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Weak convergence of iterative methods generated by strongly Fejér monotone mappings

Dedicated to the professors of mathematics
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ABSTRACT. We consider the general class of strongly Fejér monotone mappings and the subclass of strongly nonexpansive operators. We show that only a few additional assumptions suffice to obtain weak convergence of the corresponding iterative methods. These methods can be widely used to solve convex problems. Besides, we get generalizations of convergence results known from the literature.

KEY WORDS. Set-valued mappings, Fejér monotone mappings, nonexpansive operators, relaxations, convex problems, Fejér monotone sequences, iterative methods.

1 Strongly Fejér monotone mappings

Let H be a (real) Hilbert space. We consider a nonempty, convex and closed subset Q of H and set-valued mappings $g : Q \mapsto \mathbb{P}(Q)$, where $\mathbb{P}(Q)$ consists of all nonempty subsets of Q. For g we introduce sets of *weak* and *strong fixed points*, namely

$$F_-(g) := \{x \in Q : x \in g(x)\} \quad , \quad F_+(g) := \{x \in Q : \{x\} = g(x)\} ,$$

where obviously $F_+(g) \subseteq F_-(g)$. In the case $F_+(g) = F_-(g)$ we simply speak of the fixed point set $F(g)$. As usual operators (i.e. single-valued mappings) $g : Q \mapsto Q$ are integrated as embeddings. Here both kinds of fixed point sets coincide with $F(g) := \{x \in Q : x = g(x)\}$. We exclude the uninteresting special case $g = I$ (I the identity). At first we summarize some basic concepts and results given in [12] and [13].

Definition 1.1 *Let M be a nonempty (proper) subset of Q and α a positive number. The mapping $g : Q \mapsto \mathbb{P}(Q)$ is said to be α-*strongly M-Fejér monotone (*in notation:*

$g \in \mathbb{F}^{\alpha}(M)$) if

$$\|y-x\|^2 - \|z-x\|^2 \geq \alpha\, \|y-z\|^2 \quad \forall x \in M \ , \ \forall y \in Q \ , \ \forall z \in g(y) \tag{1}$$

and

$$y \notin g(y) \quad \forall y \in Q \backslash M \tag{2}$$

hold. Besides, g is called α-strongly Fejér monotone (*in notation: $g \in \mathbb{F}^{\alpha}$) if g is α-strongly M-Fejér monotone for some $M \neq \emptyset$. Moreover, g is called* strongly M-Fejér monotone (*in notation: $g \in \mathbb{F}_s(M)$) if g is α-strongly M-Fejér monotone for some $\alpha > 0$. Finally, the reference to M can also be omitted in the latter case.*

Remark 1.2 General *M-Fejér monotone* mappings $g \in \mathbb{F}(M)$ satisfy (1) with the limit value $\alpha = 0$. If also (2) is added, then g is said to be *regularly M-Fejér monotone* (in notation: $g \in \mathbb{F}_r(M)$). Thus $\mathbb{F}_r(M)$ can be regarded as the limit class $\mathbb{F}^0(M)$ of $\mathbb{F}^{\alpha}(M)$. Besides, the hierarchy relations

$$\mathbb{F}^{\beta}(M) \subset \mathbb{F}^{\alpha}(M) \subset \mathbb{F}_s(M) \subset \mathbb{F}_r(M) \quad \text{for} \quad \beta > \alpha > 0$$

are satisfied. For regularly Fejér monotone and all the more for strongly Fejér monotone mappings g the set M is uniquely determined. Namely, it is the convex and closed set

$$M = \{x \in Q : \|z-x\| \leq \|y-x\| \quad \forall y \in Q \ , \ \forall z \in g(y)\} = F(g) \ .$$

In the applications M will play the part of the solution set of a certain convex problem.

Definition 1.3 *Let be $g \in \mathbb{F}_s$. Then the number*

$$\alpha^* = \alpha^*_F(g) := \sup\{\alpha : g \in \mathbb{F}^{\alpha}\}$$

is said to be the F-index *of g. (Mappings $g \in \mathbb{F}_r \setminus \mathbb{F}_s$ obtain the F-index 0.)*

Remark 1.4 The F-index $\alpha^* = \alpha^*_F(g)$ of $g \in \mathbb{F}_s$ is the maximal number α for which (1) holds. The nonempty and pairwise disjoint sets

$$\mathbb{F}^{\alpha}_* := \mathbb{F}^{\alpha} \setminus \bigcup_{\beta > \alpha} \mathbb{F}^{\beta}$$

contain all mappings g with F-index $\alpha^*_F(g) = \alpha$.

2 Strongly nonexpansive operators

Now we turn to special classes of nonexpansive operators $g : Q \mapsto Q$. Again the basic concepts and results can be found in [12] and [13].

Definition 2.1 *The operator* $g : Q \mapsto Q$ *is said to be* α-strongly nonexpansive *for* $\alpha > 0$ *(* $g \in \mathbb{L}^\alpha$ *) if*

$$\|y - x\|^2 - \|g(y) - g(x)\|^2 \geq \alpha \, \|g'(y) - g'(x)\|^2 \quad \forall x \, , \, y \in Q \, , \tag{3}$$

where g' *denotes the complement* $I - g$ *of* g*. Besides,* g *is called* strongly nonexpansive ($g \in \mathbb{L}_s$) *iff* g *is* α*-strongly nonexpansive for some* $\alpha > 0$ *.*

Remark 2.2 The limit case $\alpha = 0$ in (3) characterizes *nonexpansive* operators g (operators with Lipschitz norm less or equal to 1). If the fixed point property is added ($F(g) \neq \emptyset$), then we speak of *regularly nonexpansive* operators g ($g \in \mathbb{L}_r$) which are also regularly M-Fejér monotone with $M = F(g)$ ($g \in \mathbb{F}_r(M)$, see Remark 1.2). This implies also that $F(g)$ has to be convex and closed , a fact which is well-known for nonexpansive operators $g : Q \mapsto Q$ (see e.g. [7, p. 64] or [15, p. 41]). The fixed point theorem of Browder says that nonexpansive operators $g : Q \mapsto Q$ are regularly nonexpansive if Q is additionally supposed to be bounded (see [1], for more general spaces see e.g. [7, p. 62] or [15, p. 39]). The 1-strongly nonexpansive operators g turn out to be just the *firmly nonexpansive* operators introduced in [6, p. 41-44] by another definition. This fact is proven in my paper [13].
The sets of strongly nonexpansive operators fulfil the hierarchy relations

$$\mathbb{L}^\beta \subset \mathbb{L}^\alpha \subset \mathbb{L}_s \subset \mathbb{L}_r \quad \text{for} \quad \beta > \alpha > 0 \, .$$

Sometimes it is useful to specify the fixed point set $M = F(g)$ of operators $g \in \mathbb{L}^\alpha$ with $F(g) \neq \emptyset$. Then we write $g \in \mathbb{L}^\alpha(M)$. Besides, we use the notation $L^0(M) := L_r(M)$. This happens in accordance with the notation $\mathbb{F}^\alpha(M)$ in Definition 1.2.

Definition 2.3 *Let be* $g \in \mathbb{L}_s$ *. Then the number*

$$\alpha^* = \alpha^*_L(g) := \sup\{\alpha : g \in \mathbb{L}^\alpha\}$$

is called L-index *of* g*. (Operators* $g \in \mathbb{L}_r \setminus \mathbb{L}_s$ *obtain the L-index* 0*.)*

Remark 2.4 The L-index $\alpha^*_L(g)$ of $g \in \mathbb{L}_s$ is the maximal number α for which (3) holds.

The nonempty and pairwise disjoint sets

$$\mathbb{L}^\alpha_* := \mathbb{L}^\alpha \setminus \bigcup_{\beta > \alpha} \mathbb{L}^\beta$$

contain all mappings g with L-index $\alpha^*_L(g) = \alpha$.

Lemma 2.5 ([13]) *For arbitrary* $\alpha \geq 0$ *we get*

$$\mathbb{L}^{\alpha}(M) \subseteq \mathbb{F}^{\alpha}(M) ,$$

that is, α*-strongly nonexpansive operators with fixed points are* α*-strongly Fejer monotone.*

3 Strongly Fejér monotone sequences

For our convergence analysis we need some facts about Fejér monotone sequences and Fejér monotone iterative methods.

Convention 3.1 *The mapping* $g : Q \mapsto \mathbb{P}(Q)$ *induces iterative sequences* $(x_k) = (x_k(g, y))$ *by the recursion*

$$x_{k+1} \in g(x_k) \quad , \quad x_0 = y \in Q .$$

We call the whole family $\{(x_k) : y \in Q\}$ *of such sequences the* ordinary iterative method *relative to* g *with the starting elements* $x_0 = y$ *. This family will shortly denoted also by* (x_k)*.*

Remark 3.2 If $g \in \mathbb{F}^{\alpha}(M)$, then the sequences (x_k) satisfy

$$\|x_k - x\|^2 - \|x_{k+1} - x\|^2 \geq \alpha \|x_k - x_{k+1}\|^2$$

for all $x \in M$ and $x_{k+1} \neq x_k$ for all $x_k \in Q \setminus M$ as the admissible substitutions $y = x_k$ and $z = x_{k+1}$ in (1) and (2) show. So (x_k) is also α-strongly M-Fejér monotone in the sense of [11]. In the same way M-Fejér monotone and regularly M-Fejér monotone mappings g induce M-Fejér monotone and regularly M-Fejér monotone sequences (x_k) , respectively. Especially, M-Fejér monotone sequences fulfil for all $x \in M$ the relations

$$\|x_{k+1} - x\| \leq \|x_k - x\| .$$

Lemma 3.3 *M-Fejér monotone sequences (x_k) have the following properties:*

a) (x_k) *is bounded and weakly precompact.*

b) (x_k) *converges weakly to an element* $x^* \in M$ *iff all weak accumulation values of* (x_k) *ly in* M*.*

Proof: It is an easy consequence of the definition that Fejér monotone sequences (x_k) are bounded. But bounded sequences are weakly precompact (see e.g. [7, p. 18]). The assertion b) is shown in [11]. ∎

4 Convergence results

We start with some definitions which are important for the following convergence statements.

Definition 4.1 *Let be* $f : Q \mapsto \mathbb{P}(H)$ *. Then* f *is said to be* demiclosed *if for all sequences* (y_k) *in* Q *and the corresponding sequences* (u_k) *of images* $u_k \in f(y_k)$ *the conditions* $y_k \rightharpoonup y' \in Q$ *and* $u_k \to u' \in H$ *imply* $u' \in f(y')$ *.*

The definition is well-known for operators f (see e.g. [7, p. 74], [15, p. 42]). But you can find the generalization for mappings already in [8, p. 202].
The mapping f is called *closed* if weak convergence of (y_k) in the above definition is replaced by strong convergence. Observe that demiclosed mappings are closed, but not vice versa. Hence, the concept in Definition 4.1 is the stronger one. In the following we consequently use convention 3.1 .

Definition 4.2 *The mapping* $g : Q \mapsto \mathbb{P}(Q)$ *is said to be* asymptotically regular *if*

$$\lim_k (x_k - x_{k+1}) = 0$$

holds for the ordinary iterative method (x_k) *r.t.* g*, that is for all induced sequences* $(x_k) = (x_k(g, y))$ *with* $y \in Q$ *(see Convention* 3.1*). Besides,* g *is said to be a* reasonable wanderer *if*

$$\sum_{k=0}^{\infty} \|x_k - x_{k+1}\|^2 < \infty$$

holds for the method (x_k) *r.t.* g*.*

The first concept is well-known in the case of operators g, where x_k can be expressed by $g^k(y)$ (see e.g. [7, p. 69] or [15, p. 42]). The second concept is defined for operators g in [15, p. 44]. Obviously reasonable wanderers g are asymptotically regular.

Now the central convergence statement and some not less important easy consequences will follow. All convergence results will contain the assumption that the solution set M (the fixed point set of g) has to be nonempty. So in general existence results for the solution are necessary. This tribute has to be paid to the generality of the assumptions. On the other hand, the considerably stronger condition that g is a contractive operator ensures even the existence of a unique solution (fixed point theorem of BANACH).

Theorem 4.3 *Let be* $g : Q \mapsto \mathbb{P}(Q)$ *and* $\emptyset \neq M \subset Q$ *. Under the assumptions*

a) g *regularly* M*-Fejér monotone* $(g \in \mathbb{F}_r(M))$ *,*

b) *g asymptotically regular ,*

c) $g' = I - g$ *demiclosed* (I *identity*)

the ordinary iterative method $(x_k) = (x_k(g,y))$ *r.t.* g *converges weakly to an element* $x^* = x^*(y)$ *in* M *for arbitrary starting elements* $y \in Q$ *. The limit element* x^* *is a fixed point of* g*.*

Proof: Let be $g : Q \mapsto \mathbb{P}(Q)$ regularly M-Fejér monotone and $y \in Q$ arbitrary. Then we choose any induced sequence $(x_k) = (x_k(g,y))$. By Remark 3.2 (x_k) is M-Fejér monotone. Hence, (x_k) is weakly precompact in view of Lemma 3.3 a). Let x' be an arbitrary weak accumulation value of (x_k). Then there is a subsequence $(x_{k'})$ of (x_k) with $x_{k'} \rightharpoonup x'$.
By assumption b) the mapping g is asymptotically regular. Thus we obtain $x_{k'} - x_{k'+1} \to 0$.
Further, because of $x_{k'+1} \in g(x_{k'})$ the relation $x_{k'} - x_{k'+1} \in (I-g)(x_{k'})$ follows.
Since g is demiclosed by assumption c), we get $0 \in (I-g)(x')$ or $x' \in g(x')$. Observing Remark 1.2 we have $x' \in F_-(g) = F(g) = M$. Hence, x' is a fixed point of g. As x' was chosen arbitrary, all weak accumulation values have to be in M. But then all weak accumulation values x' of (x_k) coincide with the weak limit x^*. Consequently we have $x_k \rightharpoonup x^* \in M$. This is the assertion. ∎

The proof shows that assumption c) is necessary only for $0 \in (I-g)Q$. A direct consequence of Theorem 4.3 is the following well-known result.

Theorem 4.4 *Let be* $g : Q \mapsto Q$ *and* $\emptyset \neq F(g) \subset Q$ *. Under the assumptions*

a) *g nonexpansive,*

b) *g asymptotically regular,*

the ordinary iterative method $(x_k) = (x_k(g,y))$ *r.t.* g *converges weakly to an element* x^* *in* $F(g)$ *for arbitrary starting elements* $y \in Q$ *.*

Proof: Following Remark 2.2 the assumption a) can be completed to

$$g \in \mathbb{L}_r(M) = \mathbb{L}^0(M) \quad , \quad M = F(g) \neq \emptyset \, .$$

By Lemma 2.5 this implies also

$$g \in \mathbb{F}^0(M) = \mathbb{F}_r(M) \, .$$

Besides, it is well-known that nonexpansive operators have a demiclosed complement $I - g$ (for a short proof see [15, p. 42]). So all assumptions of Theorem 4.3 are fulfilled. ∎

Theorem 4.4 which here occurs as a corollary is directly proven in [15, p. 42-44]). If Q is bounded, then the assumption $F(g) \neq \emptyset$ can be omitted in view of the fixed point theorem of Browder as already mentioned in Remark 2.2. In this form the above theorem is shown in [7, p. 77].

Lemma 4.5 *Strongly Fejér monotone mappings g are reasonable wanderers.*

Proof: By assumption we have $g \in \mathbb{F}^{\alpha}(M)$ for some $\alpha > 0$ and some $M \neq \emptyset$. Thus, by Remark 3.2, the induced sequences $(x_k) = (x_k(g, y))$ with $x_0 = y \in Q$ are α-strongly M-Fejér monotone. Hence, the estimate

$$\begin{aligned} \alpha \sum_{k=0}^{n} \|x_k - x_{k+1}\|^2 &\leq \sum_{k=0}^{n} (\|x_k - x\|^2 - \|x_{k+1} - x\|^2) \\ &\leq \|y - x\|^2 - \|x_{n+1} - x\|^2 \leq \|y - x\|^2 < \infty \end{aligned}$$

is valid for arbitrary $n \in \mathbb{N}$. Now, by tending with n to infinity, g turns out to be a reasonable wanderer. ■

This Lemma allows important modifications of Theorem 4.3 and Theorem 4.4.

Theorem 4.6 *Let be $g : Q \mapsto \mathbb{P}(Q)$ and $\emptyset \neq M \subset Q$. Under the assumptions*

a) *g strongly M-Fejér monotone $(g \in \mathbb{F}_s(M))$,*

b) *$g' = I - g$ demiclosed (I identity)*

the ordinary iterative method $(x_k) = (x_k(g, y))$ r.t. g converges weakly to an element x^* in M for arbitrary starting elements $y \in Q$.

Proof: By assumption a) g is strongly M-Fejér monotone. Hence, g is all the more regularly M-Fejér monotone. Besides, g is asymptotically regular in view of Lemma 4.5. So all assumptions of Theorem 4.3 are fulfilled. ■

Theorem 4.7 *Let be $g : Q \mapsto Q$ and $F(g) \neq \emptyset$. If g is strongly nonexpansive $(g \in \mathbb{L}_s)$, then the ordinary iterative method $(x_k) = (x_k(g, y))$ r.t. g converges weakly to an element x^* in $F(g)$ for arbitrary starting elements $y \in Q$.*

Proof: By assumption g is strongly nonexpansive and has a nonempty fixed point set $M = F(g)$. Hence, $g \in \mathbb{L}_s(M)$. Then g is on the one hand all the more nonexpansive and on the other hand strongly M-Fejér monotone. Namely, the relation

$$\mathbb{L}_s(M) \subseteq \mathbb{F}_s(M)$$

is an immediate corollary of Lemma 2.5. But, then g is also asymptotically regular in view of Lemma 4.5. Consequently, all assumptions of Theorem 4.4 are fulfilled. ■

Theorem 4.7 can also be reduced to Theorem 4.6. Finally, we turn to iterative methods with relaxations which are often used in the literature (see e.g. [15]). For a mapping $g : Q \mapsto \mathbb{P}(Q)$ a *relaxation* of g with the parameter λ is the affine combination

$$g_\lambda := (1 - \lambda)\, I + \lambda\, g \quad , \quad \lambda > 0 \,, \tag{4}$$

of I and g. Obviously the weak and strong fixed point sets of g and g_λ coincide. The next statement shows how relaxations of Fejér monotone mappings g influence the coefficients of strongness.

Lemma 4.8 ([13]) *For a mapping* $g : Q \mapsto \mathbb{P}(Q)$ *and parameters* $\alpha \geq 0$, $\beta \geq 0$ *and* $\lambda > 0$ *which are connected by the equation*

$$(1 + \alpha)\, \lambda = 1 + \beta$$

the following statement holds:

$$g \in \mathbb{F}^\beta(M) \quad \Longleftrightarrow \quad g_\lambda \in \mathbb{F}^\alpha(M)\,.$$

Moreover, this correspondence is also fulfilled for the F-indices:

$$\beta = \alpha_F^*(g) \quad \Longleftrightarrow \quad \alpha = \alpha_F^*(g_\lambda)\,.$$

A completely analogous result can be obtained for corresponding classes of nonexpansive operators. Especially, we have

$$g \in \mathbb{L}^\beta \quad \Longleftrightarrow \quad g_\lambda \in \mathbb{L}^\alpha$$

for the above given equation between α, β and λ.

Lemma 4.9 *Let be* $\beta \geq 0$ *and* $\lambda \in (0, 1 + \beta)$. *Then*

$$g \in \mathbb{F}^\beta(M) \quad \Longrightarrow \quad g_\lambda \in \mathbb{F}_s(M)\,.$$

Proof: Suppose $g \in \mathbb{F}^{\beta}(M)$ with $\beta \geq 0$ and $\lambda > 0$. Then $g_\lambda \in \mathbb{F}^{\alpha}(M)$ is true for $\alpha = \frac{1}{\lambda}(1+\beta-\lambda) > 0$ by Lemma 4.8. Consequently, $g_\lambda \in \mathbb{F}_s(M)$ holds for $\lambda \in (0, 1+\beta)$. ■

Analogously we have for $\beta \geq 0$ and $\lambda \in (0, 1+\beta)$ the statement

$$g \in \mathbb{L}^{\beta}(M) \quad \Longrightarrow \quad g_\lambda \in \mathbb{L}_s(M) \, .$$

By Lemma 4.5 all mappings $g \in \mathbb{F}_s(M)$ are reasonable wanderers. So g_λ is a reasonable wanderer for $g \in \mathbb{F}^{\beta}(M)$, $\beta \geq 0$ and $\lambda \in (0, 1+\beta)$. Especially, this holds for $\beta = 0$. Thus g_λ is asymptotically regular for $\lambda \in (0,1)$ if g is regularly Fejér monotone. The same is all the more true if g is regularly nonexpansive (see also [7, p. 70], [15, p. 45]).

Corollary 4.10 *Let be* $g : Q \mapsto \mathbb{P}(Q)$ *and* $\emptyset \neq M \subset Q$. *Under the assumptions*

a) $g \in \mathbb{F}^{\beta}(M) \quad , \quad \beta \geq 0 \, ,$

b) $g' = I - g$ *demiclosed* (I *identity*),

c) $\lambda \in (0, 1+\beta)$

the ordinary iterative method $(x_k) = (x_k(g_\lambda, y))$ *r.t.* g_λ *converges weakly to an element* x^* *in* M *for arbitrary starting elements* $y \in Q$.

Proof: By Lemma 4.9 the assumptions a) and c) ensure that $g_\lambda \in \mathbb{F}_s(M)$ is valid. Since $\lambda > 0$ and

$$g'_\lambda = I - g_\lambda = \lambda \, (I - g) = \lambda \, g'$$

is satisfied, g'_λ is demiclosed by assumption b). Hence Theorem 4.6 holds with g_λ instead of g. ■

Corollary 4.11 *Let be* $g : Q \mapsto Q$ *and* $F(g) \neq \emptyset$. *Under the assumptions*

a) $g \in \mathbb{L}^{\beta} \quad , \quad \beta \geq 0 \, ,$

b) $\lambda \in (0, 1+\beta)$

the ordinary iterative method $(x_k) = (x_k(g_\lambda, y))$ *r.t.* g_λ *converges weakly to an element* x^* *in* $F(g)$ *for arbitrary starting elements* $y \in Q$.

Proof: By the analogue of Lemma 4.9 the assumptions a) and b) supply

$$g_\lambda \in \mathbb{L}_s(M) \quad , \quad M = F(g) = F(g_\lambda) \neq \emptyset \, .$$

Hence Theorem 4.7 with g_λ instead of g can be applied. ■

Both foregoing theorems hold also in the case $\beta = 0$. So g itself need not to be asymptotically regular. Nevertheless the iterative method for g_λ converges weakly as long as $\lambda \in (0,1)$ (compare [2], [7, p. 78], [15, p. 45] with Corollary 4.11).
Naturally, the question arises under which conditions also strong convergence of Fejér monotone iterative methods can be reached. If a Fejér monotone sequence (x_k) satisfies

$$\rho(x_k, M) \to 0 \ \ (k \to \infty) \, ,$$

then it converges strongly to an element $x^* \in M$ (see [11]). If even the stronger condition

$$\exists l > 0 \ : \ \|x_k - x_{k+1}\| \geq l \, \rho(x_k, M)$$

holds, then the convergence rate is linear (geometrical). For special cases the existence of such constants l is known (see e.g. [10, p. 1340]). A convergence theory for Fejér monotone iterative methods based primarily on strong convergence is developed in my paper [10].

5 Applications

Let us consider a certain convex problem with nonempty solution set M. The problem can be for instance an equation with a convex solution set, a convex inequality, a convex optimization problem or a variational inequality. Often a convex problem is of intersection type, that is, M is the intersection of convex and closed sets M_i. Systems of linear equations, systems of convex inequalities or systems of convex set constraints are examples for this type. The question arises how to construct a suitable Fejér monotone mapping g for a given convex problem with solution set M. If this is done, we are able to approximate solutions by means of Fejér monotone iterative methods. We start with two standard constructions.

Let $b : Q \mapsto \mathbb{R}$ be a convex and continuous functional. Then the set $N(b) = \{x \in Q : b(x) \leq 0\}$ is convex and closed. We assume $N(b)$ to be nonempty. Further, the *subgradient* ∂b is defined on Q. If b^+ denotes the positive part of b, we define for elements $y \in Q$ and $v \in H$

$$\begin{aligned} \mu(b,y,v) &= \begin{cases} b^+(y)v/\|v\|^2 & \text{if } v \neq 0, \\ 0 & \text{if } v = 0, \end{cases} \\ t_b(y) &= \{\, \mu(b,y,v) \, : \, v \in \partial b(y)\}. \end{aligned} \tag{5}$$

Lemma 5.1 ([14]) *Let be* $t_b(y) \subseteq y - Q$ *for all* $y \in Q$. *Then the mapping* $g_b : Q \mapsto \mathbb{P}(Q)$ *defined by* $g_b(y) = y - t_b(y)$ *is at least* 1*-strongly* $N(b)$*-Fejér monotone, that is*

$$g := g_b \in \mathbb{F}^1(N(b)) \ .$$

For a convex and closed set $M \subset Q$ the *metric projector* $P_M : Q \mapsto Q$ onto M is a well-defined operator.

Lemma 5.2 ([14]) *If* M *is a convex and closed nonempty proper subset of* Q*, then* P_M *is* 1*-strongly nonexpansive. More precisly, we have even* $\alpha_F^*(P_M) = \alpha_L^*(P_M) = 1$ *, that is*

$$g \in \mathbb{L}_*^1(M) \cap \mathbb{F}_*^1(M) \ .$$

The index classes $\mathbb{L}_*^1(M)$ and $\mathbb{F}_*^1(M)$ in the above lemma are explained in Remark 1.4 and in Remark 2.4, respectively.

Corollary 5.3 *Let* $g = g_b$ *be a subgradient type mapping which satisfies the assumption of Lemma* 5.1 . *Further, let be* $\lambda \in (0, 2)$. *Then the relaxed form* g_λ *is* α*-strongly Fejér monotone with* $\alpha = (2 - \lambda)/\lambda$.
Let be $g = P_M$ *the metric projector onto* M *and* $\lambda \in (0, 2)$. *Then the relaxed projector* g_λ *is* α*-strongly nonexpansive with the index*

$$\alpha = \alpha_L^*(g_\lambda) = \alpha_F^*(g_\lambda) = (2 - \lambda)/\lambda \ .$$

Proof: Let be $g = g_b$. Then we have $g \in \mathbb{F}^1(M)$ with $M = N(b)$ in view of Lemma 5.1 . If Lemma 4.8 is used with $\beta = 1$, then $g_\lambda \in \mathbb{F}^\alpha(M)$ holds for $\alpha = (2 - \lambda)/\lambda$.
Let be $g = P_M$. Then $g \in \mathbb{L}_*^1(M) \cap \mathbb{F}_*^1(M)$ by Lemma 5.2 . The analogue of Lemma 4.8 for nonexpansive operators supplies the assertion if again $\beta = 1$ is chosen. ■

Hence, the results of section 4 can be used for this special mappings. But observe that the second construction is of small practical value if it is applied to the solution set M of the whole problem. Namely, if you know the projector P_M, then a solution is obtained in the first iterative step by $P_M y \in M$. But the determination of P_M is in the most cases a difficult task, similarly difficult as the determination of M itself. On the other hand, if the problem is of intersection type, then the sets M_i are often simple enough to calculate the projectors P_i onto M_i. This idea leads to the next question: How to construct a Fejér monotone mapping g for M if Fejér monotone mappings g_i for M_i are known. This question is discussed in the following.
Let M_i $(i = 1, \cdots, m)$ be convex and closed subsets of Q with nonempty intersection $M := \cap_{i=1}^m M_i$. If we know corresponding mappings $g_i : Q \mapsto \mathbb{P}(Q)$ $(i = 1, \cdots, m)$, then we can form two standard combinations to get a mapping $g : Q \mapsto \mathbb{P}(Q)$ for M, namely

a) a *parallel* or *simultaneous* combination

$$\begin{aligned} g &:= \gamma_1 g_1 + \gamma_2 g_2 + \cdots + \gamma_m g_m \ , \\ \gamma_i &\geq 0 \quad , \quad \gamma_1 + \gamma_2 + \cdots + \gamma_m = 1 \ , \end{aligned}$$

b) a *sequential* or *successive* combination

$$g := g_m g_{m-1} \cdots g_1 \ .$$

As expected, the combined mapping g is strongly Fejér monotone if the same is already true for the original mappings g_i.

Lemma 5.4 ([14]) *Let M_i $(i = 1, \cdots, m)$ be convex and closed subsets of Q with nonempty intersection M. Besides, let $g_i : Q \mapsto \mathbb{P}(Q)$ $(i = 1, \cdots, m)$ be corresponding mappings.*
If $g_i \in \mathbb{F}^{\alpha_i}(M_i)$ $(i = 1, \cdots, m)$ and $\alpha := \min\{\alpha_i \ : \ i = 1, \cdots, m\}$ holds, then we have $g \in \mathbb{F}^{\beta}(M)$ with $\beta := \alpha$ in the parallel case and with $\beta := \alpha/2^{m-1}$ in the sequential case.

A completely analogous result holds for strongly nonexpansive operators $g_i \in \mathbb{L}^{\alpha_i}(M_i)$. Observe the following consequence of Lemma 5.4 . The assumptions about g_i imply $M_i = F(g_i)$. The result for g involves

$$M := \bigcap_{i=1}^{m} M_i = F(g) \ ,$$

that means, M is the fixed point set of g and at the same time the set of common fixed points of g_i.
Lemma 5.4 and its analogue can be used to produce convergence theorems for convex problems of intersection type in connection with the results of section 4. I will mention only one simple example here.

Corollary 5.5 *Let M_i $(i = 1, \cdots, m)$ be convex and closed subsets of Q with nonempty intersection M. Further, let P_i be the projectors onto M_i and*

$$g_i = (1 - \lambda_i)\, I + \lambda_i\, P_i \quad , \quad 0 < \lambda_i < 2$$

be corresponding relaxations. Finally, let g be the combined mapping of parallel or sequential type. Then there is a number $\beta > 0$ such that the ordinary iterative method (x_k) r.t. a relaxation g_λ converges weakly to an element $x^* \in M$ for arbitrary starting elements $y \in Q$ as long as $\lambda \in (0, 1 + \beta)$ holds.

Proof: Under the above assumptions we have the relations

$$g_i \in \mathbb{L}^{\alpha_i}(M_i) \quad , \quad \alpha_i = (2-\lambda_i)/\lambda_i > 0$$

by Corollary 5.3 and $g \in \mathbb{L}^{\beta}(M)$ with $M = F(g) \neq \emptyset$ and with β choosen according to the analogue of Lemma 5.4 depending on wether g is the parallel or sequential combination of g_i. Since the numbers α_i are all positive, the number β is positive, too. Now the assertion follows from Corollary 4.11 .

Some of the possible corollaries for convex problems of intersection type are known from the literature (see e.g. [3]). A lot of other aspects and applications concerning Fejér monotone iterative methods are contained in [4], [5], [9], [10] and [15].

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