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G. I. Balikov

Notes on the continuous linear extension operator and the basis in some DFS–spaces of ultradifferentiable functions on interval*

ABSTRACT. We consider relation between a basis and continuous linear extension operator of the functions from outside a interval.

This work considers DFS–spaces† of ultradifferentiable functions on interval. Unlike the results of R.Meise's and A.Taylor's works [8,9], we give a method for concretely construction of continuous linear extension operator.We also considered the relations between a basis and a continuous linear extension operator of the functions outside an interval.

1 Introduction

Let $M = \{M_{pk}\}_{p,k=0}^{\infty}$ be a matrix of positives numbers satisfying the following conditions:

$$1 \leq M_{pk} \leq M_{pk+1} \quad \text{for any} \quad p, k = 0, 1, \ldots \tag{1}$$

$$M_{pk}^2 \leq M_{pk-1} M_{pk+1} \quad \text{for} \quad p = 0, 1, \ldots ; k = 1, 2, \ldots \tag{2}$$

For any p there exists q such that

$$\sup_k (k^k M_{pk}/M_{qk}) < +\infty. \tag{3}$$

For any p there exists q such that

$$\sup_k (M_{p2k}/M_{qk}) < +\infty. \tag{4}$$

An example of a matrix, satisfying the conditions (1–4) is $M = \{k^{pk}\}_{p,k=0}^{\infty}$. We shall use the following spaces, connected with the matrix M :

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†An inductive limit of F spaces with compact embedding operator.

Let K be compact set regular according [6] (further we consider only such compact sets), contained in R, and p is a non-negative number.
Let

$$E\{p,K,M\} = \{f \in C^\infty(K), \|f\|_p = \sup_{K,s} |f^{(s)}(t)|/M_{ps} < +\infty\}. \tag{5}$$

It is easy to see, that the space $E\{p,K,M\}$ is a B–space and it is known that for $p<q$, the space $E\{p,K,M\}$ is embedded in the space $E\{q,K,M\}$ with a nuclear embedding operator (see[1]).
Let

$$E\{K,M\} = \lim_{p} \; ind \; E\{p,K,M\}, \tag{6}$$

it is known that $E\{K,M\}$ is a nuclear DFS–space. Let K and L are compact sets and let $K \subset \; Int\, L \subset R$.
Let

$$E\{L,K,M\} = \{f \in E\{L,M\}, f_{|K} \equiv 0\} \tag{7}$$

considered with relative topology induced by the space $E\{L,M\}$. From the conditions, imposed on the matrix M, it follows that the space $E\{L,K,M\}$ is not trivial. From the theorems of the Whitney type for ultradifferentiable functions (see [5]), it follows that the sequence (8) is exact.

$$0 \longrightarrow E\{L,K,M\} \xrightarrow{I} E\{L,M\} \xrightarrow{J} E\{K,M\} \longrightarrow 0 \quad , \tag{8}$$

where I is the operator of embedding and J is the operator of restriction.

2 The main result.

Theorem 1 *Let in the exact sequence (8) the space $E\{K,M\}$ has an absolute basis $\{f_s\}_{s=0}^\infty$. The continuous linear right inverse of the operator J exists then $\Longleftrightarrow$ to exist functions $\{\hat{f}_s\}_{s=0}^\infty$ satisfying the following conditions:*

$$\hat{f}_s \in E\{L,M\} \text{ and } \hat{f}_{s|K} \equiv f_s, \quad s = 0,1,2,\dots ;$$

and for any q exist p and a constant C that depends on p and q, but does not depend on s, for which

$$\|\hat{f}_s\|_{E\{p,L,M\}} \leq C\,\|f_s\|_{E\{q,K,M\}}, \quad s = 0, 1, 2, \ldots \tag{9}$$

Proof: The spaces $E\{L, M\}$ and $E\{K, M\}$ as DFS–spaces are regular inductive limits (*see* [4]). Let J_r^{-1} be a continuous linear right inverse for J of the exact sequence (8) and let $\hat{f}_s = J_r^{-1} f_s$. From the regularity of the spaces $E\{L, M\}$ and $E\{K, M\}$, it follows that for any q there exists p, such that J_r^{-1} is a continuous operator.

$$J_r^{-1} : E\{q, K, M\} \longrightarrow E\{p, L, M\} \quad .$$

It easily follows that for any $q > q_0$, $f_s \in E\{q, K, M\}$.
On the other hand, let $\{f_s\}_{s=0}^{\infty}$ be an absolute basis in the space $E\{K, M\}$ and the functions $\{\hat{f}_s\}_{s=0}^{\infty}$ are with the indicated characteristics. Let $f \in E\{K, M\}$ then exists $p_0 \geq 0$, so that $f \in E\{p_0, K, M\}$ i.e. $\|f\|_{p_0} < +\infty$. Let's $f = \sum_s \xi_s f_s$. From this that the functions $\{f_s\}_{s=0}^{\infty}$ are an absolute basis in the space $E\{K, M\}$ and it is a regular inductive limit, it follows that

$$\exists q \ , \ \sum_s |\xi_s| \|f_s\|_q \leq C_1 \|f\|_{p_0}. \tag{10}$$

We define $\hat{f} = J_r^{-1} f = \sum_s \xi_s \hat{f}_s$ and $p(q)$ by the condition (9), then

$$\|\hat{f}\|_{p(q)} \leq \sum_s |\xi_s| \|\hat{f}\|_{p(q)} \leq C_2 \sum_s |\xi_s| \|f_s\|_q \leq C_3 \|f\|_{p_0}. \tag{11}$$

From (10) and (11) it follows that the operator defined in this way is a continuous linear right inverse for the operator J from (8).

3 Note I.

Upper bound of the derivatives of one useful function.
Let $0<a<1$ and $b>0$ and

$$B(a, b, t) = \begin{cases} 0 & \text{when} \quad -1 \leq t \leq -a \ , \\ \exp\left(-\frac{ba^4}{t^2(a+t)^2}\right) & \text{when} \quad -a < t \leq 0 \ , \\ 0 & \text{when} \quad 0 \leq t \leq 1 \ . \end{cases} \tag{12}$$

Let define the useful function

$$A(a, b, t) = \begin{cases} \frac{\int_{-1}^{t} B(a,b,t)dt}{\int_{-1}^{1} B(a,b,t)dt} & \text{when} \quad -1 \leq t \leq 0 \ , \\ A(a, b, -t) & \text{when} \quad 0 < t \leq 1 \ , \\ 0 & \text{when} \quad t \notin [-1, 1] \ . \end{cases} \tag{13}$$

We shall bound the derivatives of the function $A(a,b,t)$. A Function like A, for the first time was used by Dzanasija (*see* [3]).
First we shall lower bound $\int_{-1}^{1} B(a,b,t)dt$ in the follow way. From this that the function $B(a,b,t)$ is increasing when $-1 \le t < -\frac{a}{2}$ and decreasing symmetrically when $-\frac{a}{2} < t < 0$ it follows that

$$\int_{-1}^{1} B(a,b,t)dt > \frac{a}{2} B(a,b,-\frac{a}{4})$$

and

$$\int_{-1}^{1} B(a,b,t)dt > \frac{a}{2}\exp(-36b) \quad . \tag{14}$$

For the bound of $|B^{(n)}(a,b,t)|$ we shall use, by analogy with [3], that $B(a,b,t)$ is analytical in a interval $(-a,0)$ and we shall use the Cauchy's formula for contour with a center in the point $t \in [-\frac{a}{2}, 0)$ and radius ht, where $0<h<1$

$$B^{(n)}(a,b,t) = \frac{n!}{2\pi}\int_0^{2\pi} \frac{B(a,b,t-hte^{i\varphi})}{(-hte^{i\varphi})^n} d\varphi \quad , \tag{15}$$

$$|B^{(n)}(a,b,t)| \le \frac{n!}{2\pi h^n |t|^n}\int_0^{2\pi} |B(a,b,t-hte^{i\varphi})| d\varphi \quad , \tag{16}$$

$$|B(a,b,t-the^{i\varphi})| = \left| exp\left(-\frac{ba^4}{t^2(1-he^{i\varphi})(a+t-th^{i\varphi})^2}\right)\right| \tag{17}$$

$$\le exp\left(-\frac{a^4 b\cos(2\gamma+2\beta)}{t^2(1+h^2-2h\cos\varphi)((a+t+th\cos\varphi)^2+t^2h^2\sin^2\varphi)}\right) \quad ,$$

where

$$\sin\beta = \frac{h\sin\varphi}{\sqrt{(1-2h\cos\varphi+h^2)}} \quad ,$$

$$\sin\gamma = \frac{th\sin\varphi}{\sqrt{((a+t+th\cos\varphi)^2+t^2h^2\sin^2\varphi)}} \quad . \tag{18}$$

When h is small and fixed, we have:

$$\frac{\cos(2\gamma+2\beta)}{(1+h^2-2h\cos\varphi)((a+t+th\cos\varphi)^2+t^2h^2\sin^2\varphi)} > \frac{1}{18a^2} \quad . \tag{19}$$

From the last it follows that

$$|B(a,b,t(1-he^{i\varphi}))| \le exp\left(-\frac{a^2 b}{18t^2}\right) \quad . \tag{20}$$

From (16) and (20), we have:

$$|B^{(n)}(a,b,t)| \le \frac{n!}{h^n|t|^n} exp\left(-\frac{a^2 b}{18t^2}\right) \quad . \tag{21}$$

It is easy to see, that

$$\max_t \left(\frac{1}{|t|^n} exp\left(-\frac{a^2 b}{18t^2}\right)\right) \le \frac{3^n n^{\frac{n}{2}} e^{-\frac{n}{2}}}{a^n b^{\frac{n}{2}}} \quad . \tag{22}$$

From (21) and (22) we get

$$|B^{(n)}(a,b,t)| \le \frac{3^n n! n^{\frac{n}{2}} e^{-\frac{n}{2}}}{h^n a^n b^{\frac{n}{2}}} \quad . \tag{23}$$

From (13), (14) and (23) we get the following bound for the derivatives of the function $A(a,b,t)$:

$$|A^{(n)}(a,b,t)| \le 2\exp(-36b)\left(\frac{3}{h}\right)^n \frac{n^{(n-1)} n^{\frac{(n-1)}{2}}}{a^n b^{\frac{(n-1)}{2}}} \tag{24}$$

when $t \in [-1,1]$.

4 Note II.

In the exact sequence (25)

$$0 \longrightarrow E\{[-2,2],[-1,1],M\} \xrightarrow{I} E\{[-2,2],M\} \xrightarrow{J} E\{[-1,1],M\} \longrightarrow 0 \tag{25}$$

the operator J has a continuous linear right inverse and the sequence is split. In fact, in [2] it is proved that Jacobian's polynamials $J_k^{\alpha\beta}(t)$, when $\alpha, \beta \ge -\frac{1}{2}$ are fixed, form an absolute basis in the space $E\{[-1,1],M\}$. We shall define the functions

$$\hat{J}_k^{\alpha\beta}(t) = J_k^{\alpha\beta}(t) A_k(t) \quad , \quad t \in [-2,2] \quad , \tag{26}$$

where $A_0(t) = 1$ when $k = 1, 2, \ldots$

$$A_k(t) = \begin{cases} A\left(k^{-2}, 1, t+1\right) & \text{when} \quad t \in [-2,-1] \ , \\ 1 & \text{when} \quad t \in [-1,1] \ , \\ A\left(k^{-2}, 1, t-1\right) & \text{when} \quad t \in [1,2] \ , \end{cases} \tag{27}$$

where $A(.,.,.)$ is the functions from (13). We shall show that the functions $\hat{J}_k^{\alpha\beta}$ satisfy the conditions of the Theorem I. For this, we shall bound the derivatives of the functions $\hat{J}_k^{\alpha\beta}(t)$. As $A_k(t) \equiv 0$ when $t \notin \left[-1-\frac{1}{k^2}, 1+\frac{1}{k^2}\right]$, then

$$\left|\hat{J}_k^{\alpha\beta}(t)^{(n)}\right| \le \sum_{s=0}^{n} \binom{n}{s} \left|J_n^{\alpha\beta(n-s)}(t)\right| \left|A_k^{(s)}(t)\right| \le$$
$$\le \sum_{s=0}^{n} \binom{n}{s} \max_{t \in \left[-1-\frac{1}{k^2}, 1+\frac{1}{k^2}\right]} \left|J_k^{\alpha\beta(n-s)}(t)\right| \left|A_k^{(s)}(t)\right| . \tag{28}$$

To bound $\max\left|J_k^{\alpha\beta(n-s)}(t)\right|$ when $t\in\left[-1-\frac{1}{k^2},1+\frac{1}{k^2}\right]$ we shall use the Chebyshev's inequality (see [11] p.79), if $p(t)$ is a polynomial with $\deg p\le k$ and $|x_0|>1$, then

$$|p(x_0)|\le(|x_0|+\sqrt{x_0^2-1})^k\max_{t\in[-1,1]}|p(t)| \quad . \tag{29}$$

As above from Markov's inequality (see [11] p.179), i.e. if $p(t)$ is a polynomial with $\deg p\le k$ then

$$\max_{t\in[a,b]}|p'(t)|\le 2k^2\max_{t\in[a,b]}|p(t)|\,/(b-a) \tag{30}$$

and from the bound

$$\max_{t\in[-1,1]}\left|J_k^{\alpha\beta}(t)\right|\le C_1k^{\sigma+\frac{1}{2}} \tag{31}$$

where $\sigma=\max(\alpha,\beta)$ and $C_1>0$ is a constant which does not depend on k (see [12]). From (29), (30) and (31) we have

$$\begin{aligned}\max_{t\in\left[-1-\frac{1}{k^2},1+\frac{1}{k^2}\right]}\left|J_k^{\alpha\beta(s)}(t)\right|&\le\left(1+\frac{3}{k}\right)^{k-s}\max_{t\in[-1,1]}\left|J_k^{\alpha\beta(s)}(t)\right|\\&\le e^3\max_{t\in[-1,1]}\left|J_k^{\alpha\beta(s)}(t)\right|\le e^3\left\|J_k^{\alpha\beta}\right\|_{E\{p,[-1,1],M\}}M_{ps} \quad .\end{aligned} \tag{32}$$

Let be as above $a_{p0}=1$,

$$a_{pk}=\sup_s\log(k^sM_{p0}\,/M_{ps})\,,k=0,1,2,\dots \tag{33}$$

Then (see [7]) from (2) it follows that

$$\sup_k(k^q/e^{a_{pk}})\sim\frac{C_2}{M_{p0}}M_{pq}. \tag{34}$$

From (34), (30) and (31) it follows that for any natural number p_1 there exist natural number p_2 and a positive constant $C_1>0$ such that :

$$\left\|J_k^{\alpha\beta}\right\|_{E\{p_1,[-1,1],M\}}\le C_1e^{a_{p_2k}}\,,k=0,1,2,\dots \tag{35}$$

and for any natural number q_1 there exist a natural number q_2 and a positive constant C_2 such that

$$\left\|J_k^{\alpha\beta}\right\|_{E\{q_2,[-1,1],M\}}\ge C_2e^{a_{q_2k}}$$

From (28) and (32) and when $t\in[-2,2]$, we have :

$$\begin{aligned}\left|\hat{J}_k^{\alpha\beta\,(l)}(t)\right|\le e^3\sum_{s=0}^{l}\binom{l}{s}\max_{t\in[-1,1]}\left|J_k^{\alpha\beta\,(l-s)}(t)\right|\left|A_k^{(s)}(t)\right|\le\\ C_3T^lk^{2l+\sigma+\frac{1}{2}}l^{2l} \quad ,\end{aligned} \tag{36}$$

where C_3 and T are positive constants which do not depend on k and l. From last and (35), for any natural number p we have :

$$\left|\hat{J}_k^{\alpha\beta\,(l)}(t)\right| \leq C_3 T^l l^{2l} \sup_k \left(\frac{k^{pl}}{l^{a_{pk}}}\right) l^{a_{pk}} \leq C_4 T^l l^{2l} M_{ps_l} \left\|J_k^{\alpha\beta}\right\|_{E\{p,[-1,1],M\}} \ . \tag{37}$$

And from $(3),(4)$ and (37) it follows that for any natural number p there exist a natural number q and a positive constant C_5 such that :

$$\left|\hat{J}_k^{\alpha\beta\,(l)}\right| \leq C_5 M_{ql} \left\|J_k^{\alpha\beta}\right\|_{E\{p,[-1,1],M\}} \quad , i.e.$$

$$\left\|\hat{J}_k\right\|_{E\{q,[-2,2],M\}} \leq C_5 \left\|J_k^{\alpha\beta}\right\|_{E\{p,[-1,1],M\}}. \tag{38}$$

From (38) it follows that the functions $\left\{\hat{J}_k^{\alpha\beta}\right\}_{k=0}^{\infty}$ satisfy all the conditions of the Theorem I.

5 Note III.

In the exact sequence (39), the operator J has a linear and uninterupted right inverse and the sequence is split.

$$0 \longrightarrow E\{[-1,1],\{0\},M\} \xrightarrow{I} E\{[-1,1],M\} \xrightarrow{J} E\{\{0\},M\} \longrightarrow 0 \ . \tag{39}$$

In fact, the space $E\{\{0\},M\}$ is a space of sequences

$$E\{\{0\},M\} = \left\{\xi = \{\xi_k\}, \exists p : \|\xi\|_p = \sup_k |\xi_k| / M_{pk} < +\infty\right\} \ . \tag{40}$$

In the space $E\{\{0\},M\}$ the sequences $e_k = \{\delta_{i,k}\}_{i,k=0}^{\infty}$, where $\delta_{ik} = \begin{cases} 0 & i \neq k \\ 1 & i = k \end{cases}$ form a basis. Let

$$m_k = \max_{0 \leq i,j \leq k} M_{ij}.$$

We shall define the functions $\hat{e}_k$

$$\hat{e}_k = A\left(m_k^{-1}, m_k^2, t\right) t^k / k! \quad , k = 0,1,2,\dots,a \ . \tag{41}$$

$A(.,.,.)$ is the function from (13). We will bound the derivatives of the functions $\hat{e}_k(t)$. As *supp* $A\left(m_k^{-1}, m_k^2, t\right) \in \left[-m_k^{-1}, m_k^{-1}\right]$, so that

$$\left|\hat{e}_k{}^{(l)}(t)\right| \leq \sum_{s=0}^{l} \binom{l}{s} \left|A\left(m_k^{-1}, m_k^2, t\right)^{(l-s)}\right| \frac{1}{m_k^{(k-s)}(k-s)!} \leq$$

$$\leq C_5 l^{-36 m_k^2} \sum_{s=0}^{l} \binom{l}{s} \left(\frac{3}{h}\right)^{(l-s)} \frac{(l-s)^{(l-s-1)}(l-s)^{\frac{l-s-1}{2}} m_k^{(l-s)}}{m_k^{2\frac{l-s-1}{2}} m_k^{(k-s)}(k-s)!} \leq$$

$$\leq C_6 l^{-36 m_k^2} \frac{m_k^{(l+1)} k^l l^{2l} T_1^l}{m_k^k k^k} \quad , \tag{42}$$

where C_5, C_6 and T_1 are positive constants that do not depend on k and l.
If we take under consideration that $\|e_k\|_p = M_{pk}^{-1}$ then by (42) and a fixed natural number p we have :
I.When $k = l$

$$\left|\hat{e}_l{}^{(l)}(t)\right| \leq C_7 l^{-36m_l^2} m_l l^{2l} T_1^l M_{pe} / M_{pl} \leq$$
$$\leq \frac{C_8 m_l M_{pl} l^{2l} T_1^l}{m_l^2 M_{pl}} \leq C_9 M_{q_1 l} \frac{1}{M_{pl}} \quad , \tag{43}$$

where q_1 is a natural number such that

$$\sup_l \left(\frac{M_{pl} l^{2l} T_1^l}{M_{q_1 l}} \right) < \infty \quad .$$

II.When $k > l$

$$\left|\hat{e}_k{}^{(l)}(t)\right| \leq C_{10} l^{-36m_k^2} \frac{m_k^{(l)}}{m_k^k} \leq$$
$$C_{11} \frac{k^l M_{pk}}{m_k^{k-l+3} M_{pk}} \frac{l^l T_1^l}{M_{q_1 l}} M_{q_1 l} \leq$$
$$C_{12} \sup_k \frac{k^l}{m_k^{k-l+2}} \frac{C_{13}}{M_p k} M_{q_1 l} \leq C_{14} \frac{M_{q_1 l}}{M_{pk}} \quad . \tag{44}$$

III.When $k < l$

$$\left|\hat{e}_k{}^{(l)}(t)\right| \leq C_{15} l^{-36m_k^2} m_k^{l-k+1} k^{l-k} l^{2l} T_1^l \leq$$
$$C_{16} l^{-36m_k^2} m_k^l l^{3l} T_1^l \leq C_{17} \frac{l^{-36m_k^2} m_k^{l+1}}{M_{pk}} \frac{l^{3l} T_1^l}{M_{q_2 l}} M_{q_2 l} \; , \tag{45}$$

where q_2 is such that

$$\sup_l \left(l^{4l} T_1^l / M_{q_2 l} \right) < \infty \quad .$$

As $l^{-36m_k^2} m_k^{l+1} \leq C_{18} l^{\frac{l}{2}} T_2^l$ by $(43), (44)$ and (45), it follows that for any natural number p there exists a natural number q such that :

$$\|\hat{e}_k\|_{E\{q,[-1,1]\}} \leq C_{20} \|e_k\|_p \quad ,$$

i.e. the conditions of the theorem are fulfilled and it follows that the continuous linear and continuous extension operator exists.

6 Note IV.

If we use that $[E\{\{0\},M\}]^2 = E\{\{0\},M\}$ and the results of Note III, then it follows that:

$$E\{R,M\} = \lim_{N} \operatorname{ind} \mathrm{E}\{[-\mathrm{N},\mathrm{N}],\mathrm{M}\} \cong \prod_{s=1}^{\infty} \mathrm{E}\{[-1,1],\mathrm{M}\} \ ,$$

and as $E\{[-1,1],M\}$ has a basis, it follows that $E\{R,M\}$ has also a basis.

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Author:

Gurko I. Balikov
Institute of Mathematics,
Bulgarian Academy of Sciences
Acad. Bonchev str., bl. 8,
1113 Sofia
Bulgaria