R.M. Trigub [21])** *For all $f \in C^m$, $s \ge 0$ there exist a polynomial $q_n \in \Pi_n$, such that for all $x \in [-1,1]$ and for all $0 \le k \le m$*

$$|(f-q_n)^{(k)}(x)| \le C\delta_n(x)^{m-k}\omega_s(f^{(m)}, \delta_n(x)) . \tag{2}$$

**Proposition 2.3** **(Z. Ditzian, D. Jiang [2])** *Let $f \in C[-1,1]$ and $s \ge 0$ be given. If $q_n \in \Pi_n$ satisfies*

$$|(f-q_n)(x)| \le C\omega_s(f, \delta_n(x)) ,$$

*then*

$$|q_n^{(s)}(x)| \le C\delta_n(x)^{-s}\omega_s(f, \delta_n(x)) . \tag{3}$$

Furthermore we need error estimates for polynomials with additional interpolatory conditions. Therefore let

$$K_\nu = \{t_i, s_i\,;\, 1 \le i \le \nu\} \qquad (K_0 = \emptyset)$$

a set of nodes with

$$-1 \le t_1 \le \ldots \le t_\nu \le -1 + Cn^{-2}\,, \qquad 1 - Cn^{-2} \le s_\nu \le \ldots \le s_1 \le 1\,.$$

and

$$Q_{2\nu}(x) = \begin{cases} 1 & \text{for } \nu = 0\,, \\ \prod_{i=1}^{\nu}(x-t_i)(x-s_i) & \text{for } \nu > 0\,. \end{cases}$$

**Proposition 2.4** **(A.A. Privalov [15], see also [10])** *Let $m, s \in \mathbb{N}$, $n \ge s+m-1$, $m \ge 2\nu - 1$, $0 \le k \le m$ and $f \in C^m$. Then there exists a polynomial $q_n \in \Pi_n$ such that*

$$|(f-q_n)^{(k)}(x)| \le C\delta_n(x)^{m-k}\omega_s(f^{(m)}, \delta_n(x)) \qquad \textit{for all} \quad x \in [-1,1] \tag{4}$$

*and*

$$q_n(x) = f(x) \qquad \textit{for all} \quad x \in K_\nu\,.$$

*The points from $K_\nu$ may coalesce, in which case one also interpolates at the coalescent point a number of derivatives one less than the multiplicity of coalescence.*

One can easily summarize these estimates to the following result.

**Corollary 2.1** Let $m, s \in \mathbb{N}$ , $n \ge s+m-1$, $m \ge 2\nu$ and $f \in C^m$. Then there exists a polynomial $p_n \in \Pi_n$ such that

$$p_n(x) = f(x) \quad \text{ for all } x \in K_\nu\,.$$

Furthermore, for all $x \in [-1,1]$ and for all $0 \le k \le m,\ m+s \le j$,

$$|p_n^{(j)}(x)| \le C\delta_n(x)^{m-j}\omega_s(f^{(m)}, \delta_n(x)) \tag{5}$$

and

$$|(f-p_n)(x)| \le C|Q_{2\nu}(x)|n^{-m}\omega_s(f^{(m)}, \delta_n(x))\,. \tag{6}$$

**Proof of the Corollary 2.1:** Starting with the polynomial from Proposition 2.4 we obtain by the Proposition 2.3 and the triangular inequality immediately (5). To prove (6) one can follow the ideas in [1] and [10], i.e. one has to apply Rolle's theorem.

## 3 Main results

In our theorems we compare norms $\|\circ\|_{m,\alpha,s}$ and $\|\circ\|_{r,\beta,t}$ with "small" $r+\beta$ and "large" $m+\alpha$. More exactly, we assume in the sequel

$$r, m \in I\!N_0,\ s,t \in \{1;2\},\ 0 \le \alpha \le s,\ 0 \le \beta \le t, \tag{7}$$

$$r \le m \quad \text{and} \quad r+\beta \le m+\alpha\ . \tag{8}$$

Our first result is an inequality of Markov-type.

**Theorem 3.1** *Let (7) and (8) be satisfied. For $t=2$ we further assume:*

$$\text{If} \quad r = m-2, \quad \text{then} \quad \beta \le 1+\alpha. \tag{9}$$

$$\text{If} \quad r = m-1 \quad \text{and} \quad s=1, \quad \text{then} \quad \beta = 0. \tag{10}$$

$$\text{If} \quad r = m-1 \quad \text{and} \quad s=2, \quad \text{then} \quad \beta \le \alpha. \tag{11}$$

$$\text{If} \quad r = m \quad \text{and} \quad s=1, \quad \text{then} \quad \beta = 0. \tag{12}$$

*Then for all $p_n \in \Pi_n$ the estimate*

$$\|p_n\|_{m,\alpha,s} \le C n^{2(m+\alpha-r-\beta)} \|p_n\|_{r,\beta,t} \tag{13}$$

*is satisfied.*

The conditions (9)-(12) and analogous conditions in the next theorems seem to be very technical. But the observation $\omega_2(p_1,\delta)=0$ for all $p_1 \in \Pi_1$ shows that in general one cannot avoid these conditions.

Supplementary to the global Markov-type inequality we give also a local Bernstein-type inequality.

**Theorem 3.2** *Let (7) and (8) be satisfied and let $p_n \in \Pi_n$. If*

$$|p_n(x)| \le \delta_n(x)^{m+\alpha} \quad \textit{for all} \quad x \in [-1,1]\ , \tag{14}$$

*then*

$$\|p_n\|_{r,\beta,t} \le C n^{r+\beta-m-\alpha}\ .$$

Now we state the direct approximation result, namely the so-called Jackson-type inequality on best approximation.

**Theorem 3.3** *Let (7) and (8) be satisfied. For $s = 2$ we further assume:*

$$\text{If} \quad r = m \quad \text{and} \quad t = 1, \quad \text{then} \quad \beta = 0. \tag{15}$$

$$\text{If} \quad r = m-1 \quad \text{and} \quad t = 2, \quad \text{then} \quad \beta \leq 1. \tag{16}$$

*Then for arbitrary $f \in C^{m,\alpha,s}$ there exists a polynomial $p_n \in \Pi_n$ with*

$$\|f - p_n\|_{r,\beta,t} \leq C n^{r+\beta-m-\alpha} \|f^{(m)}\|_{\alpha,s} \, . \tag{17}$$

*In particular, for $f \in \tilde{C}^{m,\alpha,s}, 0 \leq \alpha < s$, the polynomial $p_n$ satisfies*

$$\|f - p_n\|_{r,\beta,t} = o(n^{r+\beta-m-\alpha}) \qquad n \to \infty \, . \tag{18}$$

Such a "small-o"– result as in (18) can be obtained for all of the following approximation theorems. For shortness we do not repeat them.

Note that we can use in the proof of Theorem 3.3 the polynomial $p_n$ from Proposition 2.2. Hence, for fixed $f \in C^{m,\alpha,s}$ the same polynomial $p_n$ gives (17)-(18) simultaneously for all $r, \beta, t$, satisfying (7)-(8) and (15)-(16). Moreover, we can also use the polynomial constructed by A.A. Privalov (see Proposition 2.4 and Corollary 2.1). Hence we obtain with the assumptions of Theorem 3.3 and for sufficiently large $n$ the following extension of a Jackson-type inequality with $2\nu$ $(2\nu \leq m+1)$ interpolatory conditions

$$\inf\{\|f - q_n\|_{r,\beta,t};\ q_n \in \Pi_n,\ f(x) = q_n(x) \text{ for } x \in K_\nu\} \leq C n^{r+\beta-m-\alpha} \|f^{(m)}\|_{\alpha,s} \, . \tag{19}$$

Also of particular interest are approximation results in Hölder norms with large parameters if one knows error estimates in Hölder norms with small paramters.

**Theorem 3.4** *Together with (7) and (8) let $\ell \in \mathbb{N}_0$ , $u \in \{1, 2\}$ , $0 \leq \gamma \leq u$, $\ell \leq r$ and $\ell + \gamma \leq r + \beta$ be satisfied. Furthermore we impose the following restrictions summarized in the following tabular:*

| $u$ | $s$ | $t$ | *assumptions* |
|---|---|---|---|
| 2 | 2 | 2 | *if* $\ell = r - 2$, *then* $\gamma \leq 1 + \beta$ <br> *if* $\ell = r - 1$, *then* $\gamma \leq \max\{1, \beta\}$ <br> *if* $r = m - 1$, *then* $\beta \leq 1$ |
| 2 | 2 | 1 | *if* $\ell = r - 2$, *then* $\gamma \leq 1 + \beta$ <br> *if* $\ell \geq r - 1$, *then* $\gamma = 0$ <br> *if* $r = m$, *then* $\beta = 0$ |

| $u$ | $s$ | $t$ | *assumptions* | |
|---|---|---|---|---|
| 2 | 1 | 2 | *if* $\ell = r-2$, | *then* $\gamma \le 1+\beta$ |
| | | | *if* $\ell = r-1$, | *then* $\gamma \le \beta$ |
| 2 | 1 | 1 | *if* $\ell = r-2$, | *then* $\gamma \le 1+\beta$ |
| | | | *if* $\ell \ge r-1$, | *then* $\gamma = 0$ |
| 1 | 2 | 2 | *if* $r = m-1$, | *then* $\beta \le 1$ |
| | | | *if* $\ell = m$, | *then* $\gamma = 0$ |
| 1 | 2 | 1 | *if* $r = m$, | *then* $\beta = 0$ |

*Now let $f \in C^{m,\alpha,s}$ and $q_n \in \Pi_n$ be given such that*

$$\|f - q_n\|_{\ell,\gamma,u} \le C n^{\ell+\gamma-m-\alpha} \|f^{(m)}\|_{\alpha,s} \,. \tag{20}$$

*Then we obtain*

$$\|f - q_n\|_{r,\beta,t} \le C n^{2r+2\beta-\ell-\gamma-m-\alpha} \|f^{(m)}\|_{\alpha,s} \,. \tag{21}$$

The convergence order in (21) can be improved by assuming Bernstein-type conditions.

**Theorem 3.5** *Let* (7) *and* (8) *be satisfied. For $s = 2$ we further assume* (15) *and* (16). *Now let $f \in C^{m,\alpha,s}$ and $q_n \in \Pi_n$ be given such that*

$$|(f - q_n)(x)| \le C\delta_n(x)^{m+\alpha} \|f^{(m)}\|_{\alpha,s} \quad \textit{for all} \quad x \in [-1,1] \,. \tag{22}$$

*Then*

$$\|f - q_n\|_{r,\beta,t} \le C n^{r+\beta-m-\alpha} \|f^{(m)}\|_{\alpha,s} \,.$$

We will end this section with the application of these results to linear approximation processes. Let us focus on Lagrange interpolation.

For $n$ distinct nodes $-1 \le x_n < x_{n-1} < \ldots < x_1 \le 1$ we define the Lagrange interpolation polynomial $L_n f \in \Pi_{n-1}$ of a given function $f \in C[-1,1]$ by

$$L_n f(x_k) = f(x_k) \qquad \text{for } k = 1, \ldots, n \,.$$

With the Lebesgue fundamental functions $\ell_k \in \Pi_{n-1}$ we have the well-known representation

$$L_n f(x) = \sum_{k=1}^{n} f(x_k) \ell_k(x)$$

with the operator norm

$$\|L_n\|_{C \to C} = \| \sum_{k=1}^{n} |\ell_k| \, \| \,.$$

From the Jackson-type theorem and the Markov-type theorem we easily obtain the following approximation result.

**Theorem 3.6** *Let* (7) *and* (8) *be satisfied. For* $s=2$ *we further assume* (15) *and* (16). *If* $f \in C^{m,\alpha,s}$, *then*

$$\|f - L_n f\|_{r,\beta,t} \le C \|L_n\|_{C\to C}\, n^{2r+2\beta-m-\alpha} \|f^{(m)}\|_{\alpha,s}\,.$$

The approximation order $n^{2r+2\beta-m-\alpha}$ cannot be improved in general (see e.g. [18]). However, we can modify the interpolation process by choosing additional nodes near $\pm 1$. Then one can achieve a better order of approximation. This is discussed for the simultaneous approximation of derivatives in a series of papers (see e.g. [19], Chap. 8, and the literature cited there). Here we obtain the results for Hölder norms with higher differences.

Therefore, we define the Lagrange-Hermite polynomial $H_n f \in \Pi_{n+2\nu-1}$ by $H_n f(x) = f(x)$ for $x = x_k, k = 1, ..., n$ and $x \in K_\nu$. As in Proposition 2.4 we allow coalescent points in $K_\nu$ which means interpolation of derivatives. Now the improved approximation result for Lagrange interpolation with additional nodes near $\pm 1$ reads as follows.

**Theorem 3.7** *Let* (7)–(8) *be satisfied. For* $s=2$ *we further assume* (15) *and* (16). *Then for arbitrary* $f \in C^{m,\alpha,s}$

$$\|(f - H_n f)(x)\|_{r,\beta,t} \le C n^{r+\beta-m-\alpha+(r+\beta-2\nu)_+} \|L_n\|_{C\to C} \|f^{(m)}\|_{\alpha,s}\,.$$

The constant $C$ depends here also on $K_\nu$, but is clearly independent on $n$ and $f$. With $(a)_+$ we denote as usual the truncated power $max(0,a)$.

## 4 Proofs

In the next proofs we have to estimate Hölder norms. Therefore it is natural to split the proofs into two parts. In (i) we consider the sum of the norms of the derivatives. The Hölder term which turns out to be the main part is discussed in (ii).

**Proof of Theorem 3.1:**

**(i)** At first we consider a single derivative of $p_n$. From (4) we conclude that there exists a polynomial $q_n \in \Pi_n$ such that for all $r \ge 0$

$$\begin{aligned} |(p_n - q_n)^{(r)}(x)| &\le C\omega_t(p_n^{(r)}, \delta_n(x)) \\ &\le C\delta_n(x)^\beta \|p_n^{(r)}\|_{\beta,t}\,. \end{aligned}$$

Then, for all $k \ge 0$ by Theorem 2.1

$$|(p_n - q_n)^{(r+k)}(x)| \le C\delta_n(x)^{\beta-k} \|p_n^{(r)}\|_{\beta,t}\,.$$

Furthermore, for $t \le k$ we have from (5)

$$\begin{aligned} |q_n^{(r+k)}(x)| &\le C\delta_n(x)^{-k}\omega_t(p_n^{(r)}, \delta_n(x)) \\ &\le C\delta_n(x)^{\beta-k}\|p_n^{(r)}\|_{\beta,t} \,. \end{aligned}$$

Hence, for $t \le k$ we obtain

$$|p_n^{(r+k)}(x)| \le C\delta_n(x)^{\beta-k}\|p_n^{(r)}\|_{\beta,t} \,,$$

and in particular

$$\|p_n^{(r+k)}\| \le Cn^{2(k-\beta)}\|p_n^{(r)}\|_{\beta,t} \,. \tag{23}$$

Now we have to sum up the norms of the derivatives. Easily we conclude

$$\sum_{k=0}^{m} \|p_n^{(k)}\| \le \|p_n\|_{r,\beta,t} + \sum_{k=r+1}^{m} \|p_n^{(k)}\| \,.$$

Therefore it is nothing to prove for $r = m$. For $r < m$ we apply the classical inequality of Markov and (23) which gives

$$\begin{aligned} \sum_{k=r+1}^{m} \|p_n^{(k)}\| &\le \|p_n^{(r+1)}\| + Cn^{2(m-r-2)}\|p_n^{(r+2)}\| \\ &\le \|p_n^{(r+1)}\| + Cn^{2(m-r-2)}n^{2(2-\beta)}\|p_n^{(r)}\|_{\beta,t} \\ &\le Cn^{2(m-r-\beta)})\|p_n\|_{r,\beta,t} + \begin{cases} n^2\|p_n^{(r)}\| & \text{if} \quad t = 2, \\ n^{2(1-\beta)}\|p_n\|_{r,\beta,t} & \text{if} \quad t = 1 \,. \end{cases} \end{aligned}$$

Taking into consideration the assumptions (9) – (11) we obtain (i).

**(ii)** Let us consider the Hölder term.

The main idea is to investigate the supremum over $h$ for small and large $h$ separately. Here we define

$$G = \{h : h > 1/n^2\} \qquad \text{and} \qquad H = \{h : 0 < h \le 1/n^2\} \,.$$

1. For $h \in H$ we apply the mean value theorem and (23) for $m + s - r \ge t$ to obtain

$$\begin{aligned} \sup_{h\in H} h^{-\alpha} \max_{x\in J_{sh}} |\Delta_h^s p_n^{(m)}(x)| &\le Ch^{s-\alpha}\|p_n^{(m+s)}\| \\ &\le Cn^{2(\alpha-s)}n^{2(m+s-r-\beta)}\|p_n^{(r)}\|_{\beta,t} \\ &\le Cn^{2(m+\alpha-r-\beta)}\|p_n\|_{r,\beta,t} \,. \end{aligned}$$

Only in the case (12), namely $m+s-r<t$, we modify this argument to

$$\begin{aligned}\sup_{h\in H} h^{-\alpha}\max_{x\in J_h}|\Delta_h^1 p_n^{(m)}(x)| &\le h^{1-\alpha}\|p_n^{(m+1)}\| \\ &\le Cn^2h^{1-\alpha}\|p_n^{(m)}\| \\ &\le Cn^{2\alpha}\|p_n^{(m)}\|\,.\end{aligned}$$

2. Let now $h\in G$. For $r\le m-t$ we argue again with (23) to obtain

$$\begin{aligned}\sup_{h\in G} h^{-\alpha}\max_{x\in J_{sh}}|\Delta_h^s p_n^{(m)}(x)| &\le 4n^{2\alpha}\|p_n^{(m)}\| \\ &\le 4n^{2\alpha}Cn^{2(m-r-\beta)}\|p_n^{(r)}\|_{\beta,t} \\ &\le Cn^{2(m+\alpha-r-\beta)}\|p_n^{(r)}\|_{\beta,t}\,.\end{aligned}$$

Now we have only to deal with three remaining cases $r>m-t$.

a) For $s\ge t, r=m$ we write

$$\begin{aligned}\sup_{h\in G} h^{-\alpha}\max_{x\in J_{sh}}|\Delta_h^s p_n^{(m)}(x)| &\le \sup_{h\in G} 2h^{-\alpha}\max_{x\in J_{th}}|\Delta_h^t p_n^{(m)}(x)| \\ &\le \sup_{h\in G} 2h^{\beta-\alpha}\|p_n^{(r)}\|_{\beta,t} \\ &\le Cn^{2(\alpha-\beta)}\|p_n^{(r)}\|_{\beta,t}\,.\end{aligned}$$

b) For $t=2, s=1, r=m$ we use the condition (12) to write

$$\sup_{h\in G} h^{-\alpha}\max_{x\in J_h}|\Delta_h^1 p_n^{(m)}(x)| \le Cn^{2\alpha}\|p_n^{(m)}\|\,.$$

c) For $t=2, r=m-1$, with the conditions (10) and (11), respectively, we conclude

$$\begin{aligned}\sup_{h\in G} h^{-\alpha}\max_{x\in J_{2h}}|\Delta_h^2 p_n^{(m)}(x)| &\le \sup_{h\in G} Ch^{-\alpha}n^2\max_{x\in J_{2h}}|\Delta_h^2 p_n^{(m-1)}(x)| \\ &\le Cn^{2\beta-2\alpha+2}\|p_n^{(r)}\|_{\beta,2}\end{aligned}$$

and

$$\sup_{h\in G} h^{-\alpha}\max_{x\in J_h}|\Delta_h^1 p_n^{(m)}(x)| \le Cn^{2\alpha}n^2\|p_n^{(m-1)}\|\,.$$

Summarizing the estimates we obtain for the Hölder norm the inequality (17) which concludes the proof. ∎

For the proof of Theorem 3.2 we need the following preliminary result.

**Lemma 4.1** *Let $x, x+h, x+2h \in [-1,1]$. If $0 < h \le \min(\delta_n(x), 1-x)$, then*

$$\delta_n(x) \le 7\delta_n(x+h) \tag{24}$$

*and*

$$\delta_n(x) \le 13\delta_n(x+2h)\,. \tag{25}$$

*If $\delta_n(x) < h < \min(\frac{1}{n}, 1-x)$, then*

$$\delta_n(x+h) < 3h \tag{26}$$

*and*

$$\delta_n(x+2h) < 4h\ . \tag{27}$$

**Proof of Lemma 4.1:** We show here (25) and (27). For (24) und (26) compare also [13].

**a)** If $\sqrt{1-x^2} \le 12/n$, then

$$\delta_n(x) = \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2} \le \frac{13}{n^2} \le 13\,\delta_n(x+2h)\ .$$

If $\sqrt{1-x^2} > 12/n$ , then

$$h < \delta_n(x) < \frac{\sqrt{1-x^2}}{n} + \frac{12}{n^2} < 2\frac{\sqrt{1-x^2}}{n} < \frac{1-x^2}{6}\ .$$

This, together with $x+2h \le 1$ gives

$$(x+2h)2h < 1-(x+2h)^2\ .$$

For $x \ge 0$ we write

$$\frac{(x+2h)2h}{\sqrt{1-(x+2h)^2}} < \sqrt{1-(x+2h)^2}\ .$$

Using the mean value theorem and $(\sqrt{1-x^2})'' \ge 0$ we obtain

$$\sqrt{1-x^2} - \sqrt{1-(x+2h)^2} < \frac{(x+2h)2h}{\sqrt{1-(x+2h)^2}} < \sqrt{1-(x+2h)^2}\ .$$

Hence

$$\sqrt{1-x^2} < 2\sqrt{1-(x+2h)^2}\ ,$$

which proves (25).

**b)** From the assumptions for (27) we have

$$1 - x^2 < h^2 n^2 \qquad \text{and} \qquad \frac{4}{n^2} < 4h\,,$$

which implies $1 - x^2 + 4h < 5h^2n^2$. Therefore, we conclude from

$$1 - (x+2h)^2 < 1 - x^2 + 4h < 5h^2 n^2$$

that

$$\sqrt{1-(x+2h)^2} < 3hn.$$

With $1/n^2 < h$ we obtain immediately (27). ∎

Now we are able to prove the Bernstein-type inequality.

**Proof of Theorem 3.2:**
**(i)** With the assumption (14) we obtain from (1) in Theorem 2.1 that

$$\begin{aligned}\sum_{k=0}^{r} \|p_n^{(k)}\| &\leq \sum_{k=0}^{r} C\|\delta_n\|^{(m+\alpha-k)} \\ &\leq \sum_{k=0}^{r} C n^{k-m-\alpha} \\ &\leq C n^{r-m-\alpha}\,.\end{aligned}$$

**(ii)** To estimate the Hölder term it is sufficient to consider $t = 2$. Let us fix $x \in [-1,1]$. Here we distinguish between $h > \delta_n(x)$ and $0 < h \leq \delta_n(x)$.

a) Let $h > \delta_n(x)$. With (1) we obtain

$$\begin{aligned} h^{-\beta}|\Delta_h^2 p_n^{(r)}(x)| &\leq h^{-\beta}(|p_n^{(r)}(x)| + 2|p_n^{(r)}(x+h)| + |p_n^{(r)}(x+2h)|) \\ &\leq Ch^{-\beta}(\delta_n(x)^{m+\alpha-r} + 2\delta_n(x+h)^{m+\alpha-r} + \delta_n(x+2h)^{m+\alpha-r}) \\ &\leq C n^{r+\beta-m-\alpha}\,.\end{aligned}$$

The last inequality follows directly for $h \geq 1/n$. For $\delta_n(x) < h < 1/n$ one has to use (26), (27).

b) Now let $0 < h \leq \delta_n(x)$. By the mean value theorem there exists an

$\xi \in [x, x+2h]$ such that

$$\begin{aligned} h^{-\beta}|\Delta_h^2 p_n^{(r)}(x)| &\leq Ch^{2-\beta}|p_n^{(r+2)}(\xi)| \\ &\leq C\delta_n(x)^{2-\beta}\delta_n(\xi)^{m+\alpha-r-2} \\ &\leq C\delta_n(\xi)^{m+\alpha-r-\beta} \\ &\leq Cn^{r+\beta-m-\alpha}\,, \end{aligned}$$

where we applied (25) to estimate $\delta_n(x)$.

To conclude the proof we form the maximum over $x \in J_{th}$ and the supremum over $h$. ∎

**Proof of Theorem 3.3:** We will show that a polynomial $p_n$, which satisfies the conditions (2), also proves Theorem 3.3.

**(i)** From Theorem 2.2 we have for $0 \leq k \leq m$ that

$$\begin{aligned} |(f-p_n)^{(k)}(x)| &\leq C\delta_n(x)^{m-k}\omega_s(f^{(m)}, \delta_n(x)) \\ &\leq C\delta_n(x)^{m-k+\alpha}\|f^{(m)}\|_{\alpha,s} \\ &\leq Cn^{k-m-\alpha}\|f^{(m)}\|_{\alpha,s}\,. \end{aligned} \tag{28}$$

Hence,

$$\sum_{k=0}^{r}\|(f-p_n)^{(k)}\| \leq Cn^{r-m-\alpha}\|f^{(m)}\|_{\alpha,s}\,.$$

**(ii)** In order to estimate

$$h^{-\beta}\max_{x\in J_{th}}|\Delta_h^t(f-p_n)^{(r)}(x)|$$

we distinguish between $h > 1/n$ and $0 < h \leq 1/n$.

1. For $h > 1/n$ we simply estimate with (28) and (2) to obtain

$$\begin{aligned} h^{-\beta}\max_{x\in J_{th}}|\Delta_h^t(f-p_n)^{(r)}(x)| &\leq 4n^{\beta}\|(f-p_n)^{(r)}\| \\ &\leq Cn^{r+\beta-m-\alpha}\|f^{(m)}\|_{\alpha,s}\,. \end{aligned}$$

2. If $0 < h \leq 1/n$ we consider separately the three possible cases $r \leq m-t$ ; $r = m-1, t = 2$ and $r = m$.

a) If $r \leq m-t$, then we use (28) to obtain

$$\begin{aligned} h^{-\beta}\max_{x\in J_{th}}|\Delta_h^t(f-p_n)^{(r)}(x)| &\leq h^{t-\beta}h^{-t}\max_{x\in J_{th}}|\Delta_h^t(f-p_n)^{(r)}(x)| \\ &\leq Cn^{\beta-t}\|(f-p_n)^{(r+t)}\| \\ &\leq Cn^{r+\beta-m-\alpha}\|f^{(m)}\|_{\alpha,s}\,. \end{aligned}$$

b) Let now $r = m - 1, t = 2$. In this case, (16) implies $\beta \le 1$. Then we obtain by (28) and the mean value theorem that

$$\begin{aligned} h^{-\beta} \max_{x \in J_{th}} |\Delta_h^t (f - p_n)^{(r)}(x)| &\le 2h^{1-\beta} h^{-1} \max_{x \in J_{th}} |\Delta_h^1 (f - p_n)^{(r)}(x)| \\ &\le 2h^{1-\beta} \|(f - p_n)^{(m)}\| \\ &\le C n^{\beta - 1 - \alpha} \|f^{(m)}\|_{\alpha,s} \,. \end{aligned}$$

c) Now let us end with the third case $r = m$. If $t < s$, then (15) implies $\beta = 0$ and there is nothing to prove. Hence we assume $s \le t$. Now we fix $x$ and distinguish between $0 < h \le \delta_n(x)$ and $\delta_n(x) < h \le 1/n$.

In the first case we estimate with a certain $\xi \in [x, x + 2h]$ from the mean value theorem

$$\begin{aligned} h^{-\beta} |\Delta_h^t (f - p_n)^{(r)}(x)| &\le C h^{-\beta} (|\Delta_h^s f^{(m)}(x)| + |\Delta_h^s p_n^{(r)}(x)|) \\ &\le C h^{\alpha - \beta} \|f^{(m)}\|_{\alpha,s} + C h^{s-\beta} |p_n^{(m+s)}(\xi)| \,. \end{aligned}$$

Then, by (2) and (3) we obtain

$$|p_n^{(m+s)}(\xi)| \le C \delta_n(\xi)^{-s} \omega_s(f^{(m)}, \delta_n(\xi)) \,.$$

Now it follows from (24) and (25) that

$$\begin{aligned} h^{-\beta} |\Delta_h^t (f - p_n)^{(r)}(x)| &\le n^{\beta - \alpha} \|f^{(m)}\|_{\alpha,s} + C h^{s-\beta} \delta_n(\xi)^{-s} \omega_s(f^{(m)}, \delta_n(\xi)) \\ &\le n^{\beta - \alpha} \|f^{(m)}\|_{\alpha,s} + C \delta_n(\xi)^{\alpha - \beta} \|f^{(m)}\|_{\alpha,s} \\ &\le C n^{\beta - \alpha} \|f^{(m)}\|_{\alpha,s} \,. \end{aligned}$$

In the remaining case $\delta_n(x) < h \le 1/n$ we conclude with the help of (26) and (27). This gives (here only written for $t = 2$) that

$$\begin{aligned} h^{-\beta} |\Delta_h^2 (f - p_n)^{(r)}(x)| &\le C h^{-\beta} (\delta_n(x)^\alpha + 2\delta_n(x + h)^\alpha + \delta_n(x + 2h)^\alpha) \|f^{(m)}\|_{\alpha,s} \\ &\le C h^{\alpha - \beta} \|f^{(m)}\|_{\alpha,s} \\ &\le C n^{\beta - \alpha} \|f^{(m)}\|_{\alpha,s} \,. \end{aligned}$$

Taking the maximum over $x \in J_{th}$ and the supremum over $h$ the case c) is proved. Summarizing the estimates we obtain the Jackson-type result. ∎

**Proof of Theorem 3.4:** Using a polynomial $p_n$ satisfying the Jackson-type inequality (17) we estimate

$$\|f - q_n\|_{r,\beta,t} \le \|f - p_n\|_{r,\beta,t} + \|p_n - q_n\|_{r,\beta,t} \,.$$

Applying the Markov-type inequality (13) to the polynomial $p_n - q_n$ we obtain

$$\begin{aligned} \|f-q_n\|_{r,\beta,t} &\leq \|f-p_n\|_{r,\beta,t} + Cn^{2(r+\beta-\ell-\gamma)}\|p_n-q_n\|_{\ell,\gamma,u} \\ &\leq \|f-p_n\|_{r,\beta,t} + Cn^{2(r+\beta-\ell-\gamma)}(\|f-p_n\|_{\ell,\gamma,u} + \|f-q_n\|_{\ell,\gamma,u}) \,. \end{aligned}$$

Now we use the assumption (20) for $\|f-q_n\|$ and the Jackson-type inequality (17) for $\|f-p_n\|$ in the different norms which gives immediately the desired result. Thus we have only to note the corresponding restrictions for the parameters.

a) The Markov-type inequality (13) needs for $u=2$ the conditions:

$$\begin{array}{llll} \text{If} & l=r-2, & \text{then} & \gamma \leq 1+\beta. \\ \text{If} & l=r-1,\ t=1, & \text{then} & \gamma = 0. \\ \text{If} & l=r-1,\ t=2, & \text{then} & \gamma \leq \beta. \\ \text{If} & l=r,\ t=1, & \text{then} & \gamma = 0. \end{array}$$

b) The Jackson-type inequality (17) needs for $s=2$ the conditions

$$\begin{array}{llll} \text{If} & r=m,\ t=1, & \text{then} & \beta = 0. \\ \text{If} & r=m-1,\ t=2, & \text{then} & \beta \leq 1. \\ \text{If} & l=m,\ u=1 & \text{,then} & \gamma = 0. \\ \text{If} & l=m-1,\ u=2, & \text{then} & \gamma \leq 1. \end{array}$$

These conditions are equivalently rewritten in the tabular of Theorem 3.4. ■

**Proof of Theorem 3.5:** Again we use a polynomial $p_n$ satisfying the Jackson-type inequality (17). Following (2) from Theorem 2.2 we obtain from our assumption (22) that

$$\begin{aligned} |(p_n-q_n)(x)| &\leq |(f-p_n)(x)| + |(f-q_n)(x)| \\ &\leq C\delta_n(x)^{m+\alpha}\|f^{(m)}\|_{\alpha,s} \,. \end{aligned}$$

Applying the Bernstein-type inequality in Theorem 3.2 yields

$$\|p_n-q_n\|_{r,\beta,t} \leq Cn^{r+\beta-m-\alpha}\|f^{(m)}\|_{\alpha,s} \,.$$

Then the Jackson-type inequality with the assumptions (15) and (16) finishes the proof, namely

$$\begin{aligned} \|f-q_n\|_{r,\beta,t} &\leq \|f-p_n\|_{r,\beta,t} + \|p_n-q_n\|_{r,\beta,t} \\ &\leq Cn^{r+\beta-m-\alpha}\|f^{(m)}\|_{\alpha,s} \,. \end{aligned}$$

■

**Proof of Theorem 3.6:** Here we work with $p_n$ from the Jackson-type Theorem 3.3 and with the Markov-type ineqality (13), namely

$$\|L_n(f-p_n)\|_{r,\beta,t} \le Cn^{2r+2\beta}\|L_n(f-p_n)\|_{0,0,1}\,. \tag{29}$$

Then we estimate

$$\begin{aligned}\|f-L_nf\|_{r,\beta,t} &\le \|f-p_n\|_{r,\beta,t}+\|L_n(f-p_n)\|_{r,\beta,t}\\ &\le Cn^{r+\beta-m-\alpha}\|f^{(m)}\|_{\alpha,s}+Cn^{2r+2\beta}\|L_n(f-p_n)\|_{0,0,1}\\ &\le Cn^{r+\beta-m-\alpha}\|f^{(m)}\|_{\alpha,s}+Cn^{2r+2\beta}\|L_n\|_{C\to C}\|f-p_n\|\\ &\le Cn^{2r+2\beta-m-\alpha}\|L_n\|_{C\to C}\|f^{(m)}\|_{\alpha,s}\,.\end{aligned}$$

Note that the Markov-type inequality (29) only requires the assumptions (7) and (8). ∎

**Proof of Theorem 3.7:** At first we state an easy consequence of Theorem 3.2, namely for $r+\beta\le 2\nu$

$$\|Q_{2\nu}p_n\|_{r,\beta,t} \le Cn^{r+\beta}\|p_n\|\,.$$

Combining this with the usual Markov-type inequality (13) we obtain for arbitrary $r,\beta$ a Markov-Bernstein-type inequality

$$\|Q_{2\nu}p_n\|_{r,\beta,t} \le Cn^{r+\beta+(r+\beta-2\nu)_+}\|p_n\|\,. \tag{30}$$

Now we use a polynomial $q_n$ from the Jackson-type theorem with interpolatory conditions (see (19)) to write

$$\|f-H_nf\|_{r,\beta,t} \le \|f-q_n\|_{r,\beta,t}+\|H_n(f-q_n)\|_{r,\beta,t}\,.$$

Hence we have only to deal with $\|H_n(f-q_n)\|_{r,\beta,t}$. From

$$q_n(x)=f(x) \qquad \text{for } x\in K_\nu$$

and (30) we obtain

$$\begin{aligned}\|H_n(f &- q_n)\|_{r,\beta,t} = \Big\|\sum_{j=1}^{n}(f-q_n)(x_j)\frac{Q_{2\nu}\ell_j}{Q_{2\nu}(x_j)}\Big\|_{r,\beta,t}\\ &\le Cn^{r+\beta+(r+\beta-2\nu)_+}\Big\|\sum_{j=1}^{n}\frac{(f-q_n)(x_j)}{Q_{2\nu}(x_j)}\ell_j\Big\|\\ &\le Cn^{r+\beta+(r+\beta-2\nu)_+}\max_{1\le j\le n}\Big|\frac{(f-q_n)(x_j)}{Q_{2\nu}(x_j)}\Big|\,\Big\|\sum_{j=1}^{n}|\ell_j|\,\Big\|\\ &\le Cn^{r+\beta-m-\alpha+(r+\beta-2\nu)_+}\Big\|\sum_{j=1}^{n}|\ell_j|\,\Big\|\,n^{-m-\alpha}\|f^{(m)}\|_{\alpha,s}\,.\end{aligned}$$

Here we used (6) to derive the last inequality. ■

## References


[1] **Balázs, K., Kilgore, T.**, and **Vértesi, P.** : *An interpolatory version of Timan's theorem on simultaneous approximation.* Acta Math. Hung. **57**, 285–290 (1991)

[2] **Ditzian, Z.**, and **Jiang, D.** : *Approximation of functions by polynomials in* [4] $C[-1,1]$. Can. J. Math. Vol. **44**, 924–940 (1992)

[3] **Dzjadyk, V. K.** : *Introduction to the theory of uniform approximation of functions by polynomials* (in Russian). Moscow 1977

[4] **Elliott, D.** : *On the Hölder semi-norm of the remainder in polynomial approximation.* Bull. Austral. Math. Soc. **49**, 421–426 (1994)

[5] **Gonska, H.**, and **Hinnemann, E.** : *Punktweise Abschätzungen zur Approximation durch algebraische Polynome.* Acta Math. Hung. **46**, 243–254 (1985)

[6] **Gopengauz, I. E.** : *A theorem of A. F. Timan on the approximation of functions by polynomials on a finite segment* (in Russian). Mat. Zametki **1**, 163–172 (1967)

[7] **Ioakimidis, N. I.** : *A simple proof of Kalandiya's theorem in approximation theory.* Serdica **9**, 414–416 (1983)

[8] **Ioakimidis, N. I.** : *An improvement of Kalandiya's theorem.* J. Approx. Theory **38**, 354–356 (1983)

[9] **Kalandiya, A. I.** : *A direct method of solving the wing theory equation and its application in elasticity theory* (in Russian). Mat. Sbornik **42**, 249–272 (1957)

[10] **Kilgore, T.**, and **Prestin, J.** : *A Theorem of Gopengauz type with added interpolatory conditions.* Numer. Funct. Anal. Optim. **15**, 859–868 (1994)

[11] **Leindler, L.** : *Generalizations of Prößdorf's theorems.* Studia Sci. Math. **14**, 431–439 (1979)

[12] **Lorentz, G. G.** : *Approximation of functions.* New York 1966

[13] **Prestin, J.** : *Best Approximation and Interpolation in Algebraic Hölder Norms.* Coll. Math. Soc. Jan. Bolyai, Approximation Theory, Kecskemét (Hungary), 1990, 583–590

[14] **Prestin, J.** : *S. Prößdorf, Error Estimates in Generalized Trigonometric Hölder-Zygmund Norms.* Z. Anal. Anwendungen, **9**, 343–349 (1990)

[15] **Privalov, A. A.** : *On the simultaneous interpolation and approximation of continuous functions* (in Russian). Mat. Zametki **35**, 381–395 (1984)

[16] **Prößdorf, S.** : *Zur Konvergenz der Fourierreihen hölderstetiger Funktionen.* Math. Nachr. **69**, 7–14 (1975)

[17] **Prößdorf, S.**, and **Silbermann, B.** : *Numerical Analysis for Integral and Operator Equations.* Berlin, Basel (Series Operator Theory), 1990

[18] **Szabados, J.** : *On the convergence of the derivatives of projection operators.* Analysis **7**, 349–357 (1987)

[19] **Szabados, J.**, and **Vértesi, P.** : *Interpolation of functions.* Singapore 1990

[20] **Timan, A. F.** : *Theory of approximation of functions of a real variable.* English translation Pergamon Press, 1963

[21] **Trigub, R. M.** : *Approximation of functions by polynomials with integral coefficients* (in Russian). Isv. Akad. Nauk SSSR Ser. Mat. **26**, 261-280 (1962)

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