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A Unified Approach to Error Estimates for Interpolation on Full and Sparse Gauß–Chebyshev Grids

Dedicated to the professors of mathematics
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ABSTRACT. We give a unified approach to error estimates for interpolation on full and sparse Gauß–Chebyshev grids. We impose Strang–Fix conditions adapted to interpolation on Gauß–Chebyshev grids on the underlying Lagrange functions. This yields error bounds with explicit constants.

KEY WORDS. Interpolation by generalized translates, Strang–Fix conditions, Sparse grids, Boolean sums.

1 Introduction

The Gauß–Chebyshev knots are quite often used as interpolation points for functions given on the interval $[-1, 1]$. Beside the well-studied polynomial interpolation, one can investigate interpolation by splines adapted to these special nodes.

Using generalized translates, the so-called Chebyshev–shifts, the interpolation of univariate functions on Gauß–Chebyshev knots can be seen as interpolation by translates. Shift–invariant spaces corresponding to such translates and their wavelet analysis are described in detail in [10]. Forming tensor products yields interpolation of bivariate functions on full grids. Interpolation on sparse grids can be realized by j-th order blending. Some very recent papers use this fact for determining the quadrature error of the Clenshaw–Curtis rule on sparse Gauß–Chebyshev grids for smooth functions [8, 9] or for investigating interpolatory wavelets for sparse Gauß–Chebyshev grids [14]. Interpolation and approximation on sparse grids is well investigated for periodic functions [5, 12, 13] and closely related to hyperbolic approximation (see e.g. [15]).

We are interested here in a unified approach to error estimates for interpolation on full and sparse Gauß–Chebyshev grids for functions from Sobolev–type spaces (cf. [2]). Therefore, we adapt the concept of Strang–Fix conditions on the univariate Lagrange function to this special situation. Furthermore, we use essentially the properties of the bivariate function spaces which can be represented as tensor products or intersections of other function spaces. Collecting all these tools, we are able to give the desired error estimates with explicit constants.

2 Notation

We denote our reference interval by $I := [-1, 1]$ and the Chebyshev weight by $w(x) := (1 - x^2)^{-1/2}\,(x \in (-1, 1))$. Let $L^2_w(I)$ be the weighted Hilbert space of all measurable functions $f : I \longrightarrow \mathbb{R}$ with

$$\int_I f(x)^2 w(x)\, dx < \infty\,.$$

For $f, g \in L^2_w(I)$, the corresponding inner product is given by

$$\langle f, g\rangle := \frac{2}{\pi}\int_I f(x)g(x)w(x)\, dx\,.$$

By Π_n we denote the set of all real valued polynomials of degree at most n restricted on I. Furthermore let $T_n \in \Pi_n$ be the Chebyshev polynomials

$$T_n(x) := \cos(n \arccos x)\,.$$

They form a complete orthogonal basis

$$\langle T_k, T_\ell\rangle = \begin{cases} 2 & \text{for } k = \ell = 0, \\ 1 & \text{for } k = \ell \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

of $L^2_w(I)$. With the help of the Chebyshev coefficients

$$a_k[f] := \langle f, T_k\rangle\,, \qquad f \in L^2_w(I),\ k \in \mathbb{N}_0\,,$$

we define Sobolev–type spaces of order $\alpha \geq 0$,

$$H^\alpha_w(I) := \left\{ f \in L^2_w(I)\,;\ \|f\|^2_\alpha := \sum_{k\in\mathbb{N}_0}{}' (1 + k^2)^\alpha\, a_k^2[f] < \infty \right\}$$

as in [2] and the Wiener algebra

$$A(I) := \Big\{ f \in L^2_w(I)\,;\ {\sum_{k\in\mathbb{N}_0}}' |a_k[f]| < \infty \Big\}\,.$$

Here and the sequel, we use the notation $\sum'$ if we halve the first term and $\sum''$ where the first and the last term in the sum are halved. The Sobolev–type spaces $H^\alpha_w(I)$ are Hilbert spaces and $H^0_w(I) = L^2_w(I)$.

By $\mathcal{G}_N$, we denote the grid of the Gauß–Chebyshev nodes

$$\mathcal{G}_N := \big\{ g_k := \cos k\tfrac{\pi}{N}\,;\ k = 0, \dots, N \big\}\,.$$

The discrete Chebyshev coefficients are given as

$$a^N_k[f] := \frac{2}{N} {\sum_{\ell=0}^{N}}'' f(g_\ell)\, T_k(g_\ell)\,, \qquad k = 0, \dots, N.$$

For functions $f \in A(I)$, one can proof the aliasing formula

$$a^N_k[f] = \sum_{\ell\in\mathbb{N}_0} a_{2\ell N+k}[f] + a_{2(\ell+1)N-k}[f]\,, \qquad k = 0, \dots, N.$$

The Chebyshev coefficients of functions on I are strongly related to the Fourier cosine coefficients of even periodic functions

$$\tilde{c}_k(g) := \frac{2}{\pi} \int_0^\pi g(\theta) \cos(k\theta)\, d\theta, \qquad k \in \mathbb{N}_0,$$

and the discrete cosine coefficients

$$\tilde{c}^N_k(g) := \frac{2}{N} {\sum_{\ell=0}^{N}}'' g\big(\tfrac{\ell\pi}{N}\big)\ \cos\big(\tfrac{k\ell\pi}{N}\big)\,, \qquad k = 0, \dots, N,$$

which we want to define also for vectors $\boldsymbol{\eta} \in \mathbb{R}^{N+1}$ as

$$\tilde{c}^N_k(\boldsymbol{\eta}) := \frac{2}{N} {\sum_{\ell=0}^{N}}'' \eta_\ell\ \cos\big(\tfrac{k\ell\pi}{N}\big)\,, \qquad k = 0, \dots, N.$$

This is the well-known discrete cosine transform of type I (DCT-I($N+1$)). Fast and numerically stable algorithms working in real arithmetic are described in [1]. We extend that definition to all $k \in \mathbb{N}_0$ and obtain $\tilde{c}^N_{r+2\ell N}(\boldsymbol{\eta}) = \tilde{c}^N_r(\boldsymbol{\eta})$ and $\tilde{c}^N_{r+(2\ell+1)N}(\boldsymbol{\eta}) = \tilde{c}^N_{N-r}(\boldsymbol{\eta})$ for $r = 0, \dots, N-1$, $\ell \in \mathbb{N}_0$.

With the transform $x = \cos\theta$, it holds that

$$a_k[f] = \tilde{c}_k(f(\cos\cdot)),\ k \in \mathbb{N}_0 \qquad \text{and} \qquad a^N_k[f] = \tilde{c}^N_k(f(\cos\cdot)),\ k = 0, \dots, N.$$

3 Interpolation by generalized translates

Now we want to describe interpolation by generalized translates of a single function and the construction of a corresponding Lagrange function.

The Chebyshev shift (cf. [4, 10]) $s_h f$ of a function f is defined by

$$(s_h f)(x) := \tfrac{1}{2} f\big(xh - \tfrac{1}{w(x)w(h)}\big) + \tfrac{1}{2} f\big(xh + \tfrac{1}{w(x)w(h)}\big), \qquad x \in I.$$

For interpolation, we use the special shifts $\sigma_k := s_{g_k}$, $k = 0, \ldots, N$. The Chebyshev shift effects the Chebyshev coefficients $a_k[\sigma_n f] = T_k(g_n) a_k[f]$ in the same multiplicative way as the usual shift effects the Fourier coefficients of periodic functions [10]. The interpolant $L_N f$

$$L_N f(g_k) = f(g_k)\,, \qquad k = 0, \ldots, N,$$

interpolating in the Gauß–Chebyshev interpolation nodes shall be given as

$$L_N f := {\sum_{k=0}^{N}}'' f(g_k)\, \sigma_k \varphi$$

with a single generating function $\varphi \in A(I)$. We first want to construct the corresponding modified Lagrange function

$$\Lambda := {\sum_{k=0}^{N}}'' \lambda_k\, \sigma_k \varphi$$

with

$$\sigma_k \Lambda(g_\ell) = \frac{1}{\varepsilon_k}\, \delta_{k,\ell}\,, \qquad k = 0, \ldots, N,$$

where here and in the following

$$\varepsilon_k := \begin{cases} \frac{1}{2} & \text{for } k = 0, N, \\ 1 & \text{for } k = 1, \ldots, N-1. \end{cases}$$

Inserting the interpolation conditions and computing the discrete Chebyshev coefficients yields

$$a_k^N[\Lambda] = \frac{2}{N} = \frac{N}{2} \tilde{c}_k^N(\boldsymbol{\lambda})\, a_k^N[\varphi], \qquad k = 0, \ldots, N.$$

So we obtain the Chebyshev coefficients of Λ as

$$\begin{aligned} a_k[\Lambda] &= \frac{N}{2}\, \tilde{c}_k^N(\boldsymbol{\lambda})\, a_k[\varphi] \\ &= \begin{cases} \dfrac{2}{N} \dfrac{a_k[\varphi]}{a_r^N[\varphi]} = \dfrac{2}{N} \dfrac{a_k[\varphi]}{\sum\limits_{\ell \in \mathbb{N}_0} a_{2\ell N + r}[\varphi] + a_{2(\ell+1)N - r}[\varphi]} & \text{for } k = r + 2sN, \\ & r = 0, \ldots, N-1, s \in \mathbb{N}_0, \\ \dfrac{2}{N} \dfrac{a_k[\varphi]}{a_{N-r}^N[\varphi]} = \dfrac{2}{N} \dfrac{a_k[\varphi]}{\sum\limits_{\ell \in \mathbb{N}_0} a_{(2\ell+1)N + r}[\varphi] + a_{(2\ell+1)N - r}[\varphi]} & \text{for } k = r + (2s+1)N, \\ & r = 0, \ldots, N-1, s \in \mathbb{N}_0. \end{cases} \end{aligned}$$

Obviously, the condition $a_k^N[\varphi] \neq 0$, $k = 0, \ldots, N$, is necessary and sufficient for the existence and uniqueness of the fundamental interpolant Λ. Otherwise one can use the method described in [11] for the periodic case to define a fundamental interpolant (which then is not in the shift-invariant space generated by φ). That's why, in the sequel we assume the existence of a Lagrange function Λ and consider the interpolation

$$L_N f = \sum_{k=0}^{N}{}'' f(g_k)\, \sigma_k \Lambda \,.$$

4 Error Estimates

We want to estimate the interpolation error in some Sobolev norm of order $\alpha \geq 0$ for the interpolation of functions from a Sobolev–type space of order $\mu \geq \alpha$. Therefore, we split

$$\|f - L_N f\|_\alpha \leq \|f - S_{N-1} f\|_\alpha + \|S_{N-1} f - L_N S_{N-1} f\|_\alpha + \|L_N(f - S_{N-1} f)\|_\alpha \tag{1}$$

with the $(N-1)$-th partial sum of the Chebyshev expansion

$$S_{N-1} f := \sum_{k=0}^{N-1}{}' := a_k[f]\, T_k \,.$$

At one hand, the order of the interpolation error depends on the smoothness properties of the interpolated function f. On the other hand, it is influenced by the approximation properties of the underlying Lagrange function Λ.

In order to characterize these approximation properties, we introduce conditions on the decay of the Chebyshev coefficients of Λ. They are the pendant for the interval of the strong cardinal Strang–Fix conditions [6] and the periodic Strang–Fix conditions [3, 11, 13].

Definition 1 *The Lagrange function $\Lambda \in A(I)$ satisfies the* Strang–Fix conditions *(for Gauß–Chebyshev grids) of order $m > 0$ for the exponent $\alpha \geq 0$ if for all $k = 0, \ldots, N$ the inequalities*

$$\begin{aligned} \left|1 - \frac{N}{2}\, a_k[\Lambda]\right| &\leq b_0\, k^m\, N^{-m}\,, & & \\ \left|\frac{N}{2}\, a_{2\ell N+k}[\Lambda]\right| &\leq b_{2\ell}\, k^m\, N^{-m-\alpha}\,, & \ell &\in \mathbb{N}, \\ \left|\frac{N}{2}\, a_{2(\ell+1)N-k}[\Lambda]\right| &\leq b_{2\ell+1}\, k^m\, N^{-m-\alpha}\,, & \ell &\in \mathbb{N}_0, \end{aligned} \tag{2}$$

hold for some sequence $\{b_\ell\}_{\ell\in\mathbb{N}_0}$ with

$$\gamma_1^2 := 2 \sum_{\ell\in\mathbb{N}_0} (1 + (2\ell)^2)^\alpha\, b_{2\ell}^2 + (1 + (2(\ell+1))^2)^\alpha\, b_{2\ell+1}^2 < \infty \,. \tag{3}$$

For Lagrange functions satisfying such conditions, we can estimate the second term in (1) in the following way.

Theorem 2 *Let the Lagrange function $\Lambda \in A(I)$ satisfy the Strang–Fix conditions* (2) *of order m for the exponent α.*

Then, for $f \in \Pi_{N-1}$, it holds that

$$\|f - L_N f\|_\alpha \;\le\; \gamma_1 \; N^{-m} \; \|f\|_{m+\alpha}$$

with the constant γ_1 as in (3).

Proof: We compute the Chebyshev coefficients of the interpolant as

$$\begin{aligned} a_{2\ell N+k}[L_N f] &= \sum_{r\in\mathbb{N}_0} \left(a_{2rN+k}[f] + a_{2(r+1)N-k}[f]\right) \frac{N}{2} a_{2\ell N+k}[\Lambda], \\ a_{(2\ell+1)N+k}[L_N f] &= \sum_{r\in\mathbb{N}_0} \left(a_{(2r+1)N+k}[f] + a_{(2r+1)N-k}[f]\right) \frac{N}{2} a_{(2\ell+1)N+k}[\Lambda], \end{aligned}$$

for $k = 0, \ldots, N-1$, $\ell \in \mathbb{N}_0$. Using $f \in \Pi_{N-1}$, we obtain

$$\begin{aligned} \|f - L_N f\|_\alpha &= \Big\| {\sum_{k\in\mathbb{N}_0}}' (a_k[f] - a_k[L_N f]) T_k \Big\|_\alpha \\ &= \Big\| {\sum_{k=0}^{N-1}}' \sum_{\ell\in\mathbb{N}_0} \Big(a_{2\ell N+k}[f] - \sum_{r\in\mathbb{N}_0} \left(a_{2rN+k}[f] + a_{2(r+1)N-k}[f]\right) \frac{N}{2} a_{2\ell N+k}[\Lambda] \Big) T_{2\ell N+k} \\ &\quad + \Big(a_{(2\ell+1)N+k}[f] - \sum_{r\in\mathbb{N}_0} \left(a_{(2r+1)N+k}[f] + a_{(2r+1)N-k}[f]\right) \frac{N}{2} a_{(2\ell+1)N+k}[\Lambda] \Big) T_{(2\ell+1)N+k} \Big\|_\alpha \\ &= \Big\| \sum_{k=0}^{N-1} a_k[f] \Big(\varepsilon_k \big(1 - \frac{N}{2} a_k[\Lambda]\big) T_k - \sum_{\ell\in\mathbb{N}} \frac{N}{2} a_{2\ell N+k}[\Lambda] T_{2\ell N+k} + \frac{N}{2} a_{2\ell N-k}[\Lambda] T_{2\ell N-k} \Big) \Big\|_\alpha . \end{aligned}$$

Computing the Sobolev norm and inserting the Strang–Fix conditions proves the theorem. $\square$

By a straight–forward calculation, we obtain the estimate for the first term in (1).

Lemma 3 *Let $f \in H_w^\mu(I)$, $\mu \ge \alpha \ge 0$, then*

$$\|f - S_{N-1}\|_\alpha \;\le\; N^{\alpha-\mu} \; \|f\|_\mu .$$

It remains to state an estimate for the last term in the triangle inequality (1).

Theorem 4 *Let $f \in H_w^\mu(I)$, $\mu > 1/2$, $\mu \ge \alpha \ge 0$. Let the Lagrange function $\Lambda \in A(I)$ satisfy the Strang–Fix conditions* (2) *of order m for the exponent α.*

Then it holds that

$$\|L_N(f - S_{N-1}f)\|_\alpha \;\leq\; \gamma_2\;\gamma_3\;N^{\alpha-\mu}\;\|f\|_\mu\,,$$

with the constants

$$\begin{aligned}
\gamma_2^2 \;&:=\; \frac{N^{2(\alpha+1)}}{4}\;\max_{k=0,\dots,N}\Big(\varepsilon_k a_k^2[\lambda] + \sum_{\ell\in\mathbb{N}}(1+(2\ell)^2)^\alpha\,(a_{2\ell N+k}^2[\Lambda] + (a_{2\ell N-k}^2[\Lambda])\Big)\,,\\
\gamma_3^2 \;&:=\; 2^{\alpha+1}\;\max_{k=0,\dots,N}\sum_{\ell\in\mathbb{N}}\Big(1+\Big(2\ell+\frac{k}{N}\Big)^2\Big)^{-\mu} + \Big(1+\Big(2\ell-\frac{k}{N}\Big)^2\Big)^{-\mu}.
\end{aligned}$$

Proof: Because of $\mu > 1/2$ we have the function $f \in A(I)$. Hence, the interpolant is well defined and the aliasing formula can be applied. The same assumption ensures the converges of the series which defines γ_3. From the Strang–Fix conditions, the constant γ_2 is well defined. We estimate as follows

$$\begin{aligned}
&\|L_N(f - S_{N-1}f)\|_\alpha = \Big\|\sum_{k\in\mathbb{N}_0}{}' a_k[L_N(f-S_{N-1}f)]\,T_k\Big\|_\alpha\\
&= \;\Big\|\sum_{k=0}^{N-1}{}'\sum_{\ell\in\mathbb{N}_0}\sum_{r\in\mathbb{N}_0}\big(a_{2rN+k}[f-S_{N-1}f] + a_{2(r+1)N-k}[f-S_{N-1}f]\big)\,\frac{N}{2}a_{2\ell N+k}[\Lambda]\,T_{2\ell N+k}\\
&\qquad + \sum_{r\in\mathbb{N}_0}\big(a_{(2r+1)N+k}[f-S_{N-1}f] + a_{(2r+1)N-k}[f-S_{N-1}f]\big)\,\frac{N}{2}a_{(2\ell+1)N+k}[\Lambda]\,T_{(2\ell+1)N+k}\Big\|_\alpha\\
&\leq\; \sum_{k=0}^{N}{}'\Big(\sum_{r\in\mathbb{N}} a_{2rN+k}[f] + a_{2(r+1)N-k}[f]\Big)^2\Big(\frac{N}{2}\Big)^2\\
&\qquad\times\sum_{\ell\in\mathbb{N}_0}(1+(2\ell N+k)^2)^\alpha a_{2\ell N+k}^2[\Lambda] + (1+(2(\ell+1)N+k)^2)^\alpha a_{2(\ell+1)N-k}^2[\Lambda]\\
&\leq\; \sum_{k=0}^{N}(1+k^2)^\alpha\Big(\sum_{r\in\mathbb{N}} a_{2rN+k}[f] + a_{2(r+1)N-k}[f]\Big)^2\Big(\frac{N}{2}\Big)^2 N^{2\alpha}\\
&\qquad\times\Big(\varepsilon_k a_k^2[\Lambda] + \sum_{\ell\in\mathbb{N}}(1+(2\ell)^2)^\alpha(a_{2\ell N+k}^2[\Lambda] + a_{2\ell N-k}^2[\Lambda])\Big)\\
&\leq\; \gamma_2^2\sum_{k=0}^{N}(1+k^2)^\alpha\Big(\sum_{r\in\mathbb{N}}(1+(2\ell N+k)^2)^{-\mu} + (1+(2\ell N-k)^2)^{-\mu}\Big)\\
&\qquad\times\Big(\sum_{r\in\mathbb{N}} a_{2rN+k}[f] + a_{2(r+1)N-k}[f]\Big)\\
&\leq\; \gamma_2^2\;\gamma_3^2\;N^{2(\alpha-\mu)}\;\|f\|_\mu^2.
\end{aligned}$$

□

Now we put together these three results and obtain the desired estimate of the error of interpolation for functions from Sobolev–type spaces.

Theorem 5 *Let $f \in H^\mu_w(I)$, $\mu > 1/2$, $\mu \geq \alpha \geq 0$. Let the Lagrange function $\Lambda \in A(I)$ satisfy the Strang–Fix conditions (2) of order m for the exponent α.*

Then for $\varrho := \min\{m+\alpha, \mu\}$, it holds that

$$\|f - L_N f\|_\alpha \;\leq\; \mathrm{C}_\varrho \; N^{\alpha-\varrho} \; \|f\|_\mu \,,$$

with

$$\mathrm{C}_\varrho := 2^{m+\alpha-\varrho}\,\gamma_1 + \gamma_2\;\gamma_3 + 1\,.$$

The constants $\gamma_1, \gamma_2, \gamma_3$ are chosen as in the Theorems 2 and 4, respectively.

5 Examples

The interpolation on Gauß–Chebyshev grids is closely related to periodic interpolation on equidistant grids. The function $\mathcal{M} := \Lambda(\cos\cdot)/2$ is an even periodic fundamental interpolant on the grid $\{\frac{2\pi k}{2N}\,;\; k = -N, \ldots, N-1\}$. If $\mathcal{M}$ satisfies the periodic Strang–Fix conditions of order m for the exponent α (see [3, 11, 12, 13]) with the constants $\{d_\ell\}_{\ell\in\mathbb{Z}}$ then Λ satisfies the Strang–Fix conditions for Gauß–Chebyshev grids of order m for the exponent α with the constants $b_0 = \pi^m d_0$, $b_{2\ell} = \pi^m 2^{-\alpha} d_\ell, \ell \in \mathbb{N}$, and $b_{2\ell+1} = \pi^m 2^{-\alpha} d_{\ell+1}, \ell \in \mathbb{N}_0$.

In this way, one obtains that the scaling functions of the multiresolution analysis for a bounded interval described in [10] fulfil Strang–Fix conditions of certain order.

So the fundamental interpolant of the transformed B–spline of even order r satisfies Strang–Fix conditions of order $r-\alpha$ for the exponent α if $r - \alpha > 1/2$. The de la Vallée Poussin means of Chebyshev polynomials also described in [10] are fundamental interpolants and satisfy Strang–Fix conditions of arbitrary order m.

The constants $\{d_\ell\}_{\ell\in\mathbb{Z}}$ for the corresponding periodic functions can be found in [11, 12, 13].

6 Sobolev–type Spaces of Bivariate Functions

We denote by $L^2_w(I^2) = L^2_w(I)\hat{\otimes}L^2_w(I)$ the weighted Hilbert space of with weight $w \otimes w$ square integrable functions with inner product $\langle\cdot,\cdot\rangle_2$. Here $\hat{\otimes}$ denotes the (complete) tensor product of Hilbert spaces [16]. The Chebyshev coefficients of bivariate functions are defined as $a_{k,\ell}[f] := \langle f, T_k \otimes T_\ell\rangle_2$.

Then the Wiener Algebra $A(I^2)$ contains again all functions with absolute convergent Chebyshev expansion. The Sobolev–type spaces of order $\alpha \geq 0$ can be defined by

$$H^\alpha_w(I^2) := \Big\{ f \in L^2_w(I^2)\;;\; \|f\|^2_\alpha := \sum_{k\in\mathbb{N}_0}{}' \sum_{\ell\in\mathbb{N}_0}{}' (1 + k^2 + \ell^2)^\alpha\, a^2_{k,\ell}[f] < \infty \Big\}\,.$$

Beside these isotropic function spaces, we want to consider functions from Sobolev–type spaces of order (α,β), $\alpha,\beta \geq 0$, with dominating mixed smoothness properties

$$\begin{aligned} S_w^{\alpha,\beta}H(I^2) &:= \Big\{ f \in L_w^2(I^2)\ ;\ \|f\|_{\alpha,\beta}^2 := \sum_{k\in\mathbb{N}_0}{}' \sum_{\ell\in\mathbb{N}_0}{}' (1+k^2)^\alpha\,(1+\ell^2)^\alpha\, a_{k,\ell}^2[f] < \infty \Big\} \\ &= H_w^\alpha(I)\hat{\otimes} H_w^\beta(I)\,. \end{aligned}$$

Obviously, we have the imbeddings

$$H_w^\alpha(I^2) \subset S_w^{\alpha,\alpha}H(I^2) \qquad \text{and} \qquad S_w^{\alpha/2,\alpha/2}H(I^2) \subset H_w^\alpha(I^2). \tag{4}$$

Furthermore, one can proof with some straight–forward calculations the equality

$$H_w^\alpha(I^2) = S_w^{\alpha,0}H(I^2) \cap S_w^{0,\alpha}H(I^2)$$

with the norm equivalence

$$\mathrm{K}_\alpha^2\ \|\cdot\|_\alpha^2\ \leq\ \|\cdot\|_{\alpha,0}^2 + \|\cdot\|_{0,\alpha}^2\ \leq\ 2\ \|\cdot\|_\alpha^2\,, \tag{5}$$

where

$$\mathrm{K}_\alpha^2 := \begin{cases} 2 & \text{for } \alpha = 0, \\ 1 & \text{for } 0 < \alpha \leq 1, \\ 2^{1-\alpha} & \text{for } \alpha > 1. \end{cases}$$

7 Bivariate Interpolation on Full Grids

The interpolation on a full grid $\mathcal{G}_M \times \mathcal{G}_N$ is realized by the interpolation operator $L_M \otimes L_N$ with the univariate fundamental interpolants Λ_M and Λ_N, respectively.

We denote the complementary projector of a projector P by $P^c := I - P$. The Boolean sum [5] of two commuting projectors is given as

$$P \oplus Q := (P^c Q^c)^c = P + Q - PQ\,.$$

First we proof an error estimate for functions with dominating mixed smoothness properties.

Theorem 6 *Let $f \in S_w^{\mu,\nu}H(I^2)$, $\mu,\nu > 1/2$, $\mu \geq \alpha \geq 0$, $\nu \geq \beta \geq 0$. Let $\Lambda_M, \Lambda_N \in A(I)$ satisfy the Strang–Fix conditions* (2) *of order k for the exponent α and of order m for the exponent β, respectively. Set $\varrho := \min\{\mu, k+\alpha\}$ and $\sigma := \min\{\nu, m+\beta\}$.*

With the constants C_ϱ corresponding to Λ_M and C_σ corresponding to Λ_N as in Theorem 5, it holds that

$$\|f - (L_M \otimes L_N)f\|_{\alpha,\beta}\ \leq\ \left(\mathrm{C}_\varrho\, M^{\alpha-\varrho} + \mathrm{C}_\sigma\, N^{\beta-\sigma} + \mathrm{C}_\varrho \mathrm{C}_\sigma\, M^{\alpha-\varrho} N^{\beta-\sigma}\right)\, \|f\|_{\mu,\nu}\,.$$

Proof: The assumptions $\mu, \nu > 1/2$ ensure $f \in A(I^2)$. Hence, the interpolant is well-defined. The proof follows immediately from Theorem 5 by using the representation of the remainder

$$(L_M \otimes L_N)^c = (L_M^c \otimes I) \oplus (I \otimes L_N^c) \tag{6}$$

and the equality

$$\|P \otimes Q\|_{H \to \tilde{H}} = \|P\|_{H_1 \to \tilde{H}_1} \; \|Q\|_{H_2 \to \tilde{H}_2} \tag{7}$$

in tensor products $H = H_1 \hat{\otimes} H_2$ and $\tilde{H} = \tilde{H}_1 \hat{\otimes} \tilde{H}_2$ of Hilbert spaces [16]. □

As we can see from that theorem it might be quite useful to choose different Lagrange functions with different approximation properties to interpolate on grids with different step size parameters M and N. This kind of interpolation is of some use also for functions with different smoothness parameters μ and ν.

But for functions from isotropic function spaces, interpolation only on equidistant grids with the same kind of Lagrange function per direction makes sense (see [12, 13]). So we formulate the next theorem for that case only.

Theorem 7 *Let $f \in H_w^\mu(I^2)$, $\mu > 1$, $\mu \geq \alpha \geq 0$. Let $\Lambda_N \in A(I)$ satisfy the Strang–Fix conditions* (2) *of order m for the exponent α. Set $\varrho := \min\{\mu, m+\alpha\}$, $\eta := \min\{\mu/2, m+\alpha\}$, and $\kappa := \min\{\mu/2, m\}$. The constants $\mathrm{C}_\varrho, \mathrm{C}_\eta, \mathrm{C}_\kappa$ and K_α are chosen as in Theorem* 5 *or in* (5)*, respectively.*

With the constant

$$\tilde{\mathrm{C}}_\varrho := 3\;\sqrt{2}\;\mathrm{K}_\alpha^{-1}\;(2\,\mathrm{C}_\varrho^2 + \mathrm{C}_\eta^2 \mathrm{C}_\kappa^2)^{1/2},$$

it holds that

$$\|f - (L_N \otimes L_N)f\|_\alpha \;\leq\; \tilde{\mathrm{C}}_\varrho\; N^{\alpha-\varrho}\; \|f\|_\mu\,.$$

Proof: Because of $\mu > 1$ we have $f \in A(I^2) \subset C(I^2)$. The proof can be done analogously to the proof of Theorem 2.20 in [13]. It uses Theorem 5, the representatation of the remainder (6), the imbeddings (4) and (5), the relation (7) and the Cauchy–Schwarz inequality. □

8 Bivariate Interpolation on Sparse Grids

In contrast to interpolation on full grids, we want to consider now interpolation on sparse grids. Therefore, we choose $d \in \mathbb{N}$ and set $N_j := d\,2^j$. Furthermore, we assume the imbeddings $\operatorname{Im} L_{N_j} \subset \operatorname{Im} L_{N_{j+1}}$, $j \in \mathbb{N}_0$. The corresponding Lagrange functions Λ_{N_j} have to satisfy the Strang–Fix conditions with the same sequence of constants $\{b_\ell\}_{\ell\in\mathbb{N}_0}$. So the supremum

$$\mathrm{A}_\varrho := \sup_{j\in\mathbb{N}_0} \mathrm{C}_\varrho(j) \tag{8}$$

of the constants $\mathrm{C}_\varrho(j)$ corresponding to Λ_{N_j} from Theorem 5 is finite. For our examples in Section 5, both assumptions are fulfiled.

Then, the interpolation operator on a sparse grid is the j-th order Boolean sum (j-th order blending operator)

$$B_j := \bigoplus_{r=0}^{j} L_{N_r} \otimes L_{N_{j-r}}\,.$$

It interpolates on the sparse grid $\bigcup_{r=0}^{j} \mathcal{G}_r \times \mathcal{G}_{j-r}$ which contains $d^2(j2^{j-1}+2^{j+1})$ interpolation points. Using the representation (cf. [5]) of the remainder of the j-th order Boolean sum

$$B_j^c = L_j^c \otimes I + I \otimes L_j^c - \sum_{r=0}^{j} L_r^c \otimes L_{j-r}^c + \sum_{r=0}^{j-1} L_r^c \otimes L_{j-r-1}^c\,,$$

one can proof the following error estimates for interpolation on sparse grids in the same way as the theorems in the previous section.

Theorem 8 *Let $f \in S_w^{\mu,\nu}H(I^2)$, $\mu,\nu > 1/2$, $\mu \ge \alpha \ge 0$, $\nu \ge \beta \ge 0$. Let $\Lambda_{N_j} \in A(I)$ satisfy the Strang–Fix conditions* (2) *of order k for the exponent α and of order m for the exponent β with the same sequence of constants for all j. Assume* $\mathrm{Im}\, L_{N_j} \subset \mathrm{Im}\, L_{N_{j+1}}$, *$j \in \mathbb{N}_0$. Set $\varrho := \min\{\mu, k+\alpha\}$, $\sigma := \min\{\nu, m+\beta\}$, and $\tau := \min\{\varrho-\alpha, \sigma-\beta\}$.*

With the constants A_ϱ, A_σ as in (8) *and*

$$\tilde{\mathrm{A}}_\tau = \tilde{\mathrm{A}}_\tau(j) := \mathrm{A}_\varrho + \mathrm{A}_\sigma + (3j-2)\, d^{-\tau}\, \mathrm{A}_\varrho\, \mathrm{A}_\sigma$$

it holds that

$$\|f - B_j f\|_{\alpha,\beta} \;\le\; \tilde{\mathrm{A}}_\tau\; N_j^{-\tau}\; \|f\|_{\mu,\nu}\,.$$

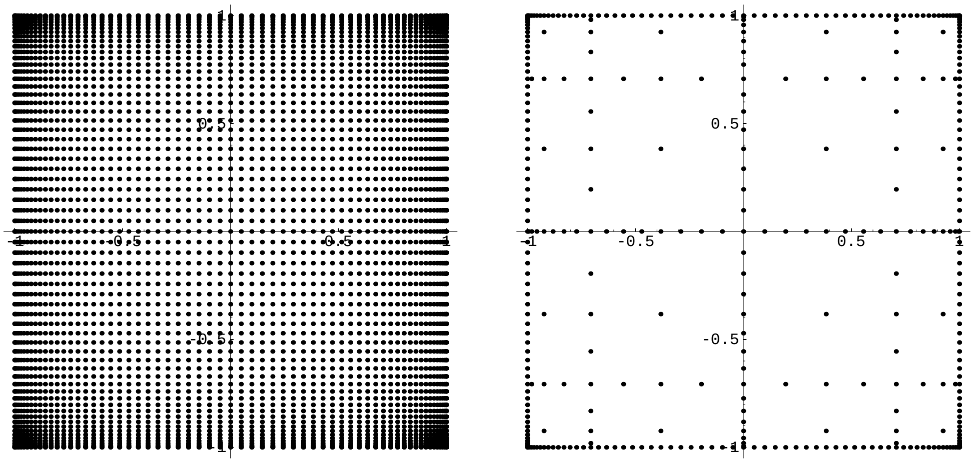
Theorem 9 *Let $f \in H_w^\mu(I^2)$, $\mu > 1$, $\mu \ge \alpha \ge 0$. Let $\Lambda_{N_j} \in A(I)$ satisfy the Strang–Fix conditions* (2) *of order m for the exponent α with the same sequence of constants for all j. Assume* $\mathrm{Im}\, L_{N_j} \subset \mathrm{Im}\, L_{N_{j+1}}$, *$j \in \mathbb{N}_0$. Set $\varrho := \min\{\mu, m+\alpha\}$, $\eta := \min\{\mu/2, m+\alpha\}$, $\kappa := \min\{\mu/2, m\}$ and $\tau := \min\{\eta-\alpha, \kappa\}$.*

With the constants $\mathrm{A}_\varrho, \mathrm{A}_\eta, \mathrm{A}_\kappa$ and K_α as in (8) *and* (5)*, respectively, and*

$$\tilde{\mathrm{A}}_\tau = \tilde{\mathrm{A}}_\tau(j) := 3\,\sqrt{2}\;\mathrm{K}_\alpha^{-1}\;(2\,\mathrm{A}_\varrho^2 + (3j-2)^2\, d^{-2\tau}\,\mathrm{A}_\eta^2\,\mathrm{A}_\kappa^2)^{1/2}$$

it holds that

$$\|f - B_j f\|_\alpha \;\le\; \tilde{\mathrm{A}}_\tau\; N_j^{-\tau}\; \|f\|_\mu\,.$$



Full (left) and sparse (right) Gauß–Chebyshev grid ($d = 1, j = 6$).

Note that the constants in the error estimates in the previous two theorems depend linearly on j. Interpolating functions from Sobolev–type spaces with dominating mixed smoothness properties on a sparse grid, the order of the interpolation error is only by this logarithmic factor worse than the order of the error of interpolation on the full grid with N_j^2 points. If we interpolate functions from the isotropic Sobolev–type spaces we lose half the order of the error of interpolation on sparse grids in contrary to interpolation on full grids (for the same parameter N_j).

9 Comparision

In the previous sections, we gave the error estimates for interpolation on full and sparse Gauß–Chebyshev grids with explicit constants in the error bounds. Now we are interested in a comparision of the two methods. For simplicity, we measure the error in the L^2_w-norm. Let $\Lambda_{N_j} \in A(I)$ satisfy the Strang–Fix conditions (2) of order m for the exponent 0 with the same sequence of constants for all j. We want to compare the order of the interpolation error in terms of the number of interpolation knots. Therefore, we set

$$K := \begin{cases} d^2\ 2^{2j} & \text{for interpolation on the full grid by } L_{N_j} \otimes L_{N_j}, \\ d^2\ (j2^{j-1} + 2^{j+1}) & \text{for interpolation on the sparse grid by } B_j. \end{cases}$$

Recomputing the estimates from the previous sections into terms of K yields the following results.

Smoothness of f	$\|(L_{N_j} \otimes L_{N_j})f - f\|_0$	$\|B_j f - f\|_0$
$f \in H^\mu_w(I^2)$, before saturation	$\mathcal{O}(K^{-\mu/2})$	$\mathcal{O}(K^{-\mu/2}\,(\log_2 K)^{\mu/2-\alpha+1})$
$f \in H^\mu_w(I^2)$, after saturation	$\mathcal{O}(K^{-m/2})$	$\mathcal{O}(K^{-m}\,(\log_2 K)^{m+1})$
$f \in S^{\mu,\mu}_w H(I^2)$, before saturation	$\mathcal{O}(K^{-\mu/2})$	$\mathcal{O}(K^{-\mu}\,(\log_2 K)^{\mu+1})$
$f \in S^{\mu,\mu}_w H(I^2)$, after saturation	$\mathcal{O}(K^{-m/2})$	$\mathcal{O}(K^{-m}\,(\log_2 K)^{m+1})$

As one can see from that table interpolation on full grids beats interpolation on sparse grids only for functions from isotropic Sobolev–type spaces. In this case, it is better only by a logorithmic term. For function with dominating mixed smoothness properties, interpolation on sparse grids is essentially better suited.

10 L_p–Estimates

In this paper, we focussed on L_2–Sobolev–type spaces only, where we used the simple structure of Hilbert spaces and their tensor products. More generally, one may want to consider L_p–Sobolev–type spaces $(1 < p < \infty)$. There, things become more difficult. As in the periodic case (see [13]), it is possible to introduce in addition function spaces which are charactized by weighted ℓ_q–summability of the Chebyshev coefficients. For these function spaces, suited Strang–Fix conditions can be considered. Moreover, one can use properties of tensor products and intersection of reflexive Banach spaces as described in [7, 13]. This again will yield error estimates with explicit constants for functions from these special spaces. Using the Riesz–Thorin theorem, one can obtain estimates with explicit constants for interpolating functions from L_p–Sobolev–type spaces $(1 < p < \infty)$.

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