Rostock. Math. Kolloq. **51**, 5–24 (1997)

Subject Classification (AMS)
35B40, 35J25, 35J55

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Spectral properties of operator pencils generated by elliptic boundary value problems for the Lamé system

Dedicated to the professors of mathematics
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0 Introduction

The present paper is concerned with the spectral properties of operator pencils generated by boundary value problems for the Lamé system

$$\Delta U + \frac{1}{\gamma}\,\mathrm{grad}\,\mathrm{div}\,U = F \tag{1}$$

in a cone $\mathcal{K}$. Here $U = (U_1, U_2, U_3)$ denotes the displacement vector and γ is a positive constant which is related to the Poisson ratio ν via the equality $\gamma = 1 - 2\nu$. The study of the spectrum of these operator pencils is of great importance for the description of the behaviour of the solutions near conical points. It is well-known (see e.g. [3], [9], [12]) that the solutions of elliptic boundary value problems in a neighbourhood of a conical point $x^{(0)}$ asymptotically behave like a linear combination of terms of the form

$$r^\lambda \sum_{k=0}^{s} \frac{1}{k!}\,(\log r)^k\, u^{(s-k)}(\omega), \tag{2}$$

where $r = |x - x^{(0)}|$ and ω are coordinates on the sphere $|x - x^{(0)}| = 1$. Here the exponents λ are the eigenvalues of some operator pencil which arises from the operators of the boundary value problem applying the Mellin transform $r \to \lambda$. Thus, the question of conic singularities is reduced to the spectral analysis of the mentioned operator pencil. This is, in fact, the subject of the present paper. Since we consider solutions which have square integrable derivatives of first order in a neighbourhood of the conical point, we are especially interested in the eigenvalues situated in the half-plane $\mathrm{Re}\,\lambda > -1/2$.

Let $\sigma(U) = \{\sigma_{i,j}(U)\}$ be the stress tensor connected with the deformation tensor

$$\Big\{\varepsilon_{i,j}(U)\Big\} = \Big\{\frac{1}{2}\,(\partial_{x_j} U_i + \partial_{x_i} U_j)\Big\}$$

by the Hooke law

$$\sigma_{i,j} = 2\mu\,\Big(\frac{1-\gamma}{2\gamma}(\varepsilon_{1,1}+\varepsilon_{2,2}+\varepsilon_{3,3})\,\delta_{i,j} + \varepsilon_{i,j}\Big),$$

where μ is the shear modulus and $\delta_{i,j}$ denotes the Kronecker symbol. In the present paper the following boundary conditions for the system (1) are considered:

(i) $U = 0$,

(ii) $U_n = 0$ and $\sigma_{n,\tau}(U) = 0$,

(iii) $U_\tau = 0$ and $\sigma_{n,n}(U) = 0$.

Here $n = (n_1, n_2, n_3)$ denotes the exterior normal to $\partial\mathcal{K}\backslash\{0\}$, $U_n = U \cdot n$ is the normal component of the vector U, U_τ is the tangential component of the vector U on the boundary (i.e., the projection of U onto the tangent plane to $\partial\mathcal{K}\backslash\{0\}$),

$$\sigma_{n,n}(U) = \sum_{i,j=1}^{3} \sigma_{i,j}(U)\, n_i\, n_j$$

is the normal component of the vector $\sigma_n(U) = \sigma(U)\cdot n$, and $\sigma_{n,\tau}(U)$ is the tangential component of the vector $\sigma_n(U)$.

The following assertions are the main results of the present paper:

1. The strip
$$\Big|\mathrm{Re}\,\lambda + \frac{1}{2}\Big|^2 \le \Big(\frac{3}{2}+\gamma\Big)^2 + \gamma^2 \tag{3}$$
contains only real eigenvalues of the pencil $\mathfrak{A}(\lambda)$.

2. There are no generalized eigenvectors corresponding to eigenvalues in the interior of the strip (3).

3. We derive a variational principle for the eigenvalues of $\mathfrak{A}$.

In the case of the Dirichlet problem the results of the present paper are contained in the paper [5] of V. A. Kozlov, V. G. Maz'ya and C. Schwab. Furthermore, we refer to the paper [10] of V. G. Maz'ya and B. A. Plamenevskiĭ who proved that the strip

$$\Big|\mathrm{Re}\,\lambda + \frac{1}{2}\Big| \le \frac{(2\gamma+1)\,\Lambda}{\Lambda + 2\gamma + 4} + \frac{1}{2}$$

does not contain eigenvalues of the operator pencil generated by the Dirichlet problem. Here Λ is that positive number for which $\Lambda(\Lambda+1)$ is the first eigenvalue of the Laplace-Beltrami operator $-\delta$ on the domain Ω which is cut out from the unit sphere by the cone $\mathcal{K}$. In particular, the strip $-1 \le \operatorname{Re}\lambda \le 0$ is free of eigenvalues. For the case of the Neumann problem it was shown by V. A. Kozlov and V. G. Maz'ya [4] that this strip contains only the eigenvalues $\lambda_0 = 0$ and $\lambda_1 = -1$.

Without proof we give a consequence of our results. Let U be a solution of the system (1) in a domain of polyhedral type satisfying one of the boundary conditions (i), (ii), (iii) on every face of the domain. We suppose, for simplicity, that the vector-function F on the right of (1) vanishes in a neighbourhood of the singular boundary point $x^{(0)}$. Then U has the asymptotics

$$U = \sum_k c_k\, r^{\lambda_k}\, u^{(k)}(\omega) + o(r^\alpha)$$

near $x^{(0)}$, where $\alpha = \sqrt{2\gamma^2 + 3\gamma + 9/4} - 1/2 - \varepsilon$, ε is an arbitrary small positive number, λ_k are the eigenvalues of the corresponding operator pencil $\mathfrak{A}$ in the interval $(-1/2, \alpha]$, and $u^{(k)}$ are eigenfunctions.

1 Formulation of the problem

Let $\mathcal{K}$ be the cone $\{(x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x/|x| \in \Omega\}$, where Ω is a domain on the unit sphere with Lipschitz boundary $\partial\Omega = \overline{\gamma}_1 \cup \cdots \cup \overline{\gamma}_N$ and $\gamma_1, \dots, \gamma_N$ are pairwise disjoint open arcs. Then

$$\partial\mathcal{K} = \overline{\Gamma}_1 \cup \cdots \cup \Gamma_N\,,$$

where $\Gamma_k = \{x : x/|x| \in \gamma_k\}$. We divide the set of the indices $1, \dots, N$ into three subsets I_0, I_n, I_τ.

The homogeneous system (1) can be written in the form

$$\sum_{j=1}^{3} \frac{\partial \sigma_{i,j}(U)}{\partial x_j} = 0 \quad \text{for } i = 1, 2, 3, \tag{4}$$

where $\{\sigma_{i,j}(U)\}$ denotes the stress tensor. Throughout the paper it will be assumed that

$$0 < \gamma < 3$$

or, what is the same, $-1 < \nu < 1/2$. Our goal is to find generalized solutions of the system (4) satisfying the boundary conditions

$$\left.\begin{array}{rll} U = 0 & \text{on } \Gamma_k & \text{for } k \in I_0, \\ U_n = 0, \quad \sigma_{n,\tau}(U) = 0 & \text{on } \Gamma_k & \text{for } k \in I_n, \\ U_\tau = 0, \quad \sigma_{n,n}(U) = 0 & \text{on } \Gamma_k & \text{for } k \in I_\tau. \end{array}\right\} \tag{5}$$

The notion of generalized solutions will be introduced by means of the Green formula

$$\int\limits_{\mathcal{K}} \sum_{i,j=1}^{3} \sigma_{ij}(U) \cdot \varepsilon_{ij}(\overline{V})\, dx = -\int\limits_{\mathcal{K}} \sum_{i,j=1}^{3} \frac{\partial \sigma_{ij}(U)}{\partial x_j} \cdot \overline{V}_i\, dx + \int\limits_{\partial\mathcal{K}\backslash\{0\}} \sum_{i,j=1}^{3} \sigma_{ij}(U)\, n_j \cdot \overline{V}_i\, d\sigma \tag{6}$$

which is satisfied for all $U, V \in C_0^\infty(\overline{\mathcal{K}}\backslash\{0\})^3$. If U is a formal solution of problem (4), (5) and $V \in H^1(\mathcal{K})^3$ is a vector-function vanishing for small and large $|x|$ which satisfies the conditions

$$V = 0 \text{ on } \Gamma_k \text{ for } k \in I_0, \quad V_n = 0 \text{ on } \Gamma_k \text{ for } k \in I_n, \quad V_\tau = 0 \text{ on } \Gamma_k \text{ for } k \in I_\tau, \tag{7}$$

then (6) implies

$$\sum_{i,j=1}^{3} \int\limits_{\mathcal{K}} \sigma_{i,j}(U) \cdot \varepsilon_{i,j}(\overline{V})\, dx = 0. \tag{8}$$

For this reason, it is natural to define generalized solutions by means of this equality. Let $\mathcal{H}$ be the space of all vector-functions $u \in H^1(\Omega)^3$ such that

- $u = 0$ on γ_k for $k \in I_0$,
- $u_n = 0$ on γ_k for $k \in I_n$,
- $u_\tau = 0$ on γ_k for $k \in I_0$,

where $u_n = u \cdot n$, n denotes the exterior normal to $\partial\mathcal{K}\backslash\{0\}$, and u_τ is the projection of the vector u onto the tangent plane to $\partial\mathcal{K}\backslash\{0\}$.

Definition *The function*

$$U(x) = r^{\lambda_0} \sum_{k=0}^{s} \frac{1}{k!} (\log r)^k\, u^{(s-k)}(\omega), \quad u^{(s-k)} \in \mathcal{H}, \tag{9}$$

is said to be a generalized solution of problem (4), (5) *if the integral identity* (8) *is valid for all* $V \in H^1(\mathcal{K})^3$ *with compact support in* $\overline{\mathcal{K}}\backslash\{0\}$ *satisfying the boundary conditions* (7).

2 The operator pencil generated by the boundary value problem

In what follows, we will systematically use the spherical coordinates r and $\omega = (\theta, \varphi)$ which are connected with the Cartesian coordinates $x = (x_1, x_2, x_3)$ by the relations

$$x_1 = r \sin\theta \cos\varphi, \qquad x_2 = r \sin\theta \sin\varphi, \qquad x_3 = r \cos\theta.$$

We write the integral identity (8) in terms of the spherical components $(U_r, U_\theta, U_\varphi)$ of the displacement vector, i.e.,

$$\begin{pmatrix} U_r \\ U_\theta \\ U_\varphi \end{pmatrix} = J \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}, \qquad J = J(\theta,\varphi) = \begin{pmatrix} \sin\theta \cos\varphi & \sin\theta \sin\varphi & \cos\theta \\ \cos\theta \cos\varphi & \cos\theta \sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix}.$$

For this we need the following representation of the components of the stress tensor in the spherical coordinate system:

$$\left.\begin{aligned} &\varepsilon_{rr} = \partial_r U_r\,, \qquad \varepsilon_{\varphi\varphi} = \frac{1}{r\sin\theta}\partial_\varphi U_\varphi + \frac{U_r}{r} + \cot\theta\,\frac{U_\theta}{r}, \\ &\varepsilon_{\theta\theta} = \frac{1}{r}\partial_\theta U_\theta + \frac{U_r}{r}, \qquad \varepsilon_{r\varphi} = \frac{1}{2}\Big(\frac{1}{r\sin\theta}\partial_\varphi U_r - \frac{U_\varphi}{r} + \partial_r U_\varphi\Big)\,, \\ &\varepsilon_{r\theta} = \frac{1}{2}\Big(\frac{1}{r}\partial_\theta U_r - \frac{U_\theta}{r} + \partial_r U_\theta\Big)\,, \\ &\varepsilon_{\theta\varphi} = \frac{1}{2}\Big(\frac{1}{r}\partial_\theta U_\varphi - \cot\theta\,\frac{U_\varphi}{r} + \frac{1}{r\sin\theta}\,\partial_\varphi U_\theta\Big) \end{aligned}\right\} \tag{10}$$

(see, for example, [7]). Furthermore, the Hooke law has the following representation in the spherical coordinate system:

$$\begin{aligned} &\sigma_{rr} = 2\mu\,\Big(\frac{1-\gamma}{2\gamma}\Theta + \varepsilon_{rr}\Big)\,, \quad \sigma_{\varphi\varphi} = 2\mu\,\Big(\frac{1-\gamma}{2\gamma}\Theta + \varepsilon_{\varphi\varphi}\Big)\,, \\ &\sigma_{\theta\theta} = 2\mu\,\Big(\frac{1-\gamma}{2\gamma}\Theta + \varepsilon_{\theta\theta}\Big)\,, \quad \sigma_{r\varphi} = 2\,\mu\,\varepsilon_{r\varphi}\,, \quad \sigma_{\theta\varphi} = 2\,\mu\,\varepsilon_{\theta\varphi}\,, \quad \sigma_{r\theta} = 2\,\mu\,\varepsilon_{r\theta}\,, \end{aligned}$$

where $\Theta = \varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{\varphi\varphi}$. Written in terms of the spherical components, the integral identity (8) has the form

$$\begin{aligned} \int_0^\infty \int_\Omega \Big(&\sigma_{rr}(U)\,\varepsilon_{rr}(\overline{V}) + \sigma_{\theta\theta}(U)\,\varepsilon_{\theta\theta}(\overline{V}) + \sigma_{\varphi\varphi}(U)\,\varepsilon_{\varphi\varphi}(\overline{V}) + 2\sigma_{r\theta}(U)\,\varepsilon_{r\theta}(\overline{V}) \\ &+2\sigma_{r\varphi}(U)\,\varepsilon_{r\varphi}(\overline{V}) + 2\sigma_{\theta\varphi}(U)\,\varepsilon_{\theta\varphi}(\overline{V})\Big)\, r^2\, d\omega\, dr = 0, \end{aligned} \tag{11}$$

where $d\omega = \sin\theta\, d\theta\, d\varphi$. We introduce the sesquilinear form

$$\begin{aligned} a(u,v;\lambda) &= \frac{1}{4\mu\,|\log\varepsilon|}\int\limits_\varepsilon^{1/\varepsilon}\int\limits_\Omega \Big(\sigma_{rr}(U)\,\varepsilon_{rr}(\overline{V}) + \sigma_{\theta\theta}(U)\,\varepsilon_{\theta\theta}(\overline{V}) + \sigma_{\varphi\varphi}(U)\,\varepsilon_{\varphi\varphi}(\overline{V}) \\ &\qquad +2\,\sigma_{r\theta}(U)\,\varepsilon_{r\theta}(\overline{V}) + 2\,\sigma_{r\varphi}(U)\,\varepsilon_{r\varphi}(\overline{V}) + 2\,\sigma_{\theta\varphi}(U)\,\varepsilon_{\theta\varphi}(\overline{V})\Big)\, r^2\, d\omega\, dr \\ &= \frac{1}{2\,|\log\varepsilon|}\int\limits_\varepsilon^{1/\varepsilon}\int\limits_\Omega \Big(\varepsilon_{rr}(U)\,\varepsilon_{rr}(\overline{V}) + \varepsilon_{\theta\theta}(U)\,\varepsilon_{\theta\theta}(\overline{V}) + \varepsilon_{\varphi\varphi}(U)\,\varepsilon_{\varphi\varphi}(\overline{V}) \\ &\qquad + 2\,\varepsilon_{r\theta}(U)\,\varepsilon_{r\theta}(\overline{V}) + 2\,\varepsilon_{r\varphi}(U)\,\varepsilon_{r\varphi}(\overline{V}) + 2\,\varepsilon_{\theta\varphi}(U)\,\varepsilon_{\theta\varphi}(\overline{V}) \\ &\qquad + \frac{1-\gamma}{2\gamma}\,\Theta(U)\,\Theta(\overline{V})\Big)\, r^2\, d\omega\, dr, \end{aligned} \tag{12}$$

where $U = r^\lambda\, u(\omega)$, $V = r^{-1-\overline{\lambda}}\, v(\omega)$, and ε is a positive real number less than 1. It can be easily verified that the expression on the right side is independent of ε. Using the formulas for the stress and deformation tensor in spherical coordinates given above, we obtain

$$\begin{aligned} a(u,v;\lambda) = [u_\omega, v_\omega] + \frac{1}{2}\int\limits_\Omega \Big(&\nabla_\omega u_r\cdot\nabla_\omega\overline{v}_r + (\lambda+2)(1-\lambda)\Big(\frac{1+\gamma}{\gamma}u_r\,\overline{v}_r + u_\omega\cdot\overline{v}_\omega\Big) \\ &+ \frac{1-\gamma}{\gamma}(\nabla_\omega\cdot u_\omega)\,\nabla_\omega\cdot\overline{v}_\omega + \Big(\frac{1-\gamma}{\gamma}(\lambda+2)+2\Big)\,u_r\,\nabla_\omega\cdot\overline{v}_\omega \\ &+ \Big(\frac{1-\gamma}{\gamma}(1-\lambda)+2\Big)\,(\nabla_\omega\cdot u_\omega)\,\overline{v}_r - (1-\lambda)\,u_\omega\cdot\nabla_\omega\overline{v}_r \\ &- (\lambda+2)\,\nabla_\omega u_r\cdot\overline{v}_\omega\Big)\, d\omega, \end{aligned} \tag{13}$$

where

$$\begin{aligned} [u_\omega, v_\omega] = \int\limits_\Omega \Big(&\partial_\theta u_\theta\cdot\partial_\theta\overline{v}_\theta + \Big(\frac{1}{\sin\theta}\partial_\varphi u_\varphi + \cot\theta\, u_\theta\Big)\cdot\Big(\frac{1}{\sin\theta}\partial_\varphi\overline{v}_\varphi + \cot\theta\,\overline{v}_\theta\Big) \\ &+\frac{1}{2}\Big(\partial_\theta u_\varphi + \frac{1}{\sin\theta}\partial_\varphi u_\theta - \cot\theta\, u_\varphi\Big)\cdot\Big(\partial_\theta\overline{v}_\varphi + \frac{1}{\sin\theta}\partial_\varphi\overline{v}_\theta - \cot\theta\,\overline{v}_\varphi\Big)\Big)\, d\omega, \end{aligned} \tag{14}$$

$$\nabla_\omega v = \begin{pmatrix} \partial_\theta v \\ (\sin\theta)^{-1}\,\partial_\varphi v \end{pmatrix},$$

$$u_\omega = \begin{pmatrix} u_\theta \\ u_\varphi \end{pmatrix}, \qquad \nabla_\omega\cdot u_\omega = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\, u_\theta) + \frac{1}{\sin\theta}\partial_\varphi u_\varphi.$$

We set

$$\begin{aligned} Q(u_\omega, u_\omega) &= \int\limits_\Omega \Big(|\partial_\theta u_\theta|^2 + |\partial_\theta u_\varphi|^2 + \Big|\frac{1}{\sin\theta}\,\partial_\varphi u_\theta - \cot\theta\, u_\varphi\Big|^2 \\ &\qquad + \Big|\frac{1}{\sin\theta}\,\partial_\varphi u_\varphi + \cot\theta\, u_\theta\Big|^2\Big)\, d\omega \end{aligned} \tag{15}$$

and define the space $h^1(\Omega)$ as the set of all vector-functions u_ω for which the quantity

$$\|u_\omega\|_{h^1(\Omega)} = \Big(Q(u_\omega, u_\omega) + \int\limits_\Omega |u_\omega|^2\, d\omega \Big)^{1/2} \tag{16}$$

is finite. Furthermore, let $\overset{\circ}{h}{}^1(\Omega)^2$ be the closure of $(C_0^\infty(\Omega))^2$ with respect to the norm (16). It can be easily shown (see [5]) that the Cartesian components of the vector-function u belong to the Sobolev space $H^1(\Omega)$ if and only if $u_r \in H^1(\Omega)$ and $u_\omega \in h^1(\Omega)$. Analogously, the Cartesian components of the vector-function u are in the space $\overset{\circ}{H}{}^1(\Omega)$ if and only if $u_r \in \overset{\circ}{H}{}^1(\Omega)$ and $u_\omega \in \overset{\circ}{h}{}^1(\Omega)$.

Lemma 1 1) *The inequalities*

$$\begin{aligned} [u_\omega, u_\omega] &\le Q(u_\omega, u_\omega)\,, && (17)\\ [u_\omega, u_\omega] &\ge \frac{1}{2} \int\limits_\Omega |\nabla_\omega \cdot u_\omega|^2\, d\omega && (18) \end{aligned}$$

are valid for arbitrary vector-functions $u_\omega \in h^1(\Omega)$.

2) *Every vector-function* $u_\omega \in \overset{\circ}{h}{}^1(\Omega)$ *satisfies the equality*

$$[u_\omega, u_\omega] + \frac{1}{2} \int\limits_\Omega |u_\omega|^2\, d\omega = \frac{1}{2}\, Q(u_\omega, u_\omega) + \frac{1}{2} \int\limits_\Omega |\nabla_\omega \cdot u_\omega|^2\, d\omega. \tag{19}$$

Proof: 1) The inequality (17) follows from the estimate

$$\Big| \partial_\theta u_\varphi + \frac{1}{\sin\theta}\, \partial_\varphi u_\theta - \cot\theta\, u_\varphi \Big|^2 \le 2 \left(|\partial_\theta u_\varphi|^2 + \Big| \frac{1}{\sin\theta}\, \partial_\varphi u_\theta - \cot\theta\, u_\varphi \Big|^2 \right),$$

while (18) follows from

$$|\nabla_\omega \cdot u_\omega|^2 \le 2 \left(|\partial_\theta u_\theta|^2 + \Big| \frac{1}{\sin\theta}\, \partial_\varphi u_\varphi + \cot\theta\, u_\theta \Big|^2 \right).$$

2) It can be easily verified that

$$\begin{aligned} [u_\omega, u_\omega] - \frac{1}{2} \int\limits_\Omega |\nabla_\omega \cdot u_\omega|^2\, d\omega &= \frac{1}{2}\, Q(u_\omega, u_\omega) \\ &+ \mathrm{Re} \int\limits_\Omega \left(\partial_\theta u_\varphi \left(\frac{1}{\sin\theta} \partial_\varphi \overline{u}_\theta - \cot\theta\, \overline{u}_\varphi \right) - \partial_\theta u_\theta \left(\frac{1}{\sin\theta} \partial_\varphi \overline{u}_\varphi + \cot\theta\, \overline{u}_\theta \right) \right) d\omega. \end{aligned}$$

Integrating by parts in the last integral, we get the equality (19) for $u_\omega \in \overset{\circ}{h}{}^1(\Omega)$. The lemma is proved. ∎

We introduce the space $\mathcal{H}_s$ which consists of all vectors (u_r, u_ω) such that their Cartesian components belong to the space $\mathcal{H}$. It is evident that $\mathcal{H}_s$ is a closed subspace of $H^1(\Omega) \times h^1(\omega)$ and

$$\overset{\circ}{H}{}^1(\Omega)\times \overset{\circ}{h}{}^1(\Omega) \subset \mathcal{H}_s \subset H^1(\Omega) \times h^1(\Omega). \tag{20}$$

Using the estimates (17), (18), it can be shown that the form $a(\cdot,\cdot;\lambda)$ is continuous on $\mathcal{H}_s \times \mathcal{H}_s$ for every complex λ. Therefore, this form generates a continuous operator

$$\mathfrak{A}(\lambda) \,:\, \mathcal{H}_s \to \mathcal{H}_s^* \tag{21}$$

which is defined by the equality

$$\Big(\mathfrak{A}(\lambda)\, u\,,\, v\Big)_{L_2(\Omega)^3} = a(u, v; \lambda)\,, \qquad u, v \in \mathcal{H}_s.$$

Clearly, $\mathfrak{A}(\lambda)$ depends polynomially on λ. Furthermore, there is the following connection between the eigen- and generalized eigenvectors of the pencil $\mathfrak{A}(\lambda)$ and the solutions (9) of the equation (8).

By the definition of the sesquilinear form $a(\cdot,\cdot;\lambda)$, the following assertion is true.

Lemma 2 *The integral identity (8) has a solution of the form (9) if and only if λ_0 is an eigenvalue of the pencil $\mathfrak{A}$ and the vector-functions $u^{(0)}, \dots, u^{(s)}$ form a Jordan chain corresponding to this eigenvalue.*

3 Basic properties of the pencil $\mathfrak{A}$

Theorem 1

1) *The operator $\mathfrak{A}(\lambda)$ is Fredholm for all $\lambda \in \mathbb{C}$.*

2) *The operator $\mathfrak{A}(-1/2 + it)$ is positive definite for all real t.*

3) *The spectrum of the pencil $\mathfrak{A}(\lambda)$ consists of isolated eigenvalues with finite algebraic multiplicities.*

4) *The number λ_0 is an eigenvalue of the pencil $\mathfrak{A}(\lambda)$ if and only if $-1-\overline{\lambda}_0$ is an eigenvalue of this pencil. The geometric, algebraic, and partial multiplicities of the eigenvalues λ_0 and $-1 - \overline{\lambda}_0$ coincide.*

Proof: 1) We prove that there exists a constant $c_1(\lambda)$ such that

$$|a(u, u; \lambda)| \geq c_0 \left(\|u_r\|^2_{H^1(\Omega)} + \|u_\omega\|^2_{h^1(\Omega)} \right) - c_1(\lambda) \int\limits_\Omega (|u_r|^2 + |u_\omega|^2)\, d\omega \tag{22}$$

for arbitrary $u \in \mathcal{H}_s$.

Using (18) and the fact that $(2\gamma)^{-1}(1-\gamma) > -1/3$ for $\gamma < 3$, we get

$$\int\limits_\Omega \frac{1-\gamma}{2\gamma}\, |\nabla_\omega \cdot u_\omega|^2\, d\omega \ge -\frac{2}{3}\,[u_\omega, u_\omega]\,.$$

Furthermore, we have

$$\begin{aligned} \Big|\int\limits_\Omega u_r\, \nabla_\omega \cdot \overline{u}_\omega\, d\omega\Big| &\le \varepsilon \int\limits_\Omega |\nabla_\omega \cdot u_\omega|^2\, d\omega + \frac{1}{4\varepsilon}\int\limits_\Omega |u_r|^2\, d\omega \\ &\le 2\varepsilon\,[u_\omega, u_\omega] + \frac{1}{4\varepsilon}\int\limits_\Omega |u_r|^2\, d\omega \end{aligned}$$

and

$$\Big|\int\limits_\Omega u_\omega \cdot \nabla_\omega \overline{u}_r\, d\omega\Big| \le \varepsilon \int\limits_\Omega |\nabla_\omega u_r|^2\, d\omega + \frac{1}{4\varepsilon}\int\limits_\Omega |u_\omega|^2\, d\omega,$$

where ε is an arbitrary positive number. Let ε be sufficiently small. Then (13) yields

$$|a(u,u;\lambda)| \ge \frac{1}{4}\Big([u_\omega, u_\omega] + \int\limits_\Omega |\nabla_\omega u_r|^2\, d\omega\Big) - c(\lambda)\int\limits_\Omega (|u_r|^2 + |u_\omega|^2)\, d\omega$$

with a constant $c(\lambda)$ depending only on λ and γ. From this, by means of the twodimensional Korn inequality, we obtain (22). This proves the Fredholm property of the operator $\mathfrak{A}(\lambda)$ for arbitrary $\lambda \in \mathbb{C}$.

2) Now let $\operatorname{Re}\lambda = -1/2$. Then, by (12), the expression $a(u,u;\lambda)$ can be written in the form

$$\begin{aligned} \frac{1}{2|\log\varepsilon|}\int\limits_\varepsilon^{1/\varepsilon}\int\limits_\Omega \Big(&|\varepsilon_{rr}(U)|^2 + |\varepsilon_{\theta\theta}(U)|^2 + |\varepsilon_{\varphi\varphi}(U)|^2 + \frac{1-\gamma}{2\gamma}\,\big|\varepsilon_{rr}(U) + \varepsilon_{\theta\theta}(U) + \varepsilon_{\varphi\varphi}(U)\big|^2 \\ &+ 2\,|\varepsilon_{r\theta}(U)|^2 + 2\,|\varepsilon_{r\varphi}(U)|^2 + 2\,|\varepsilon_{\theta\varphi}(U)|^2\Big) r^2\, d\omega\, dr, \end{aligned}$$

where $U = r^\lambda u(\omega)$. Since $0 < \gamma < 3$, we have

$$\begin{aligned} &|\varepsilon_{rr}(U)|^2 + |\varepsilon_{\theta\theta}(U)|^2 + |\varepsilon_{\varphi\varphi}(U)|^2 + \frac{1-\gamma}{2\gamma}\,\Big|\varepsilon_{rr}(U) + \varepsilon_{\theta\theta}(U) + \varepsilon_{\varphi\varphi}(U)\Big|^2 \\ &\qquad \ge \min\Big(1\,,\ \frac{3-\gamma}{2\gamma}\Big)\cdot\Big(|\varepsilon_{rr}(U)|^2 + |\varepsilon_{\theta\theta}(U)|^2 + |\varepsilon_{\varphi\varphi}(U)|^2\Big). \end{aligned}$$

Hence the form $a(u,u;\lambda)$ is nonnegative for $\operatorname{Re}\lambda = -1/2$. Moreover, the equality $a(u,u;\lambda) = 0$ implies

$$\varepsilon_{rr}(U) = \varepsilon_{\theta\theta}(U) = \varepsilon_{\varphi\varphi}(U) = \varepsilon_{r\theta}(U) = \varepsilon_{r\varphi}(U) = \varepsilon_{\theta\varphi}(U) = 0.$$

Since $\varepsilon_{rr}(U) = \lambda r^{\lambda-1} u_r$, we conclude from this that $u_r = 0$. Analogously, by (10), we obtain $u_\varphi = u_\theta = 0$, i.e, $u_\omega = 0$. Thus, we have proved assertion 2).

Assertion 3) is a consequence of the first two assertions (see [1]).

Finally, by (13), we have $a(u, v; \lambda) = \overline{a(v, u, -1-\overline{\lambda})}$ and, therefore,

$$\mathfrak{A}(\lambda) = \mathfrak{A}(-1-\overline{\lambda})^*.$$

This implies assertion 4). The proof of the theorem is complete. ■

In the sequel, the properties of the space $\mathcal{H}_s$ given in the next lemma will play a crucial role.

Lemma 3 1) *The subspace $\mathcal{H}_s$ admits the representation*

$$\mathcal{H}_s = \mathcal{H}_s^r \times \mathcal{H}_s^\omega\,, \tag{23}$$

where $\mathcal{H}_s^r$, $\mathcal{H}_s^\omega$ are subspaces of $H^1(\Omega)$ and $h^1(\Omega)$, respectively, such that $\overset{\circ}{H}{}^1(\Omega) \subset \mathcal{H}_s^r$, $\overset{\circ}{h}{}^1(\Omega) \subset \mathcal{H}_s^\omega$.

2) *The equality*

$$\int\limits_{\partial\Omega} u_n\, \overline{v}_r\, d\omega' = 0 \tag{24}$$

or, equivalently,

$$\int\limits_{\Omega} \Big((\nabla_\omega \cdot u_\omega)\, \overline{v}_r + u_\omega \cdot \nabla_\omega \overline{v}_r \Big)\, d\omega = 0 \tag{25}$$

is satisfied for all $u, v \in \mathcal{H}_s$.

Proof:

1) In order to prove (23) we have to show that $(u_r, 0) \in \mathcal{H}_s$ if $(u_r, u_\omega) \in \mathcal{H}_s$.

Let (u_r, u_ω) be an arbitrary element of $\mathcal{H}_s$. Then the Cartesian components of the vector-function $(u_r, 0)$ are

$$w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = J^* \begin{pmatrix} u_r \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\theta\, \cos\varphi \\ \sin\theta\, \sin\varphi \\ \cos\theta \end{pmatrix} u_r\,.$$

If $k \in I_0 \cup I_\tau$, then $u_r = 0$ on γ_k and, therefore, $w = 0$ on γ_k. Furthermore, since the vector $(\sin\theta\, \cos\varphi, \sin\theta\, \sin\varphi, \cos\theta) = x/|x|$ is orthogonal to n, we have $w_n = 0$ on γ_k for every

$k = 1, \dots, N$. Thus, $w \in \mathcal{H}$ and, consequently, $(u_r, 0) \in \mathcal{H}_s$.

2) If $k \in I_0 \cup I_\tau$, then $v_r = 0$ on γ_k, while $u_n = 0$ on γ_k for $k \in I_0 \cup I_n$. Hence

$$\int\limits_{\gamma_k} u_n \cdot \overline{v}_r \, d\omega' = 0$$

for $k = 1, \dots, N$. This implies (24).

It remains to prove that the left sides of (24) and (25) coincide. Using the representation of u_ω by the Cartesian components of u, we get

$$\begin{aligned}
\int\limits_\Omega \Big((\nabla_\omega \cdot u_\omega)\, \overline{v}_r + u_\omega \cdot \nabla_\omega \overline{v}_r \Big)\, d\omega &= \int\limits_\Omega (\sin\theta)^{-1} \Big(\partial_\theta(\sin\theta\, u_\theta\, \overline{v}_r) + \partial_\varphi(u_\varphi \overline{v}_r) \Big)\, d\omega \\
= \int\limits_\Omega \bigg(\cos\theta \, \big(\cos\varphi\, \partial_\theta(u_1 \overline{v}_r) + \sin\varphi\, \partial_\theta(u_2\overline{v}_r)\big) &- \sin\theta\, \partial_\theta(u_3\overline{v}_r) - \frac{\sin\varphi}{\sin\theta}\, \partial_\varphi(u_1\overline{v}_r) \\
+ \frac{\cos\varphi}{\sin\theta} \partial_\varphi(u_2\overline{v}_r) - 2\sin\theta\, \cos\varphi\, u_1\overline{v}_r &- 2\sin\theta\, \sin\varphi\, u_2\overline{v}_r - 2\cos\theta\, u_3 \overline{v}_r \bigg)\, d\omega.
\end{aligned} \tag{26}$$

Integrating by parts, we obtain

$$\begin{aligned}
\frac{1}{\log 2} \int\limits_{\substack{\mathcal{K} \\ 1<|x|<2}} \nabla_x \cdot (r^{-2} u(\omega)\, \overline{v_r(\omega)})\, dx &= \frac{1}{\log 2} \int\limits_{\substack{\partial\mathcal{K} \\ 1<|x|<2}} n \cdot r^{-2}\, u(\omega)\, \overline{v_r(\omega)}\, dx' \\
&= \int\limits_{\partial\Omega} u_n\, \overline{v}_r\, d\omega'.
\end{aligned} \tag{27}$$

The integrand $\nabla_x \cdot (r^{-2} u(\omega)\, \overline{v_r(\omega)})$ on the left side of (27) is equal to

$$\begin{aligned}
\frac{1}{r^3} \bigg(&\cos\theta\, \cos\varphi\, \partial_\theta(u_1\overline{v}_r) + \cos\theta\, \sin\varphi\, \partial_\theta(u_2\overline{v}_r) - \sin\theta\, \partial_\theta(u_3\overline{v}_r) - \frac{\sin\varphi}{\sin\theta}\, \partial_\varphi(u_1\overline{v}_r) \\
&+ \frac{\cos\varphi}{\sin\theta} \partial_\varphi(u_2\overline{v}_r) - 2\sin\theta\, \cos\varphi\, u_1\overline{v}_r - 2\sin\theta\, \sin\varphi\, u_2\overline{v}_r - 2\cos\theta\, u_3\overline{v}_r \bigg).
\end{aligned}$$

Hence the expressions (26) and (27) coincide. The lemma is proved. ■

By (25), the sesquilinear form $a(\cdot, \cdot; \lambda)$ can be written as

$$\begin{aligned}
a(u, v; \lambda) = [u_\omega, v_\omega] + \frac{1}{2} \int\limits_\Omega \bigg(&(\nabla_\omega u_r) \cdot \nabla_\omega \overline{v}_r + (\lambda+2)(1-\lambda) \Big(\frac{1+\gamma}{\gamma} u_r\, \overline{v}_r + u_\omega \cdot \overline{v}_\omega \Big) \\
+ \frac{1-\gamma}{\gamma} (\nabla_\omega \cdot u_\omega)\, \nabla_\omega \cdot \overline{v}_\omega &+ \frac{\lambda + 2 + 2\gamma}{\gamma} u_r\, \nabla_\omega \cdot \overline{v}_\omega + \frac{1 - \lambda + 2\gamma}{\gamma}\, (\nabla_\omega \cdot u_\omega)\, \overline{v}_r \bigg)\, d\omega
\end{aligned} \tag{28}$$

for $u, v \in \mathcal{H}_s$.

Let $\mathcal{M}(\lambda)$ be the matrix

$$\begin{pmatrix} \lambda+2+2\gamma & 0 & 0 \\ 0 & 1-\lambda+2\gamma & 0 \\ 0 & 0 & 1-\lambda+2\gamma \end{pmatrix}.$$

Then for $u, v \in \mathcal{H}_s$ we have $\mathcal{M}(\lambda)\, u \in \mathcal{H}_s$, $\mathcal{M}(\overline{\lambda})\, v \in \mathcal{H}_s$, and

$$a(\mathcal{M}(\lambda)u, v; -1-\lambda) = a(u, \mathcal{M}(\overline{\lambda})v; \lambda).$$

Consequently, for all $\lambda \in \mathbb{C}$ there is the equality

$$\mathcal{M}(\lambda)\, \mathfrak{A}(\lambda) = \mathfrak{A}(-1-\lambda)\, \mathcal{M}(\lambda). \tag{29}$$

Repeating the proof of Proposition 2.4 in [5], we get the following theorem.

Theorem 2 *Let λ_0 be an eigenvalue of the pencil $\mathfrak{A}(\lambda)$ and let $u^{(0)}, \dots, u^{(s)}$ be a Jordan chain corresponding to this eigenvalue. If $\lambda_0 \notin \{1+2\gamma, -2-2\gamma\}$ or $\mathcal{M}(\lambda_0)\, u^{(0)} \neq 0$, then $-1-\lambda_0$ is also an eigenvalue and the vector-functions*

$$\mathcal{M}(\lambda_0)\, u^{(0)}, \quad (-1)^k \left(\mathcal{M}(\lambda_0)\, u^{(k)} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} u^{(k-1)} \right), \quad k = 1, \dots, s,$$

form a Jordan chain to this eigenvalue.

Remark 1 If $\lambda_0 \neq 1+2\gamma$ and $\lambda_0 \neq -2-2\gamma$, then the formulas in the theorem determine a one-to-one relation between the eigenvectors and generalized eigenvectors of the pencil $\mathfrak{A}(\lambda)$ corresponding to the eigenvalues λ_0 and $-1-\lambda_0$.

4 On real eigenvalues of the pencil $\mathfrak{A}$

Theorem 3 *The strip*

$$\left|\mathrm{Re}\,\lambda + \frac{1}{2}\right|^2 \leq \left(\frac{3}{2}+\gamma\right)^2 + \gamma^2 \tag{30}$$

contains only real eigenvalues of the pencil $\mathfrak{A}(\lambda)$.

Proof: Let λ be a complex number such that $\mathrm{Re}\,\lambda \neq -1/2$, $\mathrm{Im}\,\lambda \neq 0$. We consider the sesquilinear form

$$q(u, v; \lambda) = a\Big(\begin{pmatrix} u_r \\ u_\omega \end{pmatrix}, \begin{pmatrix} \bar{c} v_r \\ v_\omega \end{pmatrix}; \lambda \Big), \tag{31}$$

where

$$c = \frac{\overline{\lambda} + 2\gamma + 2}{1 - \lambda + 2\gamma}\,.$$

By (28), we have

$$\begin{aligned} q(u,u;\lambda) \;\; &= \;\; [u_\omega, u_\omega] + \frac{1}{2} \int\limits_\Omega \Big(c\,|\nabla_\omega u_r|^2 + (\lambda+2)\,(1-\lambda)\,\Big(\frac{1+\gamma}{\gamma} c\,|u_r|^2 + |u_\omega|^2\Big) \\ &+ \; \frac{1-\gamma}{\gamma}\,|\nabla_\omega \cdot u_\omega|^2 + 2\,\mathrm{Re}\,\Big(\frac{\lambda+2+2\gamma}{\gamma}\,u_r\,\nabla_\omega\cdot\overline{u}_\omega\Big)\Big)\,d\omega. \end{aligned} \tag{32}$$

Using the formulas

$$c\,(1-\lambda)\,(\lambda+2) = |\lambda+2\gamma+2|^2 - 2\gamma(3+2\gamma)\,c,$$

$$c = \frac{(\mathrm{Re}\,\lambda + 2\gamma + 2)(1 - \mathrm{Re}\,\lambda + 2\gamma) + (\mathrm{Im}\,\lambda)^2 + i\,\mathrm{Im}\,\lambda\,(2\mathrm{Re}\,\lambda + 1)}{|1-\lambda+2\gamma|^2}\,,$$

we get

$$\begin{aligned} \mathrm{Im}\,q(u,u;\lambda) \;\; &= \;\; \frac{1}{2}\,\mathrm{Im}\,\lambda\,(2\,\mathrm{Re}\,\lambda + 1) \int\limits_\Omega \Big(\frac{1}{|1-\lambda+2\gamma|^2}\,|\nabla_\omega u_r|^2 \\ &\qquad - \frac{2(\gamma+1)(3+2\gamma)}{|1-\lambda+2\gamma|^2}\,|u_r|^2 - |u_\omega|^2\Big)\,d\omega, \end{aligned}$$

while the real part of $q(u,u;\lambda)$ is equal to

$$\begin{aligned} &[u_\omega, u_\omega] + \frac{1}{2} \int\limits_\Omega \Big(\mathrm{Re}\,c\,|\nabla_\omega u_r|^2 + \big((1-\mathrm{Re}\,\lambda)\,(\mathrm{Re}\,\lambda+2) + (\mathrm{Im}\,\lambda)^2\big)\,|u_\omega|^2 \\ &\quad + (1+\gamma)\,\Big(\frac{|\lambda+2\gamma+2|^2}{\gamma} - (6+4\gamma)\,\mathrm{Re}\,c\Big)\,|u_r|^2 \\ &\quad + \frac{1-\gamma}{\gamma}\,|\nabla_\omega\cdot u_\omega|^2 + 2\,\mathrm{Re}\,\Big(\frac{\lambda+2+2\gamma}{\gamma}\,u_r\,\nabla_\omega\cdot\overline{u}_\omega\Big)\Big)\,d\omega. \end{aligned}$$

This implies

$$\begin{aligned}
&\operatorname{Re} q(u,u;\lambda) - \frac{\operatorname{Re} c\,|1-\lambda+2\gamma|^2}{\operatorname{Im}\lambda\,(2\operatorname{Re}\lambda+1)} \operatorname{Im} q(u,u;\lambda) \\
&= [u_\omega,u_\omega] + \int\limits_\Omega \bigg(\Big((1-\operatorname{Re}\lambda)\,(\operatorname{Re}\lambda+2) + \gamma\,(3+2\gamma) + (\operatorname{Im}\lambda)^2 \Big)\, |u_\omega|^2 \\
&\qquad + \frac{1+\gamma}{2\gamma}\, |\lambda+2\gamma+2|^2\, |u_r|^2 + \frac{1-\gamma}{2\gamma}\, |\nabla_\omega\cdot u_\omega|^2 \\
&\qquad + \operatorname{Re}\Big(\frac{\lambda+2+2\gamma}{\gamma}\, u_r\, \nabla_\omega\cdot \overline{u}_\omega \Big) \bigg)\, d\omega \\
&= [u_\omega,u_\omega] + \int\limits_\Omega \frac{1+\gamma}{2\gamma} \Big| (\lambda+2\gamma+2)\, u_r + \frac{1}{1+\gamma}\, \nabla_\omega\cdot u_\omega \Big|^2\, d\omega \\
&\quad + \int\limits_\Omega \Big((1-\operatorname{Re}\lambda)\,(\operatorname{Re}\lambda+2) + \gamma\,(3+2\gamma) + (\operatorname{Im}\lambda)^2 \Big)\, |u_\omega|^2\, d\omega \\
&\quad - \frac{\gamma}{2(1+\gamma)} \int\limits_\Omega |\nabla_\omega\cdot u_\omega|^2\, d\omega.
\end{aligned}$$

If the inequality (30) is satisfied, then by (18) the right side of the last equation is positive for $u \neq 0$. Hence $q(u,u;\lambda) \neq 0$ for all nonreal λ in the strip (30), $\operatorname{Re}\lambda \neq -1/2$, and all $u \in \mathcal{H}_s$. From this we conclude the assertion of the theorem in the case $\operatorname{Re}\lambda \neq -1/2$. Since the line $\operatorname{Re}\lambda = -1/2$ does not contain eigenvalues (see Theorem 1, assertion 2), the theorem is completely proved. ■

5 On the existence of generalized eigenvectors

Now we show that the eigenvectors to eigenvalues in the interior of the strip (30) do not have generalized eigenvectors.

Lemma 4 *Let λ_0 be a real eigenvalue of the pencil $\mathfrak{A}$ in the interval*

$$-1/2 < \lambda < \sqrt{\gamma^2 + (\gamma+3/2)^2} - 1/2$$

and let $u^{(0)}$ be an eigenvector corresponding to this eigenvalue. Then

$$\frac{d}{d\lambda}\, a\Big(\begin{pmatrix} u_r^{(0)} \\ u_\omega^{(0)} \end{pmatrix}, \begin{pmatrix} \overline{c(\lambda)} u_r^{(0)} \\ u_\omega^{(0)} \end{pmatrix} ; \lambda \Big) \Big|_{\lambda=\lambda_0} < 0, \tag{33}$$

where $c(\lambda) = (\lambda+2\gamma+2)/(1-\lambda+2\gamma)$.

Proof: Let $q(\cdot,\cdot;\lambda)$ be the sesquilinear form (31) with $c(\lambda)$ as in the formulation of the lemma. Then, according to (32), the left side of (33) is equal to

$$\frac{d}{d\lambda}\, q(u^{(0)},u^{(0)};\lambda)\Big|_{\lambda=\lambda_0} = \int\limits_\Omega \Big(\frac{c'(\lambda_0)}{2}\, |\nabla_\omega u_r^{(0)}|^2 + \frac{1}{\gamma}\, \mathrm{Re}\, \big(u_r^{(0)}\, \nabla_\omega \cdot \overline{u_\omega^{(0)}}\big)$$
$$-\frac{2\lambda_0+1}{2}\, |u_\omega^{(0)}|^2 + \frac{1+\gamma}{2\gamma}\, \Big((\lambda_0+2)\,(1-\lambda_0)\, c'(\lambda_0) - (2\lambda_0+1)\, c(\lambda_0)\Big)\, |u_r^{(0)}|^2 \Big)\, d\omega,$$

where $c'(\lambda_0) = (3+4\gamma)\,(1-\lambda_0+2\gamma)^{-2}$.

Furthermore, by the first part of Lemma 3, the vector-function $(u_r^{(0)},0)$ belongs to the space $\mathcal{H}_s$. Consequently,

$$0 = a\Big(\begin{pmatrix} u_r^{(0)} \\ u_\omega^{(0)} \end{pmatrix}, \begin{pmatrix} u_r^{(0)} \\ 0 \end{pmatrix}; \lambda_0\Big) = \frac{1}{2} \int\limits_\Omega \Big(|\nabla_\omega u_r^{(0)}|^2 + \frac{1+\gamma}{\gamma}\, (\lambda_0+2)\,(1-\lambda_0)\, |u_r^{(0)}|^2$$
$$+ \frac{1-\lambda_0+2\gamma}{\gamma}\, (\nabla_\omega \cdot u_\omega^{(0)})\, \overline{u_r^{(0)}} \Big)\, d\omega.$$

From this we obtain

$$\frac{d}{d\lambda}\, q(u^{(0)},u^{(0)};\lambda)\Big|_{\lambda=\lambda_0}$$
$$= \frac{d}{d\lambda}\, q(u^{(0)},u^{(0)};\lambda)\Big|_{\lambda=\lambda_0} - \frac{2}{1-\lambda_0+2\gamma}\, \mathrm{Re}\, a\Big(\begin{pmatrix} u_r^{(0)} \\ u_\omega^{(0)} \end{pmatrix}, \begin{pmatrix} u_r^{(0)} \\ 0 \end{pmatrix}; \lambda_0\Big)$$
$$= \frac{1+2\lambda_0}{2} \int\limits_\Omega \Big(\frac{1}{(1-\lambda_0+2\gamma)^2}\, |\nabla_\omega u_r^{(0)}|^2 - \frac{2(1+\gamma)(3+2\gamma)}{(1-\lambda_0+2\gamma)^2}\, |u_r^{(0)}|^2 - |u_\omega^{(0)}|^2 \Big)\, d\omega.$$

Since $q(u^{(0)},u^{(0)};\lambda_0) = 0$, analogously to the proof of Lemma 4.1 in [5], we get

$$-\frac{(\lambda_0+2+2\gamma)(1-\lambda_0+2\gamma)}{1+2\lambda_0}\, \frac{d}{d\lambda} q(u^{(0)},u^{(0)};\lambda)\Big|_{\lambda=\lambda_0}$$
$$= q(u^{(0)},u^{(0)};\lambda_0) - \frac{(\lambda_0+2+2\gamma)(1-\lambda_0+2\gamma)}{1+2\lambda_0}\, \frac{d}{d\lambda} q(u^{(0)},u^{(0)};\lambda)\Big|_{\lambda=\lambda_0}$$
$$= [u_\omega^{(0)},u_\omega^{(0)}] + \frac{1+\gamma}{2\gamma} \int\limits_\Omega \Big| (\lambda_0+2\gamma+2) u_r^{(0)} + \frac{1}{1+\gamma} \nabla_\omega \cdot u_\omega^{(0)} \Big|^2\, d\omega$$
$$-\frac{\gamma}{2(1+\gamma)} \int\limits_\Omega |\nabla_\omega \cdot u_\omega^{(0)}|^2\, d\omega + \Big((\gamma+\frac{3}{2})^2 + \gamma^2 - (\lambda_0+\frac{1}{2})^2 \Big) \int\limits_\Omega |u_\omega^{(0)}|^2\, d\omega. \tag{34}$$

Using (18), we obtain that the right side of (34) is positive for

$$-1/2 < \lambda_0 < \sqrt{(\gamma+3/2)^2+\gamma^2} - 1/2.$$

Since $\sqrt{(\gamma+3/2)^2+\gamma^2} - 1/2 < 1+2\gamma$, we get (33). ■

Theorem 4 *The eigenvectors corresponding to eigenvalues in the strip*

$$\left|\mathrm{Re}\,\lambda + \frac{1}{2}\right|^2 < \left(\frac{3}{2}+\gamma\right)^2 + \gamma^2$$

do not have generalized eigenvectors.

Proof: By the fourth assertion of Theorem 1 we can restrict ourselves to real eigenvalues in the interval $-1/2 < \lambda < \sqrt{\gamma^2 + (\gamma+3/2)^2} - 1/2$.

Let λ_0 be an eigenvalue in this interval and let $u^{(0)}$ be an eigenvector corresponding to this eigenvalue. Furthermore, let $u^{(1)}$ be a generalized eigenvector corresponding to λ_0 and $u^{(0)}$. Then for arbitrary $v \in \mathcal{H}_s$ the following equalities are valid:

$$a(u^{(0)}, v; \lambda_0) = 0, \tag{35}$$

$$a(u^{(1)}, v; \lambda_0) + \frac{d}{d\lambda} a(u^{(0)}, v; \lambda)\Big|_{\lambda=\lambda_0} = 0. \tag{36}$$

We denote by $q(\cdot,\cdot;\lambda)$ the same sesquilinear form as in the proof of Lemma 4. From (35) it follows that

$$q(u^{(0)}, u^{(1)}; \lambda_0) = \frac{1}{1-\lambda+2\gamma}\, a(u^{(0)}\,,\, \mathcal{M}(\overline{\lambda}_0)\, u^{(1)}\,; \lambda_0) = 0.$$

Furthermore, since the form q is symmetric, we have

$$0 = q(u^{(1)}, u^{(0)}; \lambda_0) = \frac{1}{1-\lambda+2\gamma}\, a(u^{(1)}\,,\, \mathcal{M}(\overline{\lambda}_0)\, u^{(0)}\,; \lambda_0).$$

Hence we get

$$\begin{aligned}
\frac{d}{d\lambda}\, q(u^{(0)}, u^{(0)}; \lambda)\big|_{\lambda=\lambda_0} &= \frac{d}{d\lambda}\, \frac{1}{1-\lambda+2\gamma}\, a(u^{(0)}, \mathcal{M}(\overline{\lambda})\, u^{(0)}; \lambda)\Big|_{\lambda=\lambda_0} \\
&= \frac{1}{1-\lambda+2\gamma}\, \frac{d}{d\lambda}\, a(u^{(0)}, \mathcal{M}(\overline{\lambda})\, u^{(0)}; \lambda)\Big|_{\lambda=\lambda_0} \\
&= \frac{1}{1-\lambda+2\gamma} \left(a(u^{(0)}\,,\, \mathcal{M}'(\overline{\lambda}_0)\, u^{(0)}; \lambda_0) + \frac{d}{d\lambda} a(u^{(0)}, \mathcal{M}(\overline{\lambda}_0)\, u^{(0)}; \lambda)\Big|_{\lambda=\lambda_0} \right) \\
&= \frac{1}{1-\lambda+2\gamma} \left(a(u^{(0)}\,,\, \mathcal{M}'(\overline{\lambda}_0)\, u^{(0)}; \lambda_0) - a(u^{(1)}, \mathcal{M}(\overline{\lambda}_0)\, u^{(0)}; \lambda_0) \right) = 0.
\end{aligned}$$

This contradicts (33). The theorem is proved. ■

Remark 2 We have proved the results of Theorems 2–4 for the operator pencil $\mathfrak{A}$ generated by the Lamé system (4) with the boundary conditions (5). These results hold also for other boundary conditions provided the assertions of Lemma 3 are true for the space $\mathcal{H}$ which determines these boundary conditions. Note that the second assertion of Lemma 3 is not valid for the space $\mathcal{H} = H^1(\Omega) \times h^1(\Omega)$, i.e., the Neumann boundary conditions are excluded.

6 A variational principle

We consider the operator

$$A(\lambda) = \mathcal{M}(\lambda)\,\mathfrak{A}(\lambda).$$

For arbitrary $\lambda \in \mathbb{R}$, $u \in \mathcal{H}_s$ we have

$$\begin{aligned}
\Big(A(\lambda)u, u\Big)_{L_2(\Omega)^3} &= \tilde{a}(u, v; \lambda) \stackrel{def}{=} (1 - \lambda + 2\gamma)\, q(u, u; \lambda) = (1 - \lambda + 2\gamma)\, [u_\omega, u_\omega] \\
&+ \frac{1}{2} \int\limits_\Omega \Big((\lambda + 2 + 2\gamma)\, |\nabla_\omega u_r|^2 + \frac{1+\gamma}{\gamma}\, (\lambda + 2)\, (1 - \lambda)\, (\lambda + 2 + 2\gamma)\, |u_r|^2 \\
&\quad + (\lambda + 2)\, (1 - \lambda)\, (1 - \lambda + 2\gamma)\, |u_\omega|^2 + \frac{1 - \gamma}{\gamma}\, (1 - \lambda + 2\gamma)\, |\nabla_\omega \cdot u_\omega|^2 \\
&\quad + \frac{2\,(1 - \lambda + 2\gamma)\,(\lambda + 2 + 2\gamma)}{\gamma}\, \mathrm{Re}\, u_r\, \nabla_\omega \cdot \overline{u}_\omega \Big)\, d\omega.
\end{aligned}$$

The pencil $A(\lambda)$ has the following properties:

(i) there exist a positive constant C and a function h, continuous on the interval $-1/2 \leq \lambda \leq \beta$, $\beta = \sqrt{\gamma^2 + (\gamma + 3/2)^2} - 1/2$, such that $h(\lambda) > 0$ for $\lambda \in [-1/2, \beta)$ and

$$\tilde{a}(u, u; \lambda) + C\, \|u\|^2_{L_2(\Omega)^3} \geq h(\lambda)\, \|u\|^2_{\mathcal{H}_s} \qquad \text{for all } u \in \mathcal{H}_s\,,$$

(ii) $A(-1/2)$ is a positive-definite operator,

(iii) if $A(\lambda_0)u = 0$ for some $\lambda_0 \in (-1/2, \beta)$, $u \in \mathcal{H}_s \backslash \{0\}$, then

$$\frac{d}{d\lambda}\, \tilde{a}(u, u; \lambda)|_{\lambda = \lambda_0} < 0.$$

Indeed, property (i) follows from Lemma 3, Property (ii) follows from Theorem 1, and property (iii) is a consequence of Lemma 4 and of the inequality $\beta < 1 + 2\gamma$.

We denote by $\{\mu_j(\lambda)\}$ a nondecreasing sequence of eigenvalues of the operator $A(\lambda)$ counting their multiplicities. From the positivity of $\mathfrak{A}(-1/2)$ (see Theorem 1) it follows that $\mu_j(-1/2) > 0$. As a consequence of Propositions 6.3 and 6.4 in [5], we get the following assertions.

Theorem 5 *The spectrum of the pencil $\mathfrak{A}(\lambda)$ has the following properties in the strip $-1/2 \le \operatorname{Re}\lambda \le \beta$.*

1) *All eigenvalues of the pencil $\mathfrak{A}(\lambda)$ are real and may be characterized by*

$$\{\lambda_j\}_{j=1,\dots,J} = \Big\{\lambda \in [-\frac{1}{2},\beta] \,:\, \mu_j(\lambda) = 0 \ for\ j = 1,\dots,J\Big\},$$

where J is the largest index j for which the function μ_j has a zero in the interval $[-\frac{1}{2},\beta)$. For every j the function μ_j has not more than one zero in $[-\frac{1}{2},\beta)$.

2) *If $\lambda_0 \in [-\frac{1}{2},\beta)$ is an eigenvalue of multiplicity I, then I is equal to the number of function μ_j which have a zero at λ_0.*

If $\mathcal{H}_s \ne \overset{\circ}{H}{}^1(\Omega)\times \overset{\circ}{h}{}^1(\Omega)$, then the eigenvalues do not monotonically depend on the domain Ω. However, the following properties are a consequence of the variational principle

$$\mu_j(\lambda) = \max_L \min_{u\in L\backslash\{0\}} \frac{\tilde{a}(u,u;\lambda)}{\|u\|^2_{L_2(\Omega)^3}},$$

where the maximum is taken over all subspaces $L \subset \mathcal{H}_s$ of codimension $\ge j-1$.

Theorem 6 *Let $\mathcal{H}_{s,1}$, $\mathcal{H}_{s,2}$ be subspaces of $H^1(\Omega)\times h^1(\Omega)$ such that the assertions of Lemma 3 are valid. We denote by $\mathfrak{A}_1(\lambda)$, $\mathfrak{A}_2(\lambda)$ the operator pencils corresponding to these subspaces. Furthermore, let $\{\lambda_j^{(1)}\}$, $\{\lambda_j^{(2)}\}$ be the nondecreasing sequences of the eigenvalues of the pencils $\mathfrak{A}_1(\lambda)$ and $\mathfrak{A}_2(\lambda)$ in the interval $[-1/2,\beta)$ counted with their multiplicities. If $\mathcal{H}_{s,1}\subset\mathcal{H}_{s,2}$, then the number of the eigenvalues of the pencil $\mathfrak{A}_1(\lambda)$ in the interval $[-1/2,\beta)$ is not greater than the number of such eigenvalues of the pencil $\mathfrak{A}_2(\lambda)$. Furthermore, the inequality*

$$\lambda_j^{(2)} \le \lambda_j^{(1)}$$

is valid.

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received: August 10, 1997

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