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Note on hit-and-miss topologies

Dedicated to Prof. Harry Poppe on his 65th anniversary

ABSTRACT. We give necessary conditions for normality of Δ-topologies, where Δ is stable under closed subsets. We have a complete solution of metrizability of proximal topology in uniform setting.

KEY WORDS AND PHRASES: Hit-and-miss, proximal, hypertopology, bounded, totally bounded, Urysohn family, Keesling family, uniform space, compatible metric, metrizability.

1 Introduction

Hypertopologies, i.e. topologies on thr hyperset $CL(X)$ of all closed and nonempty subsets of a topological space X, are intensively studied in the last decade, since they are a fundamental tool in some aspects of optimization theory and convex analysis.

Following the papers [6, 7, 9, 10, 11, 13, 21, 24] we will continue the study of hit-and-miss hyperspace topologies, more precisely the so called Δ and proximal Δ -topologies, where Δ is a fixed subfamily of $CL(X)$. The Vietoris topology and the proximal topology are the prototypes for hit-and-miss and proximal hit-and-miss topologies.

The Vietoris topology ([19], [8]) is the largest hit-and-miss topology and was very thoroughly studied since 1951, when the seminal paper of Michael ([19]) appeared.

The proximal Δ -topologies were deeply investigated in the last years, when their applicability to convex analysis has been found.

The proximal hit-and-miss topology parallels thr Vietoris topology (i.e. is the finest proximal hit-and-miss hypertopology) and has been studied in ([11], [3]).

The Vietoris topology and the proximal topology are too strong for many applications because they fail to work properly especially in metric spaces when sets under analysis are unbounded.

Partially in response to these shortcomings have been constructed on metric spaces the bounded Vietoris ([18], [4]) and the bounded proximal topology ([22], [5]).

Furthermore, the totally bounded proximal hypertopology is now recognized as a fundamental tool in the construction of the lattice of all hypertopologies on the matric space $(X,\ d)$. In fact, its upper part is the infimum of all upper Wijsman topologies corresponding to all metrics uniformly equivalent to d ([10]).

In our paper we study a large class of Δ and proximal Δ -topologies: those determined by subfamilies Δ of $CL(X)$ stable under closed subsets, i.e. families which contain all closed subsets of every its member. Notice that all the above mentioned topologies fulfill this condition. We give necessary conditions for normality of such Δ and proximal Δ -topologies. We have the complete solution of matrizability of proximal topology in a uniform setting.

2 Preliminaries

Let X be a Hausdorff topological space. If $E \subset X$, we set:

$$E^- = \{A \in CL(X) \,:\, A \cap E \neq \emptyset\},$$
$$E^+ = \{A \in CL(X) \,:\, A \cap E\} = \{A \in CL(X) \,:\, A \cap E^c = \emptyset\},$$

where E^c denotes the cmplement of E.

The lower Vietoris topology τ_V^- on $CL(X)$ is generated by the sets of the form G^-, where G is open in X.

If Δ is a nonempty subfamily of $CL(X)$, then the upper Δ -topology τ_Δ^+ is generated by all collections of the form $(B^c)^+$, where $B \in \Delta$.

By the hit-and-miss topology determined by Δ , we will mean the topology $\tau_\Delta = \tau_V^- \vee \tau_\Delta^+$. The topologies τ_V^- and τ_Δ^+ are the lower and upper part of τ_Δ.

If $\Delta = CL(X)$ we obtain the familiar Vietoris topology τ_V, if $\Delta = K(X)$, the family of all compact subsets of X, we have the Fell topology τ_F.

Now, let $(X,\ \mathcal{U})$ be a uniform space. If $B \subset X$ put
$B^{++} = \{A \in CL(X) \,:\, \exists U \in \mathcal{U} \text{ whith } U[A] \subset B\}$.
We say that two subsets A and B are far if $A \in (B^c)^{++}$ (or what is the same $B \in (A^c)^{++}$) [20].

Let Δ be a prescribed nonempty subfamily of $CL(X)$, the upper proximal Δ-topology σ_Δ^+ has as a subbase all the sets of the form $(B^c)^{++}$, where $B \in \Delta$. The proximal Δ-topology $\sigma_\Delta = \tau_V^- \vee \sigma_\Delta^+$. If $\Delta = CL(X)$ we have the proximal topology $\sigma = \tau_V^- \vee \sigma_{CL(X)}^+$, the finest proximal hit-and-miss topology.

Since $(K^c)^+ = (K^c)^{++}$ for K compact, the Fell topology coincides with its proximal version. We refer to [1] and [17] for all undefined terms.

3 Δ-topologies

In what follows let X be a Hausdorff topological space and let Δ be a nonempty subfamily of $CL(X)$. If $A \in CL(X)$ put $\Delta(A) = \{B \in CL(X) : B \in \Delta \text{ and } B \subset A\}$ By $\tau_{\Delta(A)}$ we mean the hit-and-miss topology on $CL(A) = \{D \in CL(X) : D \subset A\}$ associated to $\Delta(A)$, i.e. the topology which has as a subbase all sets of the form U^-, where U is an open set in A, plus all the sets $(B^c)^+$, where $B \in \Delta(A)$. We declare the family Δ to be *stable under closed subsets* iff it contains each closed subset of its members, i.e. if for each $B \in \Delta$ and $A \in CL(X)$ with $A \subset B$ we have also $A \in \Delta$.

Observe that the most familiar and well studied topologies are associated to subfamilies which are stable under closed subsets:

$CL(X)$, $K(X)$, in metric spaces the family of all closed totally bounded sets $TB(X)$, and the family of all closed bounded sets $CLB(X)$.

We start with a simple but useful Lemma.

Lemma 3.1 *Let X be a Hausdorff topological space, $\Delta \subset CL(X)$ be a nonempty family stable under closed subsets and $A \in CL(X)$. Then $\left(CL(A),\ \tau_{\Delta(A)}\right)$ coincides with $CL(A)$ equipped with the relative topology of $(CL(X),\ \tau_\Delta)$.*

Proof: Let U be A-open; there ia an open set V in X with $U = V \cap A$. We have $\{F \in CL(A) : F \cap U \neq \emptyset\} = \{F \in CL(A) : F \cap V \neq \emptyset\} = V^- \cap CL(A)$.

Now, suppose $K \in \Delta(A)$. Then $\{F \in CL(A) : F \cap K = \emptyset\} = (K^c)^+ \cap CL(A)$, and $(K^c)^+ \in \tau_\Delta$, since $K \in \Delta$.

On the other hand, suppose $K \in \Delta$ and consider the set $(K^c)^+ \cap CL(A)$. If $K \cap A = \emptyset$, then $(K^c)^+ \cap CL(A) = CL(A) \in \tau_{\Delta(A)}$. If $K \cap A \neq \emptyset$, then $K \cap A \in \Delta(A)$. Thus $(K^c)^+ \cap CL(A) = \{F \in CL(A) : F \cap (K \cap A) = \emptyset\} \in \tau_{\Delta(A)}$. ■

Lemma 3.2 *Let X be a Hausdorff topological space and $\Delta \subset CL(X)$ be a nonempty family stable under closed subsets. If $(CL(X),\ \tau_\Delta)$ is normal or second countable, then every $B \in \Delta$ must be compact.*

Proof: Let $B \in \Delta$. By Lemma 3.1 $\left(CL(B),\ \tau_{\Delta(B)}\right)$ coincides with $CL(B)$ equipped with the relative topology of $(CL(X),\ \tau_\Delta)$. Hence $(CL(B),\ \tau_{\Delta(B)})$ is normal since it is a closed subspace of the normal spyce $(CL(X),\ \tau_\Delta)$. Now, notice that $\tau_{\Delta(B)}$ coincides with the

Vietoris topology on $CL(B)$ because $B \in \Delta$ and Δ is stable under closed subsets. By the result of Veličko [23], B must be compact.

Similarly, if $(CL(X),\ \tau_\Delta$ is second countable, then $(CL(B),\ \tau_{\Delta(B)})$ is second countable and again $\tau_{\Delta(B)}$ coincides with the Vietoris topology on $CL(B)$. Hence B must be compact by the result of Michael [19]. ■

The following corollary shows that in the class of Δ-topologies, where Δ is stable under closed subsets, there is one, and only one, normal element: the Fell topology.

Corollary 3.3 *Let X be a Hausdorff topological space and $\Delta \subset CL(X)$ ba a nonempty family stable under closed subsets. If $(CL(X),\ \tau_\Delta)$ is normal, then $\tau_\Delta = \tau_F$.*

Proof: By the previous Lemma $\tau_\Delta \subset \tau_F$. Since τ_Δ is Hausdorff, it is finer than Fell topology ([1]). ■

Let $(X,\ d)$ ba a metric space. If $\Delta = CLB(X)$ is the family of all closed and bounded subsets of X, the corresponding τ_Δ topology is called the bounded Vietoris Topology ([4], [18]) and is denoted by τ_{bV}. Furthermore, if $\Delta = TB(X)$, the family of all totally bounded sets of X , we have the topology τ_{TB} studied in [10].

Some efforts have been done in the literature to discribe metrizability of the bounded Vietoris topology τ_{bV} ([18]) and of the totally bounded topology τ_{TB} ([10]) in metric setting.

By applying the previous Lemma we give a deeper and more transparent description of metrizability of the above mentioned topologies: it is equivalent to normality.

Theorem 3.4 *Let $(X,\ d)$ ba a metric space. The following are equivalent:*

(1) $(CL(X),\ \tau_{bV})$ *is second countable;*

(2) $(CL(X),\ \tau_{bV})$ *is metrizable;*

(3) $(CL(X),\ \tau_{bV})$ *is paracompact;*

(4) $(CL(X),\ \tau_{bV})$ *is normal;*

(5) $(CL(X),\ \tau_{bV})$ *is Polish;*

(6) $(X,\ d)$ *is boundedly compact.*

Proof: (1)⇒(2) τ_{bV} is a Tychonoff topology (Lemma 4.4.7 in [1]), thus second countability of τ_{bV} implies its metrizability.

(2)⇒(3) and (3)⇒(4) are clear, (4)⇒(6) by Lemma 3.2.

(6)⇒(5) If $(X,\ d)$ is boundedly compact, then the bounded Vietories topology τ_{bV} coincides with the Fell topology τ_F on $CL(X)$. $(CL(X),\ \tau_F)$ is Polish since X is locally compact and second countable (Theorem 5.1.5 in [1]).

(5)⇒(1) is clear. ■

The following theorem completes a result in [10].

Theorem 3.5 *Let $(X,\ d)$ ba a metric space. The following are equivalent:*

(1) $(CL(X),\ \tau_{TB})$ *is second countable;*

(2) $(CL(X),\ \tau_{TB})$ *is metrizable;*

(3) $(CL(X),\ \tau_{TB})$ *is paracompact;*

(4) $(CL(X),\ \tau_{TB})$ *is normal;*

(5) $(CL(X),\ \tau_{TB})$ *is Polish;*

(6) $(X,\ d)$ *is complete, second countable and locally compact.*

Proof: (1)⇒(2), (2)⇒(3) and (3)⇒(4) are clear.

(4)⇒(6) by Lemma 3.2 every closed totally bounded subset must ba compact, hence $(X,\ d)$ is complete. If every $D \in TB(X)$ is compact, then $\tau_{TB} = \tau_F$ on $CL(X)$. Thus the normality of τ_{TB} guarantees the normality of τ_F which in turn implies both local compactness and Lindelöfness of X by a results of [14]. Thus X is second countable.

(6)⇒(5) If $(X,\ d)$ is complete, then $\tau_F = \tau_{TB}$ on $CL(X)$. But $(CL(X),\ \tau_F)$ is Polish since X is locally compact and second countable ([1]). ■

4 Proximal Δ-topologies

Let $(X,\ \mathcal{U})$ be a Hausdorff uniform space and $A \in CL(X)$. We denote by $\sigma_{\Delta(A)}$ the proximal topology on $CL(A)$ which has as a subbase all sets of the form U^- and $(B^c)^{++}$, where $B \in \Delta(A)$ and U is open.

Lemma 4.1 *Let $(X,\ \mathcal{U})$ be a Hausdorff uniform space, $\Delta \subset CL(X)$ be a nonempty family stable under closed subsets and $A \in \Delta$. Then $(CL(A),\ \sigma_{\Delta(A)})$ coincides with $CL(A)$ equipped with the relative topology of $(CL(X),\ \sigma_\Delta)$.*

Proof: Only the upper part of the involved topologies needs some comments. Let $B \in \Delta$ and $F \in (B^c)^{++} \cap CL(A)$. If there is $U \in \mathcal{U}$ with $U[B] \cap A = \emptyset$, then $F \in CL(A) \subset (B^c)^{++} \cap CL(A)$ and $CL(A) \in \sigma_{\Delta(A)}$.

On the contrary, suppose that we have $V[B] \cap A \neq \emptyset$ for every $V \in \mathcal{U}$. $F \in (B^c)^{++} \cap CL(A)$ implies that there exists $U \in \mathcal{U}$ with $U[B] \cap F = \emptyset$. Let $G \in \mathcal{U}$ be such that $G^4 \subset U$. Then $clG[B] \cap F = \emptyset$ and $F \in [(clG[B] \cap A)^c]^{++}$. Since Δ is stable under closed subsets and $A \in \Delta$ also $H = clG[B] \cap A \in \Delta(A)$. We claim that $\{C \in CL(A) : C \in (H^c)^{++}\} \subset (B^c)^{++} \cap CL(A)$.

By contradiction suppose that there is $L \in CL(A)$ such that $L \in (H^c)^{++}$ but $L \notin (B^c)^{++}$. Thus $G[B] \cap L \neq \emptyset$ and $G[B] \cap L \subset H = clG[B] \cap A$, a contradiction. On the other hand, let $D \in \Delta(A) \subset \Delta$. Observe that $(D^c)^{++} \cap CL(A) \subset \{C \in CL(A) : C \in (D^c)^{++}\}$, and the claim. ■

We now prove our main lemma of this section.

Lemma 4.2 *Let $(X, \mathcal{U})$ ba a Hausdorff uniform space and $\Delta \subset CL(X)$ be a notempty family stable under closed subsets. If $(CL(X), \sigma_\Delta)$ is normal or second countable, then every $B \in \Delta$ is totally bounded.*

Proof: Let $B \in \Delta$. By the previous Lemma $(CL(B), \sigma_{\Delta(B)})$ coincides with $CL(B)$ equipped with the relative topology of $(CL(X), \sigma_\Delta)$. Hence, it is normal as a closed subspace of a normal space. Suppose B is not totally bounded. There are a symmetric $U \in \mathcal{U}$ and an infinite subset $D = \{a_n : n \in Z^+\}$ of B, with $(a_n, a_m) \notin U$ for every n and $m, n \neq m$. Of course D is a closed subset of B and since Δ is stable under closed subsets also $D \in \Delta$. Thus $(CL(D), \sigma_{\Delta(D)})$ is a normal subspace. There are at most $2^{\mathcal{N}_0}$ real valued continuous functions on $CL(D)$ since it is separable. Now we use some ideas from the papers [Ke1] and [Ke2] by Keesling. For every $B \subset D$ set $\tilde{B} = \{a_{2n} : a_n \in B, n \in Z^+\} \cup \{a_{2n-1} : a_n \notin B, n \in Z^+\}$. The family $\mathcal{B} = \{\tilde{B} : \subset A\}$ is called the Keesling family relative to D. Note that $\mathcal{B}$ has cardinality $2^{\aleph_0}$ and it is $\sigma_{\Delta(D)}$-discrete and $\sigma_{\Delta(D)}$-closed. By Tietze's Theorem there are at least $2^{2^{\aleph_0}}$ real valued continuous functions on $CL(D)$, a contradiction.

To prove that also the second countability of $(CL(X), \sigma_\Delta)$ implies that every $B \in \Delta$ is totally bounded it suffices to observe that if total boundedness fails, then the constructed Keesling family has cardinality $2^{\aleph_0}$ and it is $\sigma_{\Delta(D)}$-discrete. ■

We apply the above Lemma to the families $CL(X)$ and $CLB(X)$ in metric spaces.

For $\Delta = CLB(X)$, the family of closed and bounded subsets of X, we obtain proximal bounded Vietoris topology $\sigma_{bV} = \sigma_{CLB(X)}$ ([5], [18]).

We improve results from [3] and [5] concerning the proximal topology σ and the bounded proximal topology $\sigma_{bV} = \sigma_{CLB(X)}$.

Theorem 4.3 *Let (X, d) be a metric space. The following are equivalent:*

(1) *$(CL(X), \sigma)$ is second countable;*

(2) *$(CL(X), \sigma)$ is metrizable;*

(3) *$(CL(X), \sigma)$ is paracompact;*

(4) *$(CL(X), \sigma)$ is normal;*

(5) *(X, d) is totally bounded.*

Proof: (1)⇔(2) and (2)⇔(5) are in Theorem 4.3 of [3].

(2)⇒(3) and (3)⇒(4) are clear.

(4)⇒(5) By the previous Lemma normality of the hyperspace $(CL(X),\ \sigma)$ guarantees total boundedness of X. ■

Theorem 4.4 *Let $(X,\ d)$ be a metric space. The following are equivalent:*

(1) $(CL(X),\ \sigma_{bV})$ *is second countable;*

(2) $(CL(X),\ \sigma_{bV})$ *is metrizable;*

(3) $(CL(X),\ \sigma_{bV})$ *is paracompact;*

(4) $(CL(X),\ \sigma_{bV})$ *is normal;*

(5) *each bounded subset of X is totally bounded.*

Proof: (1)⇔(2) and (2)⇔(5) are proved in [5].

(2)⇒(3) and (3)⇒(4) are clear.

(4)⇒(5) By Lemma 4.2 normality of $(CL(X),\ \sigma_{bV})$ implies that every closed bounded set is totally bounded. ■

To state an analogue of Corollary 3.3 for proximal Δ-topologies we need an extra condition on the family Δ . we recall that $\sum(\Delta)$ denotes the collection of all finite unions of elements from Δ .

If $\Delta = TB(X)$, we denote by σ_{TB} the corresponding proximal topology σ_Δ introduced and studied in [10].

Definition 4.5 *Let $(X,\ \mathcal{U})$ ba a Hausdorff uniform space. A nonempty subfamily of $CL(X)$ is declared a uniformly local family if there exists $U \in \mathcal{U}$ such that $clU[x] \in \sum(\Delta)$ for every $x \in X$.*

Corollary 4.6 *Let $(X,\ \mathcal{U})$ be a Hausdorff uniform space, Δ be a nonempty uniformly local subfamily of $CL(X)$ stable under closed subsets. If $(CL(X),\ \sigma_\Delta)$ is normal, then $\sigma_\Delta = \sigma_{TB}$.*

Proof: Since σ_Δ is normal and Δ is stable under closed sets $\sigma_\Delta \subset \sigma_{TB}$ by the provious Lemma.

To prove the opposite inclusion $\sigma_{TB} \subset \sigma_\Delta$ it suffices to show $\sigma^+_{TB} \subset \sigma^+_\Delta$. Now, let $B \in TB(X)$ and $(B^c)^{++} \in \sigma^+_{TB}$. We prove that $(B^c)^{++} \in \sigma^+_\Delta$. Let $F \in (B^c)^{++}$, then there exists $V \in \mathcal{U}$ such that $V[B] \cap V[F] = \emptyset$. Let $W \in \mathcal{U}$ such that $W^4 \subset V$ and let $\tilde{U}$ ba the entourage which guarantees that Δ is a uniformly local family. Set $U = W \cap \tilde{U}$. Since B is totally bounded, there exist $b_1, \cdots, b_n \in B$ such that $B \subset \bigcup_{i=1}^{n} U[b_i]$. Put $T = \bigcup_{i=1}^{n} clU[b_i]$. Since Δ is stable under closed sets and $U \subset \tilde{U}$, it follows $clU[x] \in \sum(\Delta)$ for every $x \in X$. Hence $T \in \sum(\Delta)$. But $U \subset W$, $b_i \in B$ for $i = 1, \cdots, n$. So $\bigcup_{i=1}^{n} W[b_i] \subset W[B]$ and $T \in W^2[B]$. So, we have $F \in (T^c)^{++} \subset (B^c)^{++}$. ■

Remark 4.7 Example 4.16 in [10] shows that $(CL(X),\ \sigma_{TB})$ can be even metrizable, without the family $TB(X)$ being a uniformly local family.

We have seen that second countability, metrizability, paracompactness and normality are all equivalent for the proximal topology σ and bounded proximal Vietoris topology σ_{bV} constructed on the hyperset of a metric space. It is very easy to verify that also Lindelöfness is equivalent to the above mentioned properties.

We do not know whether paracompactness and normality are equivalent for the proximal totally bounded topology σ_{TB}. But, from the following we can argue that Lindelöfness and paracompactness are equivalent for σ_{TB}

Lemma 4.8 *Let $(X,\ d)$ ba a metric space. If $(CL(X),\ \sigma_{TB})$ is normal, then X is separable.*

Proof: Suppose X fails to be separable. There are a positive η and an uncountable set D such that $d(a,\ b) > 2\eta$ for every $a,\ b \in D,\ a \neq b$. Of course D is a closed set. The normality of $(CL(X),\ \sigma_{TB})$ implies that $CL(D)$ is normal as a closed subspace of $(CL(X),\ \sigma_{TB})$.

We show that the relative topology on $CL(D)$ coincides with $(CL(X),\ \sigma_{TB(D)})$. Let $F \in (B^c)^{++} \cap CL(D)$, where $B \in TB(X)$. There is a positive ε such that $clS_\varepsilon[B] \cap F = \emptyset$ (where $S_\varepsilon[B] = \{x \in X\ :\ d(x,\ b) < \varepsilon$ for some $b \in B\}$). Put $\alpha = \min\left\{\frac{\varepsilon}{4},\ \frac{\eta}{4}\right\}$.

We clain that $H = clS_{\frac{\alpha}{2}}[B] \cap D$ is finite. Suppose that H is infinite. For every $a \in H$ there is $a_B \in B$ with $d(a,\ a_B) < \alpha$. Let $a,\ b \in H$ two different elements. Let us compute $d(a_B,\ b_B)$. We have $d(a,\ b) \leq d(a,\ a_B) + d(a_B,\ b_B) + d(b_B, b) < 2\alpha + d(a_B,\ b_B)$ and $\eta < 2\eta - 2\frac{\eta}{4} \leq d(a,\ b) - 2\alpha \leq d(a_B,\ b_B)$.

So there are infinitely many elements in B such that the distance between any two of them is greater than η. This is a contradiction because B is totally bounded. Thus H is finite, i.e. $H \in TB(X)$ and it is very easy to verify that $F \in \{G \in CL(D)\ :\ G \in (H^c)^{++}\} \subset (B^c)^{++} \cap CL(D)$. It is also transparent that if $L \in TB(D)$, we have $(L^c)^{++} \cap CL(D) \subset \{G \in CL(D)\ :\ G \in (L^c)^{++}\}$.

Consequently, $(CL(D),\ \sigma_{TB(D)})$ is a normal space. Realize now that $\sigma_{TB(D)}$ is just the Fell topology on $CL(D)$. By a results of [14] D must be a Lindelöf space, a contradiction. ■

Theorem 4.9 *Let $(X,\ d)$ be a metric space. The following are equivalent:*

(1) $(CL(X),\ \sigma_{TB})$ *is paracompact;*

(2) $(CL(X),\ \sigma_{TB})$ *is Lindelöf.*

Proof: (2)$\Rightarrow$(1) Is clear.

(1)$\Rightarrow$(2) By Lemma 4.8 we derive that X is separable and so the hyperspace is separable too. But every separable paracompact space is Lindelöf by Corollary 5.1.26 [12]. ∎

5 Second countability and metrizability of proximal Δ-topologies

In this part we complete results from [7] and [9] concerning second countability and metrizability of proximal Δ-topologies. Corresponding results for Δ-topologies can be found in [13]. Using some ideas and tools from [9] is possible to improve Theorem 3.10 of [7], where the attention is restricted to uniformizable hyperspace topologies.

Theorem 5.1 *Let $(X,\ \mathcal{U})$ be a Hausdorff uniform space and Δ be a subfamily of $CL(X)$. The following are equivalent:*

(1) $(CL(X),\ \sigma_\Delta)$ *is second countable;*

(2) *X is second countable anfd there is a countable family $\Delta' \subset \Delta$ such that for every $B \in \Delta$ and $U \in \mathcal{U}$ there is $S \in \sum(\Delta')$ with $B \subset S \subset U[B]$.*

We extend proposition 5.18 of [9] to uniform space.

Theorem 5.2 *Let $(X,\ \mathcal{U})$ be a Hausdorff uniform space and $\Delta \subset CL(X)$ containing the singletons. The following are equivalent:*

(1) $(CL(X),\ \sigma_\Delta)$ *is metrizable;*

(2) $(CL(X),\ \sigma_\Delta)$ *is second countable and regular.*

The condition equivalent to regularity is described in [7], where it is shown that the regularity of σ_Δ is equivalent to the condition that Δ is a uniformly Urysohn family.

Definition 5.3 ([7]) *Let $(X,\ \mathcal{U})$ ba a Hausdorff uniform space and $\Delta \subset CL(X)$ be a family containing the singletons. Δ is called a uniformly Urysohn family provided whenever $B \in \Delta$ and $A \in CL(X)$ are far there exists $S \in \sum(\Delta)$ and $U \in \mathcal{U}$ with $U[B] \subset S \subset A^c$.*

Thus we can rewrite Theorem 5.2 as follows.

Theorem 5.4 *Let $(X,\ \mathcal{U})$ be a Hausdorff uniform space and $\Delta \subset CL(X)$ be a family containing the singletons. The following are eqivalent:*

(1) *$(CL(X),\ \sigma(\Delta)$ is metrizable;*

(2) *there is a countable family $\Delta' \subset \Delta$ such that whenever $D \in \Delta$ and $A \in CL(X)$ are far, there is $S \in \sum(\Delta')$ and $V \in \mathcal{U}$ with $V[B] \subset S \subset A^c$.*

To investigate metrizability, we need some background material.

Let $\mathcal{F}$ be a family of real functionals defined on a set E. The weak topology $O(\mathcal{F})$ induced by $\mathcal{F}$on E is the weakest topology on E making each function in $\mathcal{F}$continuous. The uniformity $\mathcal{W}(\mathcal{F})$ on E determined by $\mathcal{F}$has as a subbase for its entourages all sets of the form:

$$\{(e_1,\ e_2)\ :\ |f(e_1) - f(e_2)| < \varepsilon\},$$

where $\varepsilon > 0$ and $f \in \mathcal{F}$. This uniformity not only is compatible with the weak topology $O(\mathcal{F})$ determined by $\mathcal{F}$, but also makes each element of $\mathcal{F}$uniformly continuous.

Now, given a uniform space $(X,\ \mathcal{U})$, $UC(X,\ I)$ denotes the family of uniformly continuous functions defined on $(X,\ \mathcal{U})$ with values in [0,1]. If $f \in UC(X,\ I)$ and $A \in CL(X)$, then $m_f(A) = \inf\{f(x)\ :\ x \in A\}$ is called the infimal value of f on A. For any $\alpha \in$ [0,1] we write $slv(f,\ \alpha) = \{x \in X\ :\ f(x) \leq \alpha\}$ for the sublevel set of f at height α. We may associate to every nonempty subfamily $\Delta \subset CL(X)$ a subfamily of $UC(X,\ I)$, namely $\mathcal{R}_\Delta = \{f \in UC(X,\ I)\ :\ \text{whenever } \inf f < \alpha < \beta < \sup f,\ \exists S \in \sum(\Delta) \text{ with } slv(f,\ \alpha) \subset S \subset slv(f,\ \beta)\}$. The main result of [7] shows that if Δ is uniformly Urysohn family then σ_Δ is the weak topology determined by $\mathcal{R}^*_\Delta = \{m_f\ :\ f \in \mathcal{R}_\Delta\}$. Thus, $\mathcal{W}(\mathcal{R}^*_\Delta)$ is a compatible uniformity for σ_Δ.

We analyze the uniformity $\mathcal{W}(\mathcal{R}^*_\Delta)$ to create a better picture of $(CL(X),\ \sigma_\Delta)$. First, we need an introductory Lemma.

Theorem 5.5 *Let $(X,\ \mathcal{U})$ be a Hausdorff uniform space and $\Delta \subset CL(X)$ be a family containing the singletons. If Δ is a uniformly Urysohn family, then i: $(X,\ \mathcal{U}) \to (CL(X),\ \mathcal{W}(\mathcal{R}^*_\Delta))$ defined by $i(x) = \{x\}$ is uniformly continuous.*

Proof: Let V be a subbasic element of $\mathcal{W}(\mathcal{R}^*_\Delta)$. So there are a function $f \in \mathcal{R}_\Delta$ and a positive ε such that

$$V = \{(A,\ B) \in CL(X) \times CL(X)\ :\ |m_f(A) - m_f(B)| < \varepsilon\}.$$

Since $f \in UC(X,\ I)$, there is $U \in \mathcal{U}$ with $|f(x) - f(y)| < \varepsilon$ for every $(x,\ y) \in U$. Thus for every $(x,\ y) \in U$, we have $(i(x),\ i(y)) \in V(|m_f(i(x)) - m_f(i(y))| = |f(x) - f(y)| < \varepsilon)$. Hence, the map i is uniformly continuous ([17]). ∎

Remark 5.6 Let $(X,\ \mathcal{U})$ be a Hausdorff uniform space and $\Delta \subset CL(X)$ be a uniformly Urysohn family containing the singletons. We can identify X with the set $\{\{x\} : x \in X\}$. Under this identification we can consider X as a subset of $(CL(X),\ \sigma_\Delta)$, since σ_Δ is an admissible topology. If we denote by $\mathcal{W}_X(\mathcal{R}^*_\Delta)$ the relative uniformity for X, then by Theorem 5.5 we have $\mathcal{W}_X(\mathcal{R}^*_\Delta) \subset \mathcal{U}$ and of course $\mathcal{W}_X(\mathcal{R}^*_\Delta)$ generates the orginal topology on X (induced by $\mathcal{U}$) since σ_Δ is an admissible topology.

If moreover σ_Δ is metrizable, Δ must satisfy the condition (2) from Theorem 5.4. Let Δ' be a countable subfamily of Δ described in (2) of Theorem 5.4. In the coincidence whith the proof of Theorem 3.10 in [7] put:

$$P = \left\{(S_1,\ S_2) \in \sum(\Delta') \times \sum(\Delta') : U[S_1] \subset S_2 \text{ for some } U \in \mathcal{U}\right\}.$$

Let $p = (S_1,\ S_2) \in P$. By Lemma 3.1 in [7] there is $f_p \in \mathcal{R}_\Delta$ such that $f_p(S_1) = 0$ and $f_p(clS_2^c) = 1$. Set $\mathcal{H}_\Delta = \{f_p \in \mathcal{R}_\Delta : p \in P\}$. Then $\mathcal{H}_\Delta$ is a countable subfamily of $\mathcal{R}_\Delta$ and it is shown in [7] that σ_Δ is the weak topology determined by $\mathcal{H}^*_\Delta = \{m_f : f \in \mathcal{H}_\Delta\}$. Thus $\mathcal{W}(\mathcal{H}^*_\Delta)$ is a compatible metrizable uniformity ([17]). Hence $\mathcal{W}_X(\mathcal{H}^*_\Delta)$ is a metrizable uniformity for X weaker than $\mathcal{U}$.

The following theorem shown that if $(X,\ \mathcal{U})$ is a uniform space and $\sigma(\sigma = \sigma(CL(X)))$ is metrizable, then $\mathcal{W}_X(\mathcal{H}^*_\Delta)$ is a metrizable uniformity compatible with $\mathcal{U}$. Thus if σ is a metrizable topology there is a totally bounded metric ϱ on X compatible with the uniformity $\mathcal{U}$ such that the proximal topology generated by $\varrho(\sigma_\varrho)$ and the proximal topology generated by $\mathcal{U}(\sigma_\mathcal{U})$ coincide. So, we have a complete and attractive solution to the metrization problem for the proximal topology σ.

Theorem 5.7 *Let $(X,\ \mathcal{U})$ be a Hausdorff uniform space. The following are equivalent:*

(1) $(CL(X), \sigma)$ *is metrizable;*

(2) *there is a totally bounded metric ϱ on X compatible with $\mathcal{U}$.*

Proof: Only (1)$\Rightarrow$(2) needs proof since (2)$\Rightarrow$(1) is known ([3]).

(1)$\Rightarrow$(2) Put $\mathcal{H} = \mathcal{H}_{CL(X)}$ and $\mathcal{H}^* = \mathcal{H}^*_{CL(X)}$. From Remark 5.6 we know that $\mathcal{W}_X(\mathcal{H}^*)$ is a metrizable uniformity with $\mathcal{W}_X(\mathcal{H}^*) \subset \mathcal{U}$. We show that also the opposite inclusion $\mathcal{U} \subset \mathcal{W}_X(\mathcal{H}^*)$ holds.

Let $\{B_n : n \in Z^+\}$ be a countable base of $\mathcal{W}_X(\mathcal{H}^*)$. Without any loss of generality we may suppose that $B_{n+1} \subset B_n$ for every $n \in Z^+$. Suppose by contradiction that the inclusion $\mathcal{U} \subset \mathcal{W}_X(\mathcal{H}^*)$ fails. So there is a symmetric element $U \in \mathcal{U}$ with $B_n \not\subset U$ for every $n \in Z^+$. Let $(x_n,\ y_n) \in B_n \backslash U$ for every $n \in Z^+$ nad $V \in \mathcal{U}$ be a symmetric element with $V^4 \subset U$. By Efremovic Lemma there is an infinite set $J \subset Z^+$ such that $(x_p,\ y_q) \notin V$ for every $p,\ q \in J$ ([20]).

Set $C = \{x_n \,:\, n \in J\}$ and $D = \{y_n \,:\, n \in J\}$. We claim that D and C are closed. It suffices to show that they are sequentially closed ($\mathcal{W}_X(\mathcal{H}^*)$ and $\mathcal{U}$ induce the same topology on X and $\mathcal{W}_X(\mathcal{R}^*)$ is metrizable). Suppose that C is not sequentially closed and let $\bar{x} \in cl(C)\backslash C$. Let $\{x_{n_k} \,:\, n_k \in J'\}$ be a subsequence of C converging to $\bar{x}$, where J' is an infinite subset of J. Since $(x_{n_k},\ y_{n_k}) \in B_{n_k}$ for any $n_k \in J'$, it follows that also the subsequence $\{y_{n_k} \,:\, n_k \in J'\}$ converges to $\bar{x}$, a contradiction since C and D are far sets (infact $V[C] \cap D = \emptyset$). Similarly, also D ia a closed set.

By (2) of Theorem 5.4 and Remark 5.6 there is $p = (S_1,\ S_2) \in P$ such that $C \subset S_1$ and$D \subset cl(S_2^c)$. Thus $f_p(C) = 0$ and $f_p(D) = 1$. Since the countable base $\{B_n \,:\, n \in Z^+\}$ is a nested family, there exists $n_0 \in Z^+$ such that

$$B_n \subset \left\{(A,\ B) \in CL(X) \times CL(X) \ :\ |m_{f_p}(A) - m_{f_p}(B)| < \frac{1}{4}\right\} \cap X \times X$$

for every $n \geq n_0$. This is a contradiction, since for $k \in J$ we have $(x_k,\ y_k) \in B_k$ and $|m_{f_p}(\{x_k\}) - m_{f_p}(\{y_k\})| = 1$, frequently.

Finally, we observe that if we apply Lemma 4.2 to $CL(X)$ we derive that $\mathcal{U}$must be totally bounded. Thus also $\mathcal{W}_X(\mathcal{H}^*)$ is totally bounded. ■

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