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A Limit Theorem for empirical processes from planned experiments

ABSTRACT. In this paper the data $Y_1, \ldots, Y_m$ have the following structure. The measurements are taken at $t_i \in (0,1)^q$ and there is a family of distributions Q_η, $\eta \in (a,b)$ and $g_0 : [0,1]^q \to (a,b)$ such that $\mathcal{L}(Y_i) = Q_{g_0(t_i)}$. The main results of this paper is a limit theorem for the sequence of random fields

$$W_m(t) = \frac{1}{\sqrt{m}} \sum_{j=1}^{m} (Y_j - g_0(t_j))\, I_{(0,t]}(t_j) \ .$$

Limit theorems for functionals of W_m are used to construct an asymptotic α-test for $H_0 : g = g_0$ versus $H_A : g \neq g_0$. For a regression model the asymptotic power is investigated under local alternatives.

KEY WORDS. Random fields, Limit Theorems, Wiener field, Goodness of fit test.

1 Introduction

We consider a sequence of independent random variables $Y_1, Y_2, \ldots, Y_m$ whose distributions depend on covariables $t_1, \ldots, t_m$ from $(0,1)^q$ in the following way. Assume there is a one parameter family Q_η, $\eta \in (a,b) \subseteq \mathbb{R}_1$ and some function g such that

$$\mathcal{L}(Y_i) = Q_{g(t_i)} \ .$$

We suppose that Q_η has finite second moments and is parametrized by its expectation, i. e. it holds

$$\int x\, Q_\eta(dx) = \eta \ .$$

If Q is a distribution with mean zero and finite variance then $Q_\eta(B) = Q(B - \eta)$ leads to the regression model

$$Y_i = g(t_i) + \varepsilon_i$$

where ε_i are i. i. d. with $E\varepsilon_i = 0$ and $\mathrm{V}(\varepsilon_i) = \sigma^2 < \infty$.

Let Q_η be an exponential family on $(\mathbb{R}_1, \mathcal{B}_1)$ with mean value parametrization and dominating measure ν. Then

$$\frac{dQ_\eta}{d\nu}(x) = \exp\{c(\eta)x - K(\eta)\}$$

and $\int x\, Q_\eta(dx) = \eta$. If $g_0 : [0,1]^q \to (a,b)$ is a function which links the covariables t_i with the parameter values η_i then the model

$$Q_{g_0(t_1)}, \ldots, Q_{g_0(t_m)}$$

is nearly related to the generalized linear model in which it is assumed that $g_0(t) = h(\langle \beta_0, t\rangle)$ where $h : \mathbb{R}_1 \to \mathbb{R}_1$ and β_0 is some unknown parameter vector. $\langle \cdot, \cdot \rangle$ is the scalar product, I_A is the indicator function of the set A.

We are interested in testing the hypothesis $H_0 : g = g_0$ versus $H_A : g \neq g_0$. To prove limit theorems for a suitable sequence of test statistics we introduce random fields

$$W_m(t) = \frac{1}{\sqrt{m}} \sum_{j=1}^{m} (Y_j - g_0(t_j))\, I_{(0,t]}(t_j)\ .$$

Note that W_m is a random variable taking values in $(D_q, \mathcal{D}_q)$. $D_q := D[0,1]^q$ is the Skohorod space of all functions $w : [0,1]^q \to \mathbb{R}_1$ which are continuous from above in the sense of Neuhaus [6]. $\mathcal{D}_q$ ist the σ-algebra of Borel sets generated by the Skohorod topology, which is defined in Neuhaus [6]. The main result of this paper is a limit theorem for sequences of random fields W_m. The continuous mapping theorem then implies for continuous functionals a limit theorem which is used to construct asymptotic α-tests for the hypothesis $H_0 : g = g_0$ versus $H_A : g \neq g_0$.

2 Results

To each random field

$$W(t),\ t \in [0,1]^q$$

we assign a set function on the set of all q-dimensional intervals $(s,t] \subseteq [0,1]^q$ by

$$W((s,t]) = \sum_{\varepsilon_1=0}^{1} \cdots \sum_{\varepsilon_q=0}^{1} (-1)^{q - \sum_{p=1}^{q} \varepsilon_p} W\left(s_1 + \varepsilon_1(t_1 - s_1), \ldots, s_q + \varepsilon_q(t_q - s_q)\right)$$

where $s = (s_1, \ldots, s_q), t = (t_1, \ldots, t_q)$.

We say that W vanishes on the lower boundary of $W(t) = 0$ if a least one t_i is zero. In this case we have

$$W(t) = W((0,t]) \ .$$

Assume now ν is a finite measure on the Borel sets of $[0,1]^q$. It is called continuous if the corresponding distribution function

$$B(t) = \nu([0,t])$$

is a continuous function.

Denote by $N(\mu, \sigma^2)$ the normal distribution with expectation μ and variance $\sigma^2 \geq 0$ where $N(\mu, 0) = \delta_\mu$ is the δ-distribution concentrated at μ.

A random field $W_B(t)$, $t \in [0,1]^q$ with continuous paths is called a *Wiener field* with variance B if W vanishes on the lower boundary, B is continuous, $\mathcal{L}\left(W((s,t])\right) = N\left(0, \nu((s,t])\right)$ and for every disjoint $(s_1,t_1], \ldots, (s_m,t_m]$ the random variables $W\left((s_1,t_1]\right), \ldots, W\left((s_m,t_m]\right)$ are independent.

A simple calculation shows that the covariance function

$$K(s,t) = \operatorname{cov}\left(W(s), W(t)\right)$$

of a Wiener field is given by

$$K(s,t) = B(s \wedge t)$$

where $s \wedge t = (\min(s_1,t_1), \ldots, \min(s_q,t_q))$ for $s = (s_1, \ldots, s_q), t = (t_1, \ldots, t_q)$.

Let $Y_{n,j}$, $1 \leq j \leq m_n$, $n = 1, 2, \ldots$ a double array of random variables and $t_{n,j} \in (0,1)^q$, $1 \leq j \leq m$, $n = 1, 2, \ldots$ is an associate array of covariables. We assume that for every fixed n the r. v.

$$Y_{n,1}, \ldots, Y_{n,m_n} \text{ are independent and } EY_{n,i}^2 < \infty,\ i = 1, \ldots, m_n \ . \tag{1}$$

Set $V(Y_{n,j}) = E\left(Y_{n,j} - EY_{n,j}\right)^2$. Introduce the random fields W_n by

$$W_n(t) = \frac{1}{\sqrt{m_n}} \sum_{j=1}^{m_n} \left(Y_{n,j} - EY_{n,j}\right) I_{(0,t]}(t_{n,j}) \ .$$

We suppose that there is a continuous function $B:\ [0,1]^q \to [0,\infty)$ such that

$$\frac{1}{m_n} \sum_{j=1}^{m_n} \mathrm{V}(Y_{n,j}) I_{(0,t]}(t_{n,j}) \ \stackrel{n\to\infty}{\longrightarrow} \ B(t) \tag{2}$$

for every $t \in [0,1]^q$. Let W_B be a Wiener field with variance B.

Theorem 1 *Assume the conditions* (1), (2) *are fulfilled. If the Lindeberg condition*

$$\lim_{n\to\infty} \frac{1}{m_n} \sum_{j=1}^{m_n} E\left(Y_{n,j} - EY_{n,j}\right)^2 I_{\{|Y_{n,j}-EY_{n,j}|\geq\varepsilon\sqrt{m_n}\}} = 0 \tag{3}$$

for every $\varepsilon > 0$ *is fulfilled then*

$$\mathcal{L}(W_n) \overset{n\to\infty}{\Longrightarrow} \mathcal{L}(W_B)$$

in the sense of weak convergence of distributions on $(D_q, \mathcal{D}_q)$.

Remark 3 The existence of a Wienere field, i.e. the existence of a Gaussian field with the properties formulated above follows from the proof of Theorem 1 (see Lemma 1).

We now ask under which assumptions the condition (1), (2), (3) are satisfied if the r. v. $Y_{n,j}$ belong to a model from the introduction. To be more precise we assume that

$$\mathcal{L}(Y_{n,j}) = Q_{g_0(t_{n,j})} \tag{4}$$

where Q_η is a family of distributions with

$$\int x\, Q_\eta(dx) = \eta$$

and

$$\lim_{N\to\infty} \sup_{a_1\leq\eta\leq a_2} \int\limits_{\{|x|>N\}} x^2\, Q_\eta(dx) = 0 \tag{5}$$

for every finite a_1, a_2 with $[a_1, a_2] \subseteq (a, b)$. Condition (5) implies that $EY_{n,j}^2 < \infty$ and moreover that the $Y_{n,j}^2$ are uniformly integrable. We suppose that Q_η depends continuous on η in the sense of weak convergence, i. e.

$$Q_{\eta_n} \;\Rightarrow\; Q_\eta, \text{ as } \eta_n \to \eta\ . \tag{6}$$

To describe the asymptotic behaviour of the sequence of experimental designs $\{t_{n,1}, \dots, t_{n,m_n}\}$ we introduce the corresponding empirical measure μ_n by

$$\mu_n = \frac{1}{m_n} \sum_{j=1}^{m_n} \delta_{t_{n,j}}$$

and suppose that

$$\mu_n \overset{n\to\infty}{\Longrightarrow} \mu\ . \tag{7}$$

Theorem 2 *Assume the conditions (4), (5), (6), (7) are fulfilled. If g_0 and μ are continuous then*

$$\mathcal{L}(W_n) \;\Rightarrow\; \mathcal{L}(W_B)$$

where

$$B(t) = \int\limits_{[0,t]} \left(\int (x - g_0(s))^2 \, Q_{g_0(s)}(dx) \right) \mu(ds) \; .$$

Now we give an example which shows that the conditions (5), (6), (7) are fulfilled in many situations. Let Q be a distribution on $(\mathbb{R}_1, \mathcal{B}_1)$ with

$$\int x^2 \, Q(dx) < \infty \tag{8}$$

and

$$\int x \, Q(dx) = 0 \; .$$

Set $Q_\eta(B) = Q(B - \eta)$. Then

$$\int \varphi(t) \, Q_\eta(dt) = \int \varphi(t + \eta) \, Q(dt)$$

for every bounded φ. Moreover, φ is continuous then

$$\lim_{n\to\infty} \int \varphi(t) Q_{\eta_n}(dt) = \int \varphi(t) \, Q_\eta(dt)$$

if $\eta_n \to \eta$ by the Theorem of Lebesgue. Hence (6) is fulfilled. To show (5) we note that

$$\int\limits_{\{|x|>N\}} x^2 \, Q_\eta(dx) \leq \int\limits_{\{|x|>N-|\eta|\}} (x + \eta)^2 \, Q(dx) \leq 2 \int\limits_{\{|x|>N-|\eta|\}} (x^2 + \eta^2) \, Q(dx) \; .$$

Hence with $c = \max(|a|, |b|)$

$$\sup_{a \leq \eta \leq b} \int\limits_{\{|x|>N\}} x^2 \, Q_\eta(dx) \leq 2c^2 Q_\eta \left(|x| > N - c \right) + 2 \int\limits_{\{|x|>N-c\}} x^2 \, Q(dx) \;\stackrel{N\to\infty}{\longrightarrow}\; 0$$

for every fixed a, b by condition (8) and the Theorem of Lebesgue.

3 Applications

Suppose the r. v. $Y_{n,1}, \ldots, Y_{n,m_n}$ are independent and $\mathcal{L}(Y_{n,j}) = Q_{g_0(t_{n,j})}$. Then under the assumptions formulated in Theorem 2 we have

$$\mathcal{L}(W_n) \ \Rightarrow \ \mathcal{L}(W_B)$$

where

$$\begin{aligned} W_n(t) &= \frac{1}{\sqrt{m_n}} \sum_{j=1}^{m_n} (Y_{n,j} - g_0(t_{n,j}))\, I_{(0,t]}(t_{n,j}) \\ B(t) &= \lim_{n\to\infty} \frac{1}{m_n} \sum_{j=1}^{m_n} \mathrm{V}(Y_{n,j}) I_{(0,t]}(t_{n,j}) = \int\limits_{[0,t]} \left(\int (x - g_0(s))^2\, Q_{g_0(s)}(dx) \right) \mu(ds) \ . \end{aligned}$$

We introduce the sequence of test statistics T_n by

$$T_n = \int W_n^2(t)\, dt \ .$$

The continuous mapping theorem yields

$$\mathcal{L}(T_n) \ \Rightarrow \ \mathcal{L}(T) \tag{9}$$

as $n \to \infty$ where

$$T = \int W_B^2(t)\, dt \ .$$

Note that the covariance function of W_B is given by $K(s,t) = B(s \wedge t)$. As the function B is continuous the kernel K is a Hilbert-Schmidt operator, i. e.

$$\int\int K^2(s,t)\, dsdt < \infty \ .$$

If B is not identical zero which is assumed in the sequel we have

$$\int\int K^2(s,t)\, dsdt > 0 \ .$$

Let $f_0, f_1, \ldots$ be denote the complete system of orthogonal eigenfunctions belonging to the eigenvalues $\lambda_0, \lambda_1, \lambda_2, \ldots$ of K. Then

$$W_B(t) \overset{d}{=} \sum_{l=0}^{\infty} X_l \sqrt{\lambda_l} f_l(t) \tag{10}$$

where the X_i are independent and standard normal and $\overset{d}{=}$ is the symbol for equality in distribution. The representation (10) yields

$$\int W_B^2(t)\, dt \overset{d}{=} \sum_{l=0}^{\infty} \lambda_l X_l^2 \ .$$

The characteristic function of $\int W_B^2(t)\,dt$ is given by

$$\varphi(s) = \prod_{l=0}^{\infty}(1-2i\lambda_l s)^{-\frac{1}{2}}\,.$$

Let F be the corresponding distribution function and denote by $z_{1-\alpha}$ the $(1-\alpha)$-quantil of F. Note that $z_{1-\alpha}$ is uniquely determined.

To test the hypothesis $H_0 : g = g_0$ versus $H_A : g \neq g_0$ we introduce a sequence of tests φ_n by

$$\varphi_n = I_{(z_{1-\alpha},\infty)}(T_n)\ .$$

The statement (9) yields that the sequence $\{\varphi_n\}$ is an asymptotic α-test.

Now we consider a situation in which the sequence of eigenvalues can be explicitely evaluated. Let

$$Y_i = g_0(t_i) + \varepsilon_i$$

be the nonparametric regression model with i. i. d. errors ε_i which fulfill $E\varepsilon_i = 0, \sigma^2 = \mathrm{V}(\varepsilon_i) < \infty$. Using the notations above we get

$$B(t) = \lim_{n\to\infty}\frac{1}{m_n}\sum_{i=1}^{m_n}\mathrm{V}(Y_{n,i})I_{(0,t]}(t_{n,i}) = \int_{[0,t]}\sigma^2\,\mu(ds) = \sigma^2\mu([0,t]) \tag{11}$$

and $K(s,t) = \sigma^2\mu([0, s\wedge t])$. We now assume in addition that μ is the Lebesgue measure. Then

$$K(s,t) = \sigma^2\prod_{i=1}^{q}(s_i\wedge t_i)\ .$$

Set $\widetilde{K}(x,y) = x\wedge y,\ 0\le x,y\le 1$. It is well known that the eigenvalues and eigenfunctions of $\widetilde{K}$ are given by

$$\lambda_l = \left(l+\frac{1}{2}\right)^{-2}\frac{1}{\pi^2}$$

and $\varphi_l(t) = \sqrt{2}\sin\left(l+\frac{1}{2}\right)\pi t$, respectively. Due to the product structure of K the system of eigenvalues and eigenfunctions are given by

$$\begin{aligned}\lambda_\alpha &= \sigma^2\lambda_{l_1}\cdots\lambda_{l_q}\\ \varphi_\alpha(t) &= \varphi_{l_1}(t_1)\cdots\varphi_{l_q}(t_q)\end{aligned}$$

where $\alpha = (l_1, \ldots, l_q)$. The characteristic function φ of T has the respresentation

$$\varphi(s) = \prod_{\alpha} (1 - 2i\lambda_\alpha s)^{-\frac{1}{2}} .$$

Now we investigate the asymptotic power of the test under local alternatives for the regression model. More precisely we suppose that

$$Y_{n,j} = g_0(t_{n,j}) + \frac{1}{\sqrt{m_n}} h(t_{n,j}) + \varepsilon_{n,j},\ j = 1, \ldots, m_n$$

where $h : [0,1]^q \to \mathbb{R}_1$ is a continuous function. Note that

$$\begin{aligned} W_n(t) &= \frac{1}{\sqrt{m_n}} \sum_j (Y_{n,j} - g_0(t_{n,j}))\, I_{(0,t]}(t_{n,j}) \\ &= \frac{1}{\sqrt{m_n}} \sum_{j=1}^{m_n} \left(Y_{n,j} - g_0(t_{n,j}) - \frac{1}{\sqrt{m_n}} h(t_{n,j}) \right) I_{(0,t]}(t_{n,j}) + \int_{[0,t]} h(s)\, \mu_n(ds) \\ &= \frac{1}{\sqrt{m_n}} \sum_j \varepsilon_{n,j} I_{(0,t]}(t_{n,j}) + \int_{[0,t]} h(s)\, \mu_n(ds) \ . \end{aligned}$$

As μ is supposed to be continuous we get that for the boundary $\partial[0,t]$ of $[0,t]$ holds $\mu(\partial[0,t]) = 0$. Hence for every fixed t the function $h(s)I_{[0,t]}(s)$ is continuous μ-a. s. . Hence by $\mu_n \Rightarrow \mu$

$$\int_{[0,t]} h(s)\, \mu_n(ds) \ \longrightarrow \ \int_{[0,t]} h(s)\, \mu(ds) =: H(t) \ .$$

Assume the $\varepsilon_{n,j}$ are i. i. d. with $E\varepsilon_{n,j} = 0, \sigma^2 = E\varepsilon_{n,j}^2 < \infty$. Then the application of Theorem 1 yields that $\mathcal{L}(W_n) \Rightarrow \mathcal{L}(W_B + H)$, where B is given by (11). To investigate the asymptotic power we note that

$$W_B + H \stackrel{d}{=} \sum_{l=0}^{\infty} \sqrt{\lambda_l}(X_l + \delta_l)\varphi_l$$

where $\delta_l = \frac{1}{\sqrt{\lambda_l}} \int H(t)\varphi_l(t)\mu(dt)$. Consequently

$$T \stackrel{d}{=} \sum_{l=0}^{\infty} \lambda_l (X_l + \delta_l)^2$$

and the asymptotic power of the sequence of tests φ_n is given by

$$\lim_{n\to\infty} E_h \varphi_n = P\left(T > z_{1-\alpha}\right) \ .$$

4 Proofs

Given $t_1, \dots, t_l \in [0,1]^q$ we denote by $\pi_{t_1,\dots,t_l} : D_q \to \mathbb{R}^l$ the projection defined by

$$\pi_{t_1,\dots,t_l}(x) = (x(t_1), \dots, x(t_l)) \ .$$

If P is a distribution on $(D_q, \mathcal{D}_q)$ and $t_1, \dots, t_l,\ l = 1, 2, \dots$ are arbitraryly choosen then the system of distributions $P \circ \pi_{t_1,\dots,t_l}$ is called the family of all finite dimensional distributions. For $x \in D_q$ the modul of continuity is defined by

$$w_x(\delta) = \sup_{|s-t|<\delta} |x(s) - x(t)|,\ 0 < \delta \leq 1 \ .$$

The following criteria for weak convergence can be found in Neuhaus [6]. Given $x \in D_q$ we set

$$\|x\| := \sup_{\tau \in [0,1]^q} |x(\tau)| \ .$$

Lemma 1 *Let $P_1, P_2, \dots$ be a sequence of distributions on $(D_q, \mathcal{D}_q)$ such that for every fixed $t_1, \dots, t_l$ the distributions $P_{n;t_1,\dots,t_l}$ converge weakly say to $P_{t_1,\dots,t_l}$. If*

$$\lim_{\delta \downarrow 0} \limsup_{n\to\infty} P_n\left(\{x :\ w_x(\delta) > \varepsilon\}\right) = 0$$

for every $\varepsilon > 0$ then there exists a uniquely determined distribution P with finite dimensional distributions $P_{t_1,\dots,t_l}$ such that

$$P_n \ \Rightarrow \ P \quad \text{as } n \to \infty \ .$$

Moreover $P(C_q) = 1$.

Given $t = (t_1, \dots, t_q) \in [0,1]^q$ and $0 \leq \tau_i \leq 1$ we set

$$\begin{aligned} x^{(p)}_{\tau_i}(t) &= x(t_1, \dots, t_{p-1}, \tau_i, t_{p+1}, \dots, t_q) \\ \left\|x^{(p)}_{\tau_1} - x^{(p)}_{\tau_2}\right\| &= \sup_{\substack{0\leq t_j \leq 1 \\ j \neq p}} \left|x^{(p)}_{\tau_1}(t) - x^{(p)}_{\tau_2}(t)\right| \end{aligned} \tag{12}$$

and for $0 \leq \tau_1 < \tau_2 < \tau_3 \leq 1$

$$\begin{aligned} m_p(\tau_1, \tau_2, \tau_3, x) &= \min\left(\left\|x^{(p)}_{\tau_1} - x^{(p)}_{\tau_2},\right\|, \left\|x^{(p)}_{\tau_2} - x^{(p)}_{\tau_3}\right\|\right) \\ \mathcal{M}(x) &= \max_{1\leq p \leq q} \sup_{0 \leq \tau_1 \leq \tau_2 \leq \tau_3 \leq 1} m_p(\tau_1, \tau_2, \tau_3, x) \ . \end{aligned}$$

Suppose that $x(t_1, \dots, t_q) = 0$ if at least one at the t_l is zero. Set $u_p = (1, \dots, 1, t_{p+1}, \dots, t_q)$, $\widetilde{u}_p = (1, \dots, 1, 0, t_{p+2}, \dots, t_q)$ and note that

$$\begin{aligned} |x(u_{p-1})| &\leq \min\left(|x(u_{p-1}) - x(\widetilde{u}_{p-1})|, |x(u_p) - x(u_{p-1})|\right) + |x(u_p)| \\ &\leq \mathcal{M}(x) + |x(u_p)| \ . \end{aligned}$$

Hence

$$\|x\| \le q\mathcal{M}(x) + |x(1,\dots,1)| \ . \tag{13}$$

To each $x \in D_q$ we associate a set function on the system of q-dimensional intervals $(s,t] = (s_1,t_1] \times \cdots \times (s_q,t_q]$ (where $s = (s_1,\dots,s_q), t = (t_1,\dots,t_q)$) by

$$x((s,t]) = \sum_{\varepsilon_1=0,1} \cdots \sum_{\varepsilon_q=0,1} (-1)^{q-\sum\limits_p \varepsilon_p} x\left(s_1+\varepsilon_1(t_1-s_1),\dots,s_q+\varepsilon_q(t_q-s_q)\right) \ .$$

Assume now S is a stochastic process with paths in D_q. The estimation of $\mathcal{M}(S)$ in the next Lemma is crucial for all further considerations. The statement formulated below is a special case of a more general result established in Bickel, Wischura [1].

Lemma 2 *If $\gamma > 0$ and*

$$E\left|S((s,t])S((u,v])\right|^\gamma \le \mu_{\gamma,S}((s,t])\mu_{\gamma,S}((u,v]) \tag{14}$$

for every disjoint $(s,t],(u,v]$ with some finite measure $\mu_{\gamma,S}$ on the Borel sets of $[0,1]^q$ then there is a constant K depending only on γ such that

$$P\left(\mathcal{M}(S) \ge \lambda\right) \le K\lambda^{2\gamma}\left(\mu_{\gamma,S}([0,1]^q)\right)^2 \ . \tag{15}$$

Assume $\xi_{n,1},\dots,\xi_{n,m_n}$, $n = 1,2,\dots$, is a double array of random variables which are independent for every fixed n and fulfill $E\xi_{n,i} = 0, \sigma^2_{n,i} = E\xi^2_{n,i} < \infty$. If in addition $E\xi^4_{n,i} < \infty$ then we set $\varkappa_{n,i} = E\xi^4_{n,i}$. Let $t_{n,1},\dots,t_{n,m_n}$, $n = 1,2,\dots$, $t_{n,j} \in [0,1]^q$ the corresponding double array of covariables. Denote by $t_{n,j,1},\dots,t_{n,j,q}$ the components of $t_{n,j}$. Introduce the sequence of stochastic processes W_n by

$$W_n(t) = \frac{1}{\sqrt{m_n}} \sum_{i=1}^{m_n} \xi_{n,i} I_{(0,t]}(t_{n,i}) \ . \tag{16}$$

We suppose that $t_{n,j} \in (0,1)^q$ for every n,j,i. Hence $W_n(t) = 0$ if a least one component of t is zero and W_n is random element of D_q. The definition of W_n yields

$$W_n((s,t]) = \frac{1}{\sqrt{m_n}} \sum_{i=1}^{m_n} \xi_{n,i} I_{(s,t]}(t_{n,i}) \ .$$

Note that by the independence of $\xi_{n,1},\dots,\xi_{n,m_n}$ the random variables $W_n((s,t])$ and $W_n((u,v])$ are independent again for disjoint $(s,t],(u,v]$. To describe the moments of $W_n((s,t])$ we introduce the following measures on the Borel sets of $(0,1]^q$

$$\nu_{W_n} = \frac{1}{m_n} \sum_{i=1}^{m_n} \sigma^2_{n,i}\delta_{t_{n,i}} \tag{17}$$

$$\varkappa_{W_n} = \frac{1}{m_n^2} \sum_{i=1}^{m_n} \varkappa_{n,i}\delta_{t_{n,i}} \ . \tag{18}$$

Note that for every independent random variables $X_1, \ldots X_m$ with $EX_i = 0$ it holds

$$E\left(\sum_{i=1}^{m} X_i\right)^2 = \sum_{i=1}^{m} EX_i^2 \tag{19}$$

and for $EX_i^4 < \infty$

$$E\left(\sum_{i=1}^{m} X_i\right)^4 = \sum_{i=1}^{m} EX_i^4 + 3\left(\sum_{i\neq j} EX_i^2 EX_j^2\right) . \tag{20}$$

The application of (19) and (20) to $W_n((s,t])$ yields

$$\begin{aligned} EW_n^2((s,t]) &= \nu_{W_n}((s,t]) \\ EW_n^4((s,t]) &\leq \varkappa_{W_n}((s,t]) + 3\left(\nu_{W_n}((s,t])\right)^2 . \end{aligned} \tag{21}$$

Lemma 3 *Let W_n be defined by (16) and assume $E\xi_{n,i}^2 < \infty, E\xi_{n,i} = 0$. Suppose that $0 < t_{n,j,i} < 1$ and*

$$c_1 = \sup_n \nu_{W_n}\left((0,1]^q\right) < \infty. \tag{22}$$

For every $\lambda > 0$ there is a constant $d_1(c_1, \lambda)$ such that

$$P\left(\|W_n\| \geq \lambda\right) \leq d_1(c_1,\lambda)\nu_{W_n}\left((0,1]^q\right)$$

If in addition $E\xi_{n,i}^4 < \infty$ and

$$c_2 = \sup_n \varkappa_{W_n}\left((0,1]^q\right) < \infty \tag{23}$$

then there is $d_2(c_1, c_2, \lambda)$ such that

$$P\left(\|W_n\| \geq \lambda\right) \leq d_2(c_1,c_2,\lambda)\left(\varkappa_{W_n}\left((0,1]^q\right) + 3\left(\nu_{W_n}\left((0,1]^q\right)\right)^2\right) .$$

Proof: We obtain from inequality (13) that

$$\|W_n\| \leq q\mathcal{M}(W_n) + |W_n(1,\ldots,1)| .$$

Hence

$$P\left(\|W_n\| \geq \lambda\right) \leq P\left(\mathcal{M}(W_n) \geq \frac{\lambda}{q+1}\right) + P\left(|W_n(1,\ldots,1)| \geq \frac{\lambda}{q+1}\right) . \tag{24}$$

Set $\gamma = 2$. Due to the independence of $W_n((s,t]), W_n((u,v])$ for disjoint $(s,t], (u,v]$ the condition (14) in Lemma 2 is fulfilled with

$$\mu_{2,W_n} = \nu_{W_n} .$$

Hence by (22)

$$P\left(\mathcal{M}(W_n) \geq \frac{\lambda}{q+1}\right) \leq K\left(\frac{\lambda}{q+1}\right)^4 \left(\nu_{W_n}\left((0,1]^q\right)\right)^2$$
$$\leq K\left(\frac{\lambda}{q+1}\right)^4 c_1 \nu_{W_n}\left((0,1]\right)^q \ .$$

Moreover, Chebyshev's inequality yields

$$P\left(|W_n(1,\dots,1)| \geq \frac{\lambda}{q+1}\right) \leq \frac{(q+1)^2}{\lambda^2}\nu_{W_n}\left((0,1]^q\right) \ .$$

The last two inequalities yield the first statement. To prove the second statement we set $\gamma = 4$ and note that by (21) the condition (14) is fulfilled with

$$\mu_{4,W_n}((s,t]) = \varkappa_{W_n}((s,t]) + 3\nu_{W_n}\left((0,1]^q\right)\nu_{W_n}((s,t]) \ .$$

The rest of the proof is similar as for the first statement. ■

To investigate the modul of continuity of W_n we study differences $W_n(\widetilde{u}_1) - W_n(\widetilde{u}_2)$ if the two vectors $\widetilde{u}_1, \widetilde{u}_2$ are different only in one component. More precisely we suppose that $0 < s_1 < s_2 < 1$ are fixed. Then

$$\begin{aligned}
&\sup_{\substack{s_1<t\leq s_2\\ 0<u_j\leq 1, j\neq i}} |W_n(u_1,\dots,u_{i-1},t,u_{i+1},\dots,u_q) - W_n(u_1,\dots,u_{i-1},s_1,u_{i+1},\dots,u_q)| \qquad (25)\\
= &\sup_{\substack{s_1<t\leq s_2\\ 0<u_j<1, j\neq i}} \left|\frac{1}{\sqrt{m_n}}\sum_{l=1}^{m_n}\xi_{n,l} I_{(s_1,t]}(t_{n,l,i})\prod_{j\neq i} I_{(0,u_j]}(t_{n,l,j})\right|\\
= &\sup_{\substack{s_1<t\leq s_2\\ 0<u_j<1, j\neq i}} \left|\frac{1}{\sqrt{m_n}}\sum_{l=1}^{m_n}\xi_{n,l} I_{(s_1,s_2]}(t_{n,l,i}) I_{(0,t]}(t_{n,l,i})\prod_{j\neq i} I_{(0,u_j]}(t_{n,l,j})\right|\\
= &\sup_{\substack{0<u_j<1\\ j=1,\dots,q}} \left|\frac{1}{\sqrt{m_n}}\sum_{l=1}^{m_n}\widetilde{\xi}_{n,l}\prod_{j=1}^{q} I_{(0,u_j]}(t_{n,l,j})\right|\\
= &\|\Delta_i(W_n,s_1,s_2)\|
\end{aligned}$$

where $\widetilde{\xi}_{n,l} = \xi_{n,l} I_{(s_1,s_2]}(t_{n,l,i})$ and the process $\Delta_i(W_n,s_1,s_2)$ is defined by

$$\Delta_i(W_n,s_1,s_2)(u) = \frac{1}{\sqrt{m_n}}\sum_{l=1}^{m_n}\widetilde{\xi}_{n,l} I_{(0,u]}(t_{n,l}) \ .$$

If ϱ is a measure on the Borel sets of $(0,1]^q$ we denote by $\varrho^{(i)}$ the i-th marginal measure defined by

$$\varrho^{(i)}((a,b]) = \varrho\left((0,1]\times\dots\times(0,1]\times(a,b]\times(0,1]\times\dots\times(0,1]\right)$$

where $0 \leq a < b \leq 1$.

Now we estimate the measures $\nu_{\Delta_i(W_n,s_1,s_2)}$ and $\varkappa_{\Delta_i(W_n,s_1,s_2)}$. The definitions (17), (18) and the definition of $\widetilde{\xi}_{n,l}$ provide

$$\begin{aligned} \nu_{\Delta_i(W_n,s_1,s_2)}\left((0,1]^q\right) &= \nu^{(i)}_{W_n}\left((s_1,s_2]\right) \\ \varkappa_{\Delta_i(W_n,s_1,s_2)}\left((0,1]^q\right) &= \varkappa^{(i)}_{W_n}\left((s_1,s_2]\right) . \end{aligned}$$

The application of Lemma 3 to the process $\Delta_i(W_n, s_1, s_2)$ yields

Lemma 4 *If $E\xi^4_{n,i} < \infty$, (22) and (23) are fulfilled then*

$$P\left(\|\Delta_i(W_n,s_1,s_2)\| \geq \lambda\right) \leq d_2(c_1,c_2,\lambda)\left(\varkappa^{(i)}_{W_n}((s_1,s_2]) + 3\left(\nu^{(i)}_{W_n}((s_1,s_2])\right)^2\right) .$$

To get precise estimates of the modul of continuity we apply a truncation technique. To be more precise we set for $a > 0$

$$\overline{\xi}_{n,i} = \xi_{n,i} I_{\{|\xi_{n,i}| \leq a\sqrt{m_n}\}} - E\xi_{n,i} I_{\{|\xi_{n,i}| \leq a\sqrt{m_n}\}} .$$

Note that $E\overline{\xi}_{n,i} = 0$,

$$E\overline{\xi}^2_{n,i} \leq E\xi^2_{n,i} = \sigma^2_{n,i}$$

and by $(a+b)^4 \leq 8(a^4+b^4), |EX|^4 \leq EX^4$

$$\begin{aligned} E\overline{\xi}^4_{n,i} &\leq 8\left(a^2 m_n E\xi^2_{n,i} + E\xi^4_{n,i} I_{\{|\xi_{n,i}| \leq a\sqrt{m_n}\}}\right) \\ &\leq 8\left(a^2 m_n \sigma^2_{n,i} + a^2 m_n \sigma^2_{n,i}\right) = 16a^2 m_n \sigma^2_{n,i} . \end{aligned}$$

Set

$$W_{n,a}(t) = \frac{1}{\sqrt{m_n}} \sum_{i=1}^{m_n} \overline{\xi}_{n,i} I_{(0,t]}(t_{n,i}).$$

Then

$$\nu_{W_{n,a}}((s,t]) \leq \nu_{W_n}((s,t])$$

and

$$\varkappa_{W_{n,a}}((s,t]) \leq \frac{1}{m_n^2} \sum_{i=1}^{m_n} 16a^2 m_n \sigma^2_{n,i} \delta_{t_{n,i}}((s,t]) = 16a^2 \nu_{W_n}((s,t])$$

and $c_2 = \sup\limits_n \varkappa_{W_{n,a}}\left((0,1]^q\right) \leq 16c_1$ if $a \in (0,1]$. The application of Lemma 4 to $W_{n,a}$ yields

Lemma 5 *If (22) is fulfilled then for $a \in (0,1]$*

$$P\left(\|\Delta_i(W_{n,a}, s_1, s_2)\| \geq \lambda\right) \leq d_2(c_1, 16c_1, \lambda)\left(16a^2 \nu_{W_n}^{(i)}((s_1, s_2]) + 3\left(\nu_{W_n}^{(i)}((s_1, s_2])\right)^2\right) .$$

We now show that under the Lindeberg condition the modul of continuity of the processes W_n fulfills the assumptions in Lemma 1.

Lemma 6 *Assume that $\nu_{W_n} \Rightarrow \nu$ as $n \to \infty$ where ν is some continuous measure on $(0,1]^q$. If the Lindeberg condition*

$$\lim_{n\to\infty} \frac{1}{m_n} \sum_{i=1}^{m_n} E\xi_{n,i}^2 I_{\{|\xi_{n,i}| > \varepsilon\sqrt{m_n}\}} = 0 \tag{26}$$

is fulfilled for every $\varepsilon > 0$ then

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} P\left(w_{W_n}(\delta) > \varepsilon\right) = 0$$

for every $\varepsilon > 0$.

Proof: The proof is devided into several steps
1) We show that W_n can be approximated by the process $W_{n,a}$ in the uniform metrik. To this end we note that by $E\xi_{n,i} = 0$

$$\begin{aligned} \overline{W}_n(t) &= W_n(t) - W_{n,a}(t) = \frac{1}{\sqrt{m_n}} \sum_{i=1}^{m_n} \left(\xi_{n,i} I_{\{|\xi_{n,i}| > a\sqrt{m_n}\}} - E\xi_{n,i} I_{\{|\xi_{n,i}| > a\sqrt{m_n}\}}\right) \\ &= \frac{1}{\sqrt{m_n}} \sum_{i=1}^{m_n} \overline{\overline{\xi}}_{n,i} . \end{aligned}$$

The inequality

$$E(\overline{\overline{\xi}}_{n,i})^2 \leq E\xi_{n,i}^2 I_{\{|\xi_{n,i}| > a\sqrt{m_n}\}}$$

implies

$$\nu_{\overline{W}_n}((0,1]^q) \leq \frac{1}{m_n} \sum_{i=1}^{m_n} E\xi_{n,i}^2 I_{\{|\xi_{n,i}| > a\sqrt{m_n}|\}} \overset{n\to\infty}{\longrightarrow} 0$$

by the Lindeberg condition. Hence by the first statement in Lemma 3

$$\limsup_{n\to\infty} P\left(\|W_n - W_{n,a}\| > \lambda\right) = 0 . \tag{27}$$

2) To estimate the modul of continuity of $W_{n,a}$ we remark that

$$\begin{aligned} |W_{n,a}(s) - W_{n,a}(t)| &= |W_{n,a}(s_1, \ldots, s_q) - W_{n,a}(t_1, \ldots, t_q)| \\ &\leq |W_{n,a}(s_1, \ldots, s_q) - W_{n,a}(t_1, s_2, \ldots, s_q)| \\ &\quad + \ldots + |W_{n,a}(t_1, \ldots, t_{q-1}, s_q) - W_{n,a}(t_1, \ldots, t_q)| . \end{aligned}$$

Hence by the definition of $\Delta_i(W_{n,a}, s_1, s_2)$: for fixed $s = (s_1, \ldots, s_q)$ and $\delta > 0$

$$\sup_{t:|s-t|<\delta} |W_{n,a}(s) - W_{n,a}(t)| \le \sum_{i=1}^{q} \|\Delta_i\left(W_{n,a}, (s_i - \delta) \vee 0, (s_i + \delta) \wedge 1\right)\| \tag{28}$$

where $a \wedge b = \min(a, b), a \vee b = \max(a, b)$. From the representation of $\|\Delta_i(W_{n,a}, s_1, s_2)\|$ in (25) we see that

$$\|\Delta_i(W_{n,a}, u_1, u_2)\| \ge \|\Delta_i(W_{n,a}, v_1, v_2)\| \tag{29}$$

if $u_1 < v_1, u_2 > v_2$. Then for every $m = 1, 2, \ldots$

$$\sup_{0\le s_i\le 1} \|\Delta_i\left(W_{n,a}, (s_i - \delta) \vee 0, (s_i + \delta) \wedge 1\right)\| \le \sup_{l=0,1,\ldots,m} \left\|\Delta_i\left(W_{n,a}, \frac{l}{m}, \frac{l + k(m,\delta)}{m}\right)\right\|$$

and

$$\begin{aligned} &P\left(\sup_{0\le s_i\le 1} \|\Delta_i\left(W_{n,a}, (s_i - \delta) \vee 0, (s_i + \delta) \wedge 1\right)\| > \alpha\right) \\ &\le \sum_{l=0}^{m} P\left(\left\|\Delta_i\left(W_{n,a}, \frac{l}{m}, \frac{l + k(m,\delta)}{m}\right)\right\| > \alpha\right) \ , \end{aligned}$$

where $k(m, \delta)$ is the integer which fulfills the inequality

$$\frac{k(m,\delta)}{m} \ge 2\delta > \frac{k(m,\delta) - 1}{m} \ .$$

Applying this inequality to (28) we get

$$P\left(\sup_{|s-t|<\delta} |W_{n,a}(s) - W_{n,a}(t)| > \lambda\right) \le \sum_{i=1}^{q} \sum_{l=0}^{m} P\left(\left\|\Delta_i\left(W_{n,a}, \frac{l}{m}, \frac{m + k(m,\delta)}{m}\right)\right\| > \frac{\lambda}{q}\right) \ .$$

The assumption $\nu_{W_n} \;\Rightarrow\; \nu$ implies

$$c_1 = \sup_n \nu_{W_n}\left((0,1]^q\right) < \infty \ .$$

Hence by Lemma 5

$$\begin{aligned} &P\left(\sup_{|s-t|<\delta} |W_{n,a}(s) - W_{n,a}(t)| > \lambda\right) \\ &\le \sum_{i=1}^{q} \sum_{l=0}^{m} d_2(c_1, 16c_1, \lambda)\left(16a^2 \nu_{W_n}^{(i)}\left(\left(\frac{l}{m}, \frac{l + k(m,\delta)}{m}\right]\right)\right. \\ &\quad \left. +3\left(\nu_{W_n}^{(i)}\left(\left(\frac{l}{m}, \frac{l + k(m,\delta)}{m}\right]\right)\right)^2\right) \ . \end{aligned}$$

The assumption $\nu_{W_n} \Rightarrow \nu$ in conjunction with the continuity of ν yields

$$\lim_{n\to\infty} \nu^{(i)}_{W_n}((a,b]) = \nu^{(i)}((a,b])$$

for every $0 \le a < b \le 1$. Hence

$$\begin{aligned}
&\limsup_{a\downarrow 0} \limsup_{n\to\infty} P\left(\sup_{|s-t|<\delta} |W_{n,a}(s) - W_{n,a}(t)| > \lambda \right) \\
\le\ & 3d_2\left(c_1, 16c_1, \frac{\lambda}{q}\right) \sum_{i=1}^{q} \sum_{l=0}^{m} \left(\nu^{(i)} \left(\left(\frac{l}{m}, \frac{l+k(m,\delta)}{m} \right] \right) \right)^2 .
\end{aligned}$$

Taking $\delta \downarrow 0$ we get

$$\begin{aligned}
&\lim_{\delta\downarrow 0} \limsup_{a\downarrow 0} \limsup_{n\to\infty} P\left(\sup_{|s-t|<\delta} |W_{n,a}(s) - W_{n,a}(t)| > \lambda \right) \\
\le\ & 3d_2\left(c_1, 16c_1, \frac{\lambda}{q}\right) \sum_{i=1}^{q} \sum_{l=0}^{m} \left(\nu^{(i)} \left(\left(\frac{l}{m}, \frac{l+1}{m} \right] \right) \right)^2 \\
\le\ & 3d_2\left(c_1, 16c_1, \frac{\lambda}{q}\right) \max_{i=1,\dots,q} \max_{1\le l\le m-1} \nu^{(i)} \left(\left(\frac{l}{m}, \frac{l+1}{m} \right] \right) \sum_{i=1}^{q} \nu^{(i)}((0,1]) .
\end{aligned}$$

The continuity of the $\nu^{(i)}$ implies that

$$\sup_{1\le l\le m-1} \nu^{(i)} \left(\left(\frac{l}{m}, \frac{l+1}{m} \right] \right) \overset{m\to\infty}{\longrightarrow} 0 ,$$

which yields

$$\lim_{\delta\downarrow 0} \lim_{a\downarrow 0} \limsup_{n\to\infty} P\left(w_{W_{n,a}}(\delta) > \lambda \right) = 0 . \tag{30}$$

3) It holds

$$P\left(w_{W_n}(\delta) \ge \lambda\right) \le P\left(\|W_n - W_{n,a}\| \ge \frac{\lambda}{3} \right) + P\left(w_{W_{n,a}}(\delta) \ge \frac{\lambda}{3} \right) .$$

Taking in this inequality at first $n \to \infty$ then $a \downarrow 0$ and finally $\delta \downarrow 0$ we get from (27) and (30) that

$$\lim_{\delta\downarrow 0} \limsup_{n\to\infty} P\left(w_{W_n}(\delta) \ge \lambda\right) = 0$$

which completes the proof.

Lemma 7 *If the Lindeberg condition (26) in Lemma 6 is fulfilled and $\nu_{W_n} \Rightarrow \nu$ then the finite dimensional distributions of W_n converge weakly to the corresponding finite dimensional distributions of a Wiener field W with variance B where $B(t) = \nu((0,t])$.*

Proof: Note that both W_n and W vanishes on the lower boundary. Hence $W_n(t) = W_n((0,t]), W(t) = W((0,t])$ and instead of proving the weak convergence of the finite dimensional distributions of W_n we may equivalently prove that for any disjoint $(s_1,t_1], \ldots, (s_m,t_m]$ the distribution of

$$W_n((s_1,t_1]), \ldots, W_n((s_m,t_m])$$

converges weakly to the distribution of

$$W((s_1,t_1]), \ldots, W((s_m,t_m]) \ .$$

Due to the independence of $W_n((\cdot,\cdot])$ and $W((\cdot,\cdot])$, respectively, on disjoint intervals it remains to prove that for any fixed s,t

$$\mathcal{L}\left(W_n((s,t])\right) \stackrel{n\to\infty}{\Longrightarrow} \mathcal{L}\left(W((s,t])\right) \ . \tag{31}$$

Consider first the case $\nu((s,t]) = 0$. Then

$$V\left(W_n((s,t])\right) = \nu_{W_n}((s,t]) \ \to \ \nu((s,t]) = 0.$$

$W_n((s,t])$ converges in probability to $0 = W((s,t])$ and hence (31).

Consider now $\nu((s,t]) > 0$. Set

$$b_n^2 = \sum_{i=1}^{m_n} \sigma_{n,i}^2 I_{(s,t]}(t_{n,i})$$

and $\eta_{n,i} = \xi_{n,i} I_{(s,t]}(t_{n,i})$. We obtain

$$\frac{b_n^2}{m_n} = \frac{1}{m_n}\mathrm{V}\left(\sum_{i=1}^{m_n} \eta_{n,i}\right) = \nu_{W_n}((s,t]) \ \stackrel{n\to\infty}{\longrightarrow} \ \nu((s,t]) \ . \tag{32}$$

There are $\alpha > 0$ and n_0 such that for $n \geq n_0$

$$b_n \geq \alpha\sqrt{m_n} > 0 \ .$$

We note that

$$W_n((s,t]) = \frac{b_n}{\sqrt{m_n}} \frac{1}{b_n} \sum_{i=1}^{m_n} \eta_{n,i} \ . \tag{33}$$

It holds

$$\begin{aligned} & \frac{1}{b_n^2} \sum_{i=1}^{m_n} E\left|\eta_{n,i}\right|^2 I_{\{|\eta_{n,i}| > \varepsilon b_n\}} \\ \leq \ & \frac{m_n}{b_n^2} \frac{1}{m_n} \sum_{i=1}^{m_n} E\xi_{n,i}^2 I_{\{|\xi_{n,i}| > \varepsilon\alpha\sqrt{m_n}\}} \ \stackrel{n\to\infty}{\longrightarrow} \ 0 \ . \end{aligned}$$

The central limit theorem implies

$$\mathcal{L}\left(\frac{1}{b_n}\sum_{i=1}^{m_n}\eta_{n,i}\right) \;\Rightarrow\; N(0,1)\ . \tag{34}$$

(32), (33), (34) yield (31).

Proof of Theorem 2:
Set

$$a_1 = \inf_{s\in[0,1]^q} g_0(s),\ \ a_2 = \sup_{s\in[0,1]^q} g_0(s)\ .$$

By the continuity of g_0 we get $-\infty < a_1 \le a_2 < \infty$. Set

$$\psi(s) = \int (x-g_0(s))^2\, Q_{g_0(s)}(dx) = \int x^2\, Q_{g_0(s)}(dx) - g_0^2(s)\ .$$

Denote by $\chi_N(x)$ a picewise linear function with $0 \le \chi_N(x) \le 1$ such that $\chi_N(x) = 1, -N \le x \le N; \chi_N(x) = 0, x < -(N+1), x > N+1$. As

$$\sup_{s\in[0,1]^q}\int x^2\, Q_{g_0(s)}(dx) \le \sup_{a_1\le\eta\le a_2}\int x^2\, Q_\eta(dx)$$

we see from (5) that ψ is bounded. Furthermore for $s_n \to s$

$$\begin{aligned}|\psi(s)-\psi(s_n)| \;\;&\le\;\; 2\sup_{a_1\le\eta\le a_2}\int x^2\,(1-\chi_N(x))\,Q_\eta(dx)\\ &\quad + \left|\int x^2\chi_N(x)\,Q_{g_0(s)}(dx) - \int x^2\chi_N(x)\,Q_{g_0(s_n)}(dx)\right|\ .\end{aligned}$$

Given $\varepsilon > 0$ we choose N sucht that the first term does not exceed $\frac{\varepsilon}{2}$. By condition (6) there is n_0 such that

$$\left|\int x^2\chi_N(x)\,Q_{g_0(s)}(dx) - \int x^2\chi_N(x)\,Q_{g_0(s_n)}(dx)\right| < \frac{\varepsilon}{2}$$

for even $n \ge n_0$. Hence

$$|\psi(s)-\psi(s_n)| < \varepsilon$$

for every $n \ge n_0$ which proves the continuity of ψ. As μ is continuous the function $f_t(s) = I_{[0,t]}(s)\psi(s)$ is μ-a. e. continuous for every t and bounded. This implies the continuity of B. We have

$$\begin{aligned}\frac{1}{m_n}\sum_{j=1}^{m_n} V(Y_{n,j})I_{(0,t]}(t_{n,j}) \;&=\; \int\limits_{[0,t]}\left(\int (x-g_0(s))^2\, Q_{g_0(s)}(dx)\right)\mu_n(ds)\\ &=\; \int I_{(0,t]}(s)\psi(s)\,\mu_n(ds)\\ &=\; \int f_t(s)\,\mu_n(ds) \;\stackrel{n\to\infty}{\longrightarrow}\; \int f_t(s)\,\mu(ds)\end{aligned}$$

where the last statement follows from assumption (7).

To prove the Lindeberg condition we note that

$$\{|Y_{n,j} - EY_{n,j}| > \varepsilon\sqrt{m_n}\} \subseteq \{|Y_{n,j}| > \varepsilon\sqrt{m_n} - \max(|a_1|, |a_2|)\} \ .$$

Set $b_n(\varepsilon) = \varepsilon\sqrt{m_n} - \max(|a_1|, |a_2|)$. Then

$$\begin{aligned}
&\sup_{1\le j\le m_n} E|Y_{n,j} - EY_{n,j}|^2 I_{\{|Y_{n,j}-EY_{n,j}|>\varepsilon\sqrt{m_n}\}} \\
\le &\sup_{1\le j\le m_n} EY_{n,j}^2 I_{\{|Y_{n,j}|>b_n(\varepsilon)\}} + \sup_{1\le j\le m_n} |EY_{n,j}|^2 I_{\{|Y_{n,j}|>b_n(\varepsilon)\}} \\
\le &\ 2 \sup_{1\le j\le m_n} EY_{n,j}^2 I_{\{|Y_{n,j}|>b_n(\varepsilon)\}} \\
\le &\ 2 \sup_{a_1\le\eta\le a_2} \int_{\{|x|>b_n(\varepsilon)\}} x^2\, Q_\eta(dx) \xrightarrow{n\to\infty} 0
\end{aligned}$$

by assumption (5). Hence the Lindeberg condition is established.

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