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On the chromatic number of order-interval hypergraphs

ABSTRACT. Let P a finite poset. We consider the hypergraph $\mathfrak{H}(P)$ whose vertices are the points of P and whose edges are the maximal intervals in P. We verify in the present paper the NP-completeness of the determination of the (strong) chromatic number $\gamma(\mathfrak{H}(P))$. We give some exact or asymptotic values of $\gamma(\mathfrak{H}(P))$ if P is the subposet of the Boolean lattice induced by consecutive levels $P_l \cup \cdots \cup P_u$. Moreover, we determine $\gamma(\mathfrak{H}(P))$ for interval orders P.

KEY WORDS. Partial order, Boolean lattice, interval, chromatic number, packing system, Steiner system, large set, NP-completeness

1 Preliminaries

Let P be a finite poset. A subset I of P of the form $I = \{v \in P : p \leq v \leq q\}$ (denoted $[p, q]$) is called an *interval*. If p resp. q is a minimal resp. maximal element of P, then $[p, q]$ is called *maximal interval*. Let $\mathfrak{I}(P)$ be the family of maximal intervals of P. The hypergraph $\mathfrak{H}(P) = (P, \mathfrak{I}(P))$, briefly denoted $\mathfrak{H} = (P, \mathfrak{I})$, whose vertices are the elements of P and whose edges are the maximal intervals of P is said to be the *order-interval hypergraph of P*.

Several interesting results exist about the matching, covering and independence numbers of $\mathfrak{H}$ such as the algorithmic complexity and min-max relations (see [2, 3, 6]).

In this paper, we investigate another parameter, the chromatic number, and study it in the case of the order-interval hypergraph $\mathfrak{H}_{n;l,u}$ of the subposet $P_{n;l,u}$ induced by consecutive levels of the Boolean lattice B_n, where more precisely $P_{n;l,u} = \{X \subseteq [n] : \ l \leq |X| \leq u\}$ with $0 \leq l < u \leq n$ and $[n] = \{1, \ldots, n\}$.

Let $k \geq 2$ be an integer. Recall that a *(strong vertex) k-colouring* of a hypergraph $\mathfrak{H} = (V, \mathcal{E})$ is a partition $\{C_1, \ldots, C_k\}$ of V into k (colour) classes (or independent sets) such that no colour appears twice in the same edge. The (strong) chromatic number of a hypergraph $\mathfrak{H}$, denoted by $\gamma(\mathfrak{H})$, is the smallest integer k for which $\mathfrak{H}$ admits a k-colouring. For the special order-interval hypergraph $\mathfrak{H}_{n;l,u}$ of the poset $P_{n;l,u}$, we denote the chromatic number by $\gamma_{n;l,u}$.

The design theory enables us in some cases to get results for the colouring problem. Let us recall some notions. A *packing system* $P(t,k,n)$ of order n (briefly *packing*) is a pair (Q,q), where Q is an n-set and q is a collection of k-subsets of Q called blocks, having the property that every t-subset of Q is a subset of at most one block of q. In particular, a $P(2,3,n)$ is called a *packing triple system* PT of order n (briefly $PT(n)$). A PT is *optimal* if there is no PT of the same order with more blocks. Two PTs, (Q,q_1) and (Q,q_2), are said to be disjoint if $q_1 \cap q_2 = \emptyset$. If we replace "at most" with "exactly" in the definition of the packing system, we have a *Steiner system* $S(t,k,n)$. In particular an $S(2,3,n)$ is called a Steiner triple system STS of order n (briefly $STS(n)$). Such a system exists iff $n \equiv 1$ or $3 \pmod 6$. It is easy to show that the maximum number of pairwise disjoint $STSs$ of order n is at most $n-2$. If this number is exactly $n-2$, i.e., the set of all triples can be partitioned into $n-2$ $STS(n)$, this partition will be called a *large set of disjoint Steiner triple system* (briefly $LTS(n)$). An $LTS(n)$ exists iff $n \equiv 1$ or $3 \pmod 6$, $n \neq 7$ (see [13, 14, 18]).

The objective of the paper is to begin the study of the chromatic number of the order-interval hypergraph of P where P is either the subposet induced by consecutive levels of the Boolean lattice B_n or an interval order. We also show that the problem of the determination of the chromatic number $\gamma(\mathfrak{H}(P))$ is NP-complete if there is no restriction on P.

2 NP-completeness of the colouring problem

Let us mention that the order-interval hypergraphs of chromatic number 2 can be recognized in polynomial time since $\gamma(\mathfrak{H}(P)) \leq 2$ iff P is a bipartite poset. Let us denote the problem of deciding $\gamma(\mathfrak{H}(P)) \leq k$ (with input P and k, with $k > 2$) the *colouring problem for the order-interval hypergraph.*

Theorem 1 *The colouring problem for the order-interval hypergraph is NP-complete.*

Proof: It is clear that this problem belongs to the class NP. We prove the completeness by a polynomial reduction of the colouring problem in a graph to our problem. For a graph $G = (V,E)$, the chromatic number $\gamma(G)$ is given by

$$\gamma(G) = \min\{k, \text{ there exists a partition of } G \text{ into } k \text{ sets of mutually nonadjacent vertices}\}.$$

It is known (see [10]) that the decision problem $\gamma(G) \leq k$ is NP-complete for $k > 2$. Now we present a construction which leads to the relation $2\gamma(G) = \gamma(\mathfrak{H})$. Label the vertices of G

by $v_1, \dots, v_n$ and the edges by $e_1, \dots, e_m$. We associate with G a poset $P = Q^* \oplus Q$ where Q is the incidence poset of G, i.e., $P = E \cup V \cup V' \cup E'$, where $V' = \{v' : v \in V\}$ (copy of V) and $E' = \{e' : e \in E\}$ (copy of E) and the ordering is given in the following way:

$$\begin{array}{ll} \text{for } e \in E, v \in V & \text{we have } e < v \text{ iff } v \text{ is an endpoint of } e \text{ in } G, \\ \text{for } v \in V, v' \in V' & \text{we have } v < v', \\ \text{for } v' \in V', e' \in E' & \text{we have } v' < e' \text{ iff } v \text{ is an endpoint of } e \text{ in } G. \end{array}$$

We show that, G is t-colourable iff $\mathfrak{H}(P)$ is $2t$-colourable whenever $t > 2$. Indeed, for a colouring $\{C_1, \dots, C_t\}$ of G whose colours are chosen among the set $\mathcal{C} = \{\alpha_1, \dots, \alpha_t\}$, we shall assign to the vertices of $\mathfrak{H}$ the following colours: For $e \in E$, e will get any colour $\alpha \in \mathcal{C}$ provided that it is different to the colours of its adjacent vertices in G (note $t > 2$). For $v \in V$, v will get the same colour as in G. For $v' \in V'$, v' will get the colour $\alpha' \in \mathcal{C}' = \{\alpha'_1, \dots, \alpha'_t\}$ (with $\alpha'_i \neq \alpha'_j$ for all $i \neq j$) if α is the colour of v. For $e' \in E'$, e' will get the colour $\alpha' \in \mathcal{C}'$ if e has colour α. In this way, we obtain a $2t$-colouring of $\mathfrak{H}$, that is $\gamma(\mathfrak{H}) \leq 2\gamma(G)$.

Conversely, if $\mathfrak{H}$ admits a $2t$-colouring, then there are in the first or in the second level of P at most t colours. Thus there is a k-colouring of the vertices of G with $k \leq t$. Consequently, $2\gamma(G) \leq \gamma(\mathfrak{H})$. □

3 Some results for the level induced subposet of the Boolean lattice

Proposition 1 *We have* $\gamma_{n;1,n} = 2^{n-1}$.

Proof: The inequality "$\geq$" follows by observing that each edge of $\mathfrak{H}_{n;1,n}$ has 2^{n-1} elements. Now, if we assign the same colour i to both elements, p_i and its complement $[n] \backslash p_i$ ($p_i \neq [n]$), for all $i = 1, \dots, 2^{n-1} - 1$ and the colour 2^{n-1} to the element $[n]$, we obtain the inequality $\gamma_{n;1,n} \leq 2^{n-1}$. □

We need two preliminary lemmas for the next results.

Lemma 1 *We have*

$$\max\{n - l + 1, \frac{\binom{n}{l+1}}{\lfloor \frac{n}{l+1} \lfloor \frac{n-1}{l} \lfloor \dots \lfloor \frac{n-l+1}{2} \rfloor \dots \rfloor}\} \leq \gamma_{n;l,l+2},$$

where, as usual, $\lfloor x \rfloor$ *denotes the largest integer not exceeding* x.

Proof: Proceeding by induction on l, Schönheim [17] proved that a packing system $P(l, l+1, n)$ does not contain more than $\Psi(n, l) = \lfloor \frac{n}{l+1} \lfloor \frac{n-1}{l} \lfloor \dots \lfloor \frac{n-l+1}{2} \rfloor \dots \rfloor$ $(l+1)$-subsets. Since each colour-class of the level $N_{l+1} = \{X \subseteq [n] : |X| = l + 1\}$ is obviously a packing system,

$$\frac{\binom{n}{l+1}}{\Psi(n, l)} \leq \gamma_{n;l,l+2}.$$

Besides this inequality, we have also $\gamma_{n;l,l+2} \geq n-l+1$ since the elements $\{1,\dots,l\}$ and $\{1,\dots,l,i\}$, $i=l+1,\dots,n$, must have pairwise different colours. $\square$

Lemma 2 *Suppose that there is a partition of the $(l+1)$-subsets of an n-set into $\varphi(n)$ disjoint packings $P(l,l+1,n)$.*

If $\varphi(n) > \max\{n-l, \frac{(l+1)(l+2)}{2} + l + 2$, then $\gamma_{n;l,l+2} \leq \varphi(n)$.
If $\varphi(n) = n-l$ and $n > 2l+2$, then $\gamma_{n;l,l+2} \leq \varphi(n)+1$.

Proof: Let $\{C_1,\dots,C_{\varphi(n)}\}$ be the partition of $\binom{[n]}{l+1}$ into disjoint packings $P(l,l+1,n)$. It is easy to see that $\varphi(n) \geq n-l$. Assign to each element of C_i the colour i. If $\varphi(n) > n-l$, then each l-subset has at least $\varphi(n)-(n-l)$ choices for its colour among the elements of the set $[\varphi(n)]$. After that, we observe that each $(l+2)$-subset can get at least $\varphi(n) - \binom{l+2}{l} - \binom{l+2}{l+1}$ colours from $[\varphi(n)]$. Hence, we obtain a $\varphi(n)$-colouring of $\mathfrak{H}_{n;l,l+2}$ for $\varphi(n) > \frac{(l+2)(l+1)}{2} + l + 2$. If $\varphi(n) = n-l$, then all l-subsets get a new colour $\varphi(n)+1$ and as above, since each $(l+2)$-subset has at least $\varphi(n) - \binom{l+2}{l+1}$ choices for its colour in $[\varphi(n)]$, we obtain a $(\varphi(n)+1)$-colouring of $\mathfrak{H}_{n;l,l+2}$ for $\varphi(n) > l+2$. $\square$

Now we study the values of $\gamma_{n;l,l+2}$ for some $l \leq n$. The following theorem is proved for $l=1$ and $n>5$, nevertheless, it is easy to verify that $\gamma_{3;1,3}=4$, $\gamma_{4;1,3}=5$ and $\gamma_{5;1,3}=6$.

Theorem 2 *For $n>5$, we have $\gamma_{n;1,3}=n$.*

Proof: The inequality "$\geq$" follows by Lemma 1 ($l=1$). Let n be even. The well known one-factorization $F=\{F_1,\dots,F_{n-1}\}$ of the complete graph K_n is a partition of the 2-subsets of $[n]$ into $n-1$ disjoint packings $P(1,2,n)$.

Let n be odd. It suffices to consider a one-factorization $F'=\{F'_1,\dots,F'_n\}$ of K_{n+1} and to delete all 2-subsets which contain the element $n+1$. Hence, we obtain from Lemma 2 the inequality "$\leq$". $\square$

It is not difficult to verify that $\gamma_{5;2,4}=5$, $\gamma_{6;2,4}=6$ and $\gamma_{7;2,4}=7$.

Theorem 3 *We have*

$$\gamma_{n;2,4} = \begin{cases} n-1 & \textit{if } n \equiv 0,1,2 \textit{ or } 3 \pmod 6 \quad n \neq 6,7,8 \\ n & \textit{if } n \equiv 4 \pmod 6 \qquad\qquad\quad n \neq 10 \end{cases}$$

and

$$\gamma_{n;2,4} \in \{n-1, n\} \textit{ if } n \equiv 5 \pmod 6,\ n \neq 5.$$

Proof: By Lemma 1 with $l = 2$ we have

$$\gamma_{n;2,4} \geq \begin{cases} n-1 & \text{if } n \equiv 0,1,2,3,5 \pmod 6 \\ n & \text{if } n \equiv 4 \pmod 6. \end{cases}$$

Let us determine upper bounds for $\gamma_{n;2,4}$. To this end, we need, using Lemma 2, only a partition of the 3-subsets into disjoint packings $P(2,3,n)$.

Case 1. $n \equiv 1$ or $3 \pmod 6$ and $n \neq 7$.
From the partition which is defined by the large set of disjoint Steiner triple system $LTS(n)$, we deduce an $(n-1)$-colouring of $\mathfrak{H}_{n;2,4}$ and the inequality $\gamma_{n;2,4} \leq n-1$ follows.

Case 2. $n \equiv 0$ or $2 \pmod 6$ and $n \neq 6, 8$.
In this case $n+1 \equiv 1$ or $3 \pmod 6$. Therefore all triples of order $n+1$ can be partitioned into $(n+1)-2 = n-1$ pairwise disjoint STSs whenever $n \neq 6$. By deleting all 3-subsets containing the element $n+1$, we obtain a partition into $n-1$ packings $P(2,3,n)$ and then an $(n-1)$-colouring of $\mathfrak{H}_{n;2,4}$ for $n > 11$.

Case 3. $n \equiv 5 \pmod 6$ and $n \neq 5$.
The condition $n \equiv 5 \pmod 6$ is equivalent to $n+1 \equiv 0 \pmod 6$. Hence, $\gamma_{n+1;2,4} = (n+1)-1 = n$ for $n \neq 5$. If we delete all subsets containing $n+1$, we obtain an n-colouring of $\mathfrak{H}_{n;2,4}$, i.e, $\gamma_{n;2,4} \leq n$.

Let us mention that the above inequality can be improved if the weak conjecture of Etzion [8] is proved. It states that for every $n \equiv 5 \pmod 6$, $n > 5$, there exists a partition of triples of order n into $n-1$ PTs.
More precisely, we shall have, $\gamma_{n;2,4} = n-1$ if $n \equiv 5 \pmod 6$, $n > 11$.

Case 4. $n \equiv 4 \pmod 6$ and $n \neq 10$.
In Etzion's papers [7, 8] it is proved for $n \equiv 4 \pmod 6$ that there is a partition of the triples into $n-1$ optimal PTs and one PT of size $(n-1)/3$. Hence, we can infer an n-colouring of $\mathfrak{H}_{n;2,4}$ for $n > 10$. $\square$

In the case $l = 3$, it is well known [4] that an optimal packing $P(3,4,n)$ has size $\frac{n(n-1)(n-3)}{24}$ if $n \equiv 1$ or $3 \pmod 6$ and $\frac{n(n^2-3n-6)}{24}$ if $n \equiv 0 \pmod 6$. For $n \equiv 5 \pmod 6$, it has, by the Johnson bound [12], at most size $\frac{(n-1)(n^2-3n-4)}{24} + \lfloor \frac{n-5}{12} \rfloor$. Consequently, the minimal number $\theta(n)$ of pairwise disjoint packings $P(3,4,n)$ which partition all quadruples of an n-set satisfies $\theta(n) \geq n-1$ if $n \equiv 5 \pmod 6$ and $\theta(n) \geq n-2$ otherwise.

In [9], Etzion proved that $\theta(n) \leq n-2$ if $n = 2^i$ $(i \geq 3)$. With Lemma 2, we infer the equality $\gamma_{n;3,5} = n-2$ for this value of n. Graham and Sloane [11] proved that $\theta(n) \leq n$ for all n. Hence, we obtain $\gamma_{n;3,5} \in \{n-1, n\}$ if $n \equiv 5 \pmod 6$ and $\gamma_{n;3,5} \in \{n-2, n-1, n\}$ otherwise. In particular, it was proved by Van Pul and Etzion [19] that $\theta(n) \leq n-1$ if $n = 3 \cdot 2^i$ and by Brouwer et al [5] that $\theta(n) \leq n-1$ if $n = 5 \cdot 2^i$ or $n = 7 \cdot 2^i$, therefore, we have again $\gamma_{n;3,5} \in \{n-2, n-1\}$ if $n = 3 \cdot 2^i$, $\gamma_{n;3,5} = n-1$ if $n = 5 \cdot 2^i$ and $\gamma_{n;3,5} \in \{n-2, n-1\}$ if $n = 7 \cdot 2^i$.

Theorem 4 *For every fixed l, we have $\gamma_{n;l,l+2} = n(1+o(1))$ as $n \to \infty$.*

In order to prove the theorem, we need the following result due to Spencer and Pippenger [16]. Let $\mathfrak{H} = (V, \mathcal{E})$ be a hypergraph. The *degree* of x in $\mathfrak{H}$ (denoted *deg(x)*) is the number of edges of $\mathfrak{H}$ containing x and the *codegree* of x and y in $\mathfrak{H}$ (denoted *codeg(x,y)*) is the number of edges of $\mathfrak{H}$ containing both x and y. A packing in $\mathfrak{H}$ is a set P of edges of $\mathfrak{H}$ such that each vertex of $\mathfrak{H}$ is in at most one edge of P. The chromatic index of $\mathfrak{H}$, denoted by $\chi(\mathfrak{H})$, is the smallest number of packings into which the edges of $\mathfrak{H}$ may be partitioned.

Theorem 5 *[16]. Let k, $k \geq 2$, be a fixed natural number. Let $\mathfrak{H}_n = ([n], E_n)$ be a sequence of k-uniform hypergraphs such that for $n \to \infty$*

$$\min_{x \in [n]} deg(x) = (\max_{x \in [n]} deg(x))(1+o(1)),$$

$$\max_{x,y \in [n], x \neq y} codeg(x,y) = (\max_{x \in [n]} deg(x))o(1).$$

Then

$$\chi(\mathfrak{H}_n) = (\max_{x \in [n]} deg(x))(1+o(1)).$$

Proof of Theorem 4: Let us consider the uniform hypergraph $\mathfrak{H} = (V, \mathcal{E})$ whose vertices are all l-subsets of $[n]$ and whose edges are the $(l+1)$ l-subsets of an $(l+1)$-set. $\mathfrak{H}$ satisfies both hypotheses of the above theorem since we have for all $x, y \in V$, $x \neq y$, $deg(x) = n-l$ and $codeg(x,y) = 1$ if $|x \cap y| = l-1$ and 0 otherwise. It follows that $\chi(\mathfrak{H})$, which is nothing else than the minimum number of disjoint packings $P(l, l+1, n)$ which partition the set of all $(l+1)$-subsets, is asymptotically equal to n. Hence, from Lemma 1, we have for $n \to \infty$

$$n - l \leq \gamma_{n;l,l+2} \leq n(1+o(1))$$

i.e., $\gamma_{n;l,l+2} = n(1+o(1))$. □

Theorem 6 *We have for $n > 7$*

$$\gamma_{n;1,4} = \begin{cases} 2n-1 & \textit{if } n \equiv 0 \textit{ or } 2 \pmod 6 \\ 2n-2 & \textit{if } n \equiv 1 \textit{ or } 3 \pmod 6 \end{cases}$$

and

$$\gamma_{n;1,4} \in \begin{cases} \{2n-2, 2n-1, 2n\} & \textit{if } n \equiv 5 \pmod 6 \\ \{2n-1, 2n\} & \textit{if } n \equiv 4 \pmod 6. \end{cases}$$

Proof: Let A be a subset of the vertex set of $\mathfrak{H}_{n;1,4}$ whose elements contain 1. We claim that A has not less than $2n-2$ colours if n is odd and $2n-1$ if n is even. Indeed, to show this, it suffices to observe that the set $A_1 = \{\{1\},\{1,i\}, i = 2,\dots,n\}$ needs exactly n colours and the set $A_2 = \{\{1,i,j\}, i,j \in \{2,\dots,n\}, i \neq j\}$, which is isomorphic to $\binom{[n-1]}{2}$, needs at least $n-2$ colours if $n-1$ is even (from the factorization of the complete graph on $\{2,\dots,n\}$) and $n-1$ colours if $n-1$ is odd. Moreover, each colour in A_1 cannot appear in A_2 because otherwise, there would exist an edge containing both vertices having these colours. Thus, $\gamma_{n;1,4} \geq 2n-2$ if n is odd and $\gamma_{n;1,4} \geq 2n-1$ if n is even.

Now, let us determine upper bounds for $\gamma_{n;1,4}$. We only must obtain a colouring of $\binom{[n]}{1} \cup \binom{[n]}{2} \cup \binom{[n]}{3}$ since in all cases the colour of each quadruple may be chosen among colours of triples.

Case 1. $n \equiv 1$ or $3 \pmod 6$, $n \neq 7$.
Consider a near-one-factorization $\{F_1,\dots,F_n\}$ of K_n (i.e., a one-factorization of K_{n+1} where the pairs $\{i, n+1\}$ are substituted by isolated vertices $\{i\}$) and a large set of disjoint Steiner triple systems of order n $\{S_1,\dots,S_{n-2}\}$ (which exists since $n \equiv 1$ or $3 \pmod 6$, $(n \neq 7)$. If we choose the colour i (resp. $n+j$) for the elements of F_i (resp. S_j), we have immediately $\gamma_{n;1,4} \leq 2n-2$.

Case 2. $n \equiv 0$ or $2 \pmod 6$, $n \neq 6$.
Using the parity of n and the fact that $n+1 \equiv 1$ or $3 \pmod 6$, $n+1 \neq 7$, we get on the one hand a factorization of $\binom{[n]}{2}$ into $n-1$ factors and on the other hand the existence of a large set $LTS(n+1)$. The deletion of the $n/2$ triples containing $n+1$ in each $STS(n+1)$, and one additional colour for all singletons yield $\gamma_{n;1,4} \leq (n-1)+(n-1)+1 = 2n-1$.

Case 3. $n \equiv 5 \pmod 6$, $n \neq 5$.
From Etzion's papers [7, 8], we know that $\binom{[n]}{3}$ may be partitioned into n PTs, and with a near-one-factorization of K_n we infer that $\mathfrak{H}_{n;1,4}$ is $2n$-colourable for $n > 5$.

Case 4. $n \equiv 4 \pmod 6$, $n \neq 4$.
The factorization of $\binom{[n]}{2}$ and the partition of $\binom{[n]}{3}$ into n PTs [7, 8] produce $2n-1$ colours. If we add one new colour for the singletons, we have $\gamma_{n;1,4} \leq 2n-1$ whenever $n > 4$. □

4 The chromatic number for interval orders

P is called an *interval order* if there is a mapping f from P into the set of closed intervals on the real line, such that for all $p, q \in P$:

$$p < q \text{ in } P \iff \sup f(p) < \inf f(q).$$

Let us denote the right (resp. left) endpoint of the interval $f(p)$ by $r(p)$ (resp. $l(p)$). Let $\min(P) = \{p_1, \dots, p_s\}$ and $\max(P) = \{q_1, \dots, q_t\}$ be the minimal and maximal elements of P, respectively. Assume without loss of generality that $r(p_1) \leq \cdots \leq r(p_s)$ and $l(q_1) \leq \cdots \leq l(q_t)$. In [3], we studied the order-interval hypergraph $\mathfrak{H}$ of the interval order and in particular, we presented polynomial-time algorithms which lead min-max relations $\alpha = \rho$ and $\nu = \tau$ where α, τ, ν and ρ are the independence number, the point covering number, the matching number, and the edge covering number of $\mathfrak{H}$, respectively.

Proposition 2 *We have* $\gamma(\mathfrak{H}(P)) = n - s - t + 2$.

Proof: Since the interval $[p_1, q_t]$ in P contains $n - s - t + 2$ different colours, we obtain $n - s - t + 2 \leq \gamma(\mathfrak{H})$. If we assign to the remaining elements $p_2, \dots, p_s, q_1, \dots, q_{t-1}$ the colours $c(p_i) = c(p_1)$ for all $i \geq 2$ and $c(q_j) = c(q_t)$ for all $j \leq t - 1$, we obtain the inequality $\gamma(\mathfrak{H}) \leq n - s - t + 2$. Hence, $\gamma(\mathfrak{H}) = n - s - t + 2$.

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References

[1] **Baranyai, Z. :** *On the factorization of the complete uniform hypergraph.* Colloq. Math. Soc. Janos. Bolyai **10**, 91–108 (1973)

[2] **Bouchemakh, I.**, and **Engel, K. :** *Interval stability and interval covering property in finite posets.* Order **9**, 163–175 (1992)

[3] **Bouchemakh, I.**, and **Engel, K. :** *The order-interval hypergraph of a finite poset and the König property.* Discrete Math. **170**, 51–61 (1997)

[4] **Brouwer, A.E. :** *On the packing of quadruples without common triples.* Ars Combin. **3**, 271–276 (1977)

[5] **Brouwer, A.E., Shearer, J.B., Sloane, N.J.A.**, and **Smith, W.D. :** *A new table of constant weight codes.* IEEE Trans. on Information Theory **IT-36**, 1334–1380 (1990)

[6] **Engel, K. :** *Interval packing and covering in the Boolean lattice.* Comb. Probab. Comput. **5**, 373–384 (1996)

[7] **Etzion, T. :** *Optimal partitions of triples.* J. Combin. Theory Ser. A **59**, 161–176 (1992)

[8] **Etzion, T. :** *Partitions of triples into optimal packings.* J. Combin. Theory Ser. A **59**, 269–284 (1992)

[9] **Etzion, T. :** *Partitions of quadruples.* Ars. Combin. **36**, 296–308 (1993)

[10] **Garey, M.R.**, and **Johnson, D.S. :** *Computers and Intractability: A guide to the Theory of N.P.-Completeness.* W.H. Freeman and Company, San Francisco (1979)

[11] **Graham, R.L**, and **Sloane, N.J.A. :** *Lower bounds for constant weight codes.* IEEE Trans. on Information Theory **IT-26**, 37–43 (1980)

[12] **Hanani, H. :** *On quadruple systems.* Canad. J. Math. **15**, 702–722 (1960)

[13] **Lu, J.X. :** *On large sets of disjoint Steiner triple systems I-III.* J. Combin. Theory Ser. A **34**, 140–183 (1983)

[14] **Lu, J.X. :** *On large sets of disjoint Steiner triple systems IV-VI.* J. Combin. Theory Ser. A **37**, 136–192 (1984)

[15] **Spencer, J. :** *Maximal consistent families of triples.* J. Combin. Theory **5**, 1–8 (1968)

[16] **Pippenger, N.**, and **Spencer, J. :** *Asymptotic behaviour of the chromatic index for hypergraphs.* J. Combin. Theory Ser. A **51**, 24–42 (1989)

[17] **Schönheim, J. :** *On maximal systems of k-tuples.* Studia Sci. Math. Hungar. **1**, 363–368 (1966)

[18] **Teirlinck, L. :** *A completion of Lu's determination of the spectrum for large sets of disjoint Steiner systems.* J. Combin. Theory Ser. A **57**, 302–305 (1991)

[19] **van Paul, C.L.M.**, and **Etzion, T. :** *New lower bounds for constant weight codes.* IEEE Trans. on Information Theory **IT-35**, 1324–1329 (1989)

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