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Approximation of the Bochner integral by means of Riemann sums

Dedicated to professor Harry Poppe on his 65th birthday

The natural way to approximate the integral (of some kind) of a function f on a measure space T into a Banach space E via the values of f is - in the author's opinion - that by Riemann sums. The Bochner integral does not offer this possible way of approximation in an immediate way. The goal of this paper is to investigate in which sense the desired approximation is nevertheless possible for the Bochner integral of f.

More precisely, we consider a Banach space E endowed with its norm topology and a measure space (T, μ) with a finite measure μ. There are essentially two ways to introduce the Bochner integral: one extends the natural integral of step functions having *finite* range (i.e. of simple functions), the other extends the natural integral of step functions having *countable* range. We denote the first type of Bochner integral by ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu$, the second type by ${}^{\mathrm{B}}\!\!\int \cdot\, d\mu$. Each integral defined on a subset of E^T and ranging in E is interpreted as T-ary partial operation in E, thus as a subset of $E^T \times E$. (Every function is considered as a relation.) Obviously, one has ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu \subseteq {}^{\mathrm{B}}\!\!\int \cdot\, d\mu$ (see Theorem 2.1) and, if μ is complete, one obtains ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu = {}^{\mathrm{B}}\!\!\int \cdot\, d\mu$ (see Theorem 3.2). (Although these facts must be well-known, the author could not find them explicitly in the literature.)

In the papers [6] and [7], Riemann sums were used to introduce, for each "μ-partition system" $\mathfrak{Y}$ (consisting of certain subdivisions of T), the $(\mu, \mathfrak{Y})$-integral ${}^{\mathfrak{Y}}\!\!\int \cdot\, d\mu$, called now "$(\mu, \mathfrak{Y})$-subdivision integral", which maps $\mathfrak{E}^T$ into a set $\mathfrak{E}$, having just one element more than E. (The role of this additional element is played by the empty set $\emptyset$.) ${}^{\mathfrak{Y}}\!\!\int \cdot\, d\mu$ is a T-ary operation in $\mathfrak{E}$. Certain T-ary operations in $\mathfrak{E}$, for instance ${}^{\mathfrak{Y}}\!\!\int \cdot\, d\mu$, are algebraic completions of T-ary partial operations in E, and every T-ary partial operation in E can be extended to a T-ary operation in $\mathfrak{E}$. There is a one-to-one correspondance $(\cdot)^\wedge$ between all T-ary partial operations in E and certain T-ary operations in $\mathfrak{E}$ its inverse being denoted by $(\cdot)^\vee$. For instance, $({}^{\mathfrak{Y}}\!\!\int \cdot\, d\mu)^\vee$ is a T-ary partial operation in E such that $(({}^{\mathfrak{Y}}\!\!\int \cdot\, d\mu)^\vee)^\wedge = {}^{\mathfrak{Y}}\!\!\int \cdot\, d\mu$.

(Actually, we shall consider even an extension of the mapping $(\cdot)^\wedge$ (denoted the same way) which assigns, especially, to the generalized partial operation lim defined on *certain* filtered families in E a generalized (algebraic) operation $\lim^\wedge$ in $\mathfrak{E}$ defined on *all* filtered families in $\mathfrak{E}$. For details, see Section 1 and [11].)

The most important μ-partition systems are: (a) that consisting of all *finite* "μ-partitions" (intuitively: subdivisions) of T, which we denote (later, beginning with Lemma 1.4) by $\mathfrak{X}$, and (b) that consisting of all *countable* μ-partitions of T, denoted by Ω. (Because of the implicit occurence of ${}^{\Omega}\!\!\int \cdot\, d\mu$ in [2], we call this integral *Birkhoff integral.*) In every case, one has

$$ {}^{\mathrm{B}}\!\!\int \cdot\, d\mu \subseteq \left({}^{\Omega}\!\!\int \cdot\, d\mu \right)^{\vee} \tag{1} $$

(see Theorem 2.1); for bounded $f : T \longrightarrow E$, one has

$$ {}^{\mathrm{D}}\!\!\int^{\wedge} f^{\sim} d\mu \subseteq {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu \text{ if } \mathfrak{X} \subseteq \mathfrak{Y} \tag{2} $$

(see Theorem 2.2); for bounded measurable f and complete μ, one obtains

$$ \emptyset \neq {}^{\mathrm{D}}\!\!\int^{\wedge} f^{\sim} d\mu = {}^{\mathrm{B}}\!\!\int^{\wedge} f^{\sim} d\mu = {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu \text{ if } \mathfrak{X} \subseteq \mathfrak{Y} \tag{3} $$

(see Theorem 3.4), where $^\sim$ denotes the natural injective mapping on E^T into $\mathfrak{E}^T$. The interpretation for instance of the latter assertion (3) in the traditional language is the following one: f is D-Bochner integrable, B-Bochner integrable, and $\mathfrak{Y}$-integrable, and the values of the corresponding integrals of f coincide (if $\mathfrak{X} \subseteq \mathfrak{Y}$). Each of the three assertions contains the statement that the Bochner integral of Bochner integrable functions can be approximated by means of Riemann sums either such having countably many summands (assertion (1)) or such having finitely many summands (assertions (2) and (3)). In the assertion (1), generally, the inclusion is a proper inclusion; the validity of the sign "=" implies that E is finite-dimensional provided there exists an infinite μ-partition of T into sets having positive measure (see Theorem 3.3/b).

As a byproduct of this comparison of several approaches to integration in Banach spaces we obtain, for complete μ, a characterization of the "μ-summation" in the case $E = \mathbb{R}$, i.e. of the restriction of classical μ-integration to μ-summable functions $f : T \longrightarrow \mathbb{R}$ (see Theorem 3.1). This allows us to interpret, for complete μ, the $(\mathfrak{Y}, \mu)$-subdivision integration as a strict extension of μ-summation, no matter which approach to the Lebesque integration one might choose (while originally, in [6], we (the authors of [6]) have restricted ourselves to the approach to integration presented in [13]).

Finally, we discuss the relationship of the Bochner integral and the $(\mu, \mathfrak{Y})$-subdivision integral with the Riemann integral redefined in the expected way in the Banach space E with a compact metric space as domain T being endowed with a measure μ compatible in a precise sense with the metric of T. As an example for the possible transfer of classical statements for the Riemann integral into the language of the subdivision integral in Banach spaces we discuss the rearrangement of the order of integration and uniform convergence of generalized sequences in E^T (compare Proposition 4.7 with Theorem 2.3).

In addition to the references given in [6], the author should mention HENSTOCK's books [15] and [16] (both containing numerous references to the Riemann-type integration), his paper [14], furthermore, for $T = [a, b] \subseteq \mathbb{R}$ and a suitable μ-partition system $\mathfrak{Y} \subseteq \mathfrak{X}$, GORDON's article [8] on Riemann-type integration in Banach spaces (see Example 4.2 below), where especially GRAVES' article [9] occurs under the references.

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This paper is a continuation of the papers [6] and [7]. In order to make this paper selfcontained as much as possible, we collect in Section 1 definitions and some results (used here) introduced or discussed there already.

1 Terminology and some lemmas

a) *Set theory.* Sets are special classes. Functions are special (binary) relations. If A and B are classes, a function (= mapping) $h \subseteq A \times B$ is called a function *from* A into B , a function *on* A into B, if $\operatorname{Dmn} h = A$ (symbol $h : A \longrightarrow B$). If h is a function, then $\underset{i \in \operatorname{Dmn} h}{\mathrm{P}} h(i)$ or $\mathrm{P}h$ denotes the cartesian product of h. If M is a set, $\mathcal{P}M$ denotes $\mathrm{P}h$ with $h = \{(m, m) \mid m \in M\}$, while $\mathfrak{P}M$ denotes the powerset of M. For the whole paper, we fix a set E (denoting a Banach space later) and define $\mathfrak{E}$ to be the set $\{X \mid X \subseteq E$ *and* $\operatorname{card} X \leq 1\}$. For each set I and each $h : I \longrightarrow E$, we define by $h^{\sim}$ the mapping defined by $h^{\sim}(i) = \{h(i)\}$ for all $i \in I$. Then $h^{\sim} : I \longrightarrow \mathfrak{E}$. For all $X \in \mathfrak{E}\backslash\{\emptyset\}$, $\mathrm{e}X$ denotes the element of X. Then, e^{-1} maps E one-to-one onto $\mathfrak{E}\backslash\{\emptyset\}$, and $h^{\sim} = \mathrm{e}^{-1} \circ h$ holds for all $h : I \longrightarrow E$.

b) *Ordered sets, directed sets.* If $(E, \leq)$ is an ordered set, then the relation $\leq^\star$ is defined by letting, for all $X, Y \in \mathfrak{E}$, $X \leq^\star Y$, if $x \leq y$ holds for all $(x, y) \in X \times Y$. In the case $Y = \{y\}$, we write also $X \leq^\star y$ instead of $X \leq^\star Y$. Finally, we write $X \leq Y$ and $X \leq y$ instead of $X \leq^\star Y$ and $X \leq^\star y$, respectively. – Similar agreements hold for $<$ instead of $\leq$. – If $(M, \preceq)$ is a directed set, then $\mathcal{F}(M, \preceq)$ or $\mathcal{F}M$ denotes the filter on M generated by the sections $\{y \in M \mid x \preceq y\}$ with $x \in M$ (*filter of perfinality of* M). The filter $\mathcal{F}\mathbb{N}$ on $\mathbb{N}(= \{1, 2, \cdots\})$ is called the *Fréchet filter.*

c) *Universal algebra.* Let I be a non-empty set. Each mapping Θ from E^I into E is called an *I-ary partial operation in E* (resp. an *I-ary operation in E* if $\operatorname{Dmn} \Theta = E^I$). Let Θ be an

I-ary partial operation in E (resp. an I-ary operation in E). Then (E, Θ) is called an *I-ary partial algebra* (resp. *I-ary algebra*). We denote by $\Theta^\wedge$ the I-ary operation in $\mathfrak{E}$ defined by

$$\Theta^\wedge f = \{x \in E \mid \textit{ for some } \phi \in \mathrm{P}\ f,\ x = \Theta\phi\} \textit{ for all } f \in \mathfrak{E}^I .$$

The I-ary algebra $(\mathfrak{E}, \Theta^\wedge)$ is a natural extension of (E, Θ), called *1-point-completion of* (E, Θ).The mapping e^{-1} is an isomorphism on (E, Θ) into $(\mathfrak{E}, \Theta^\wedge)$, namely onto the *trace* $(F,\ \Theta^\wedge \cap (F^I \times F))$ of $(\mathfrak{E}, \Theta^\wedge)$ in the set $F = \mathfrak{E}\backslash\{\emptyset\}$. For each $g \in E^I$,

$$g \in \operatorname{Dmn}\Theta \text{ if and only if } \Theta^\wedge g^\sim \neq \emptyset ,$$

furthermore

$$\text{if } g \in \operatorname{Dmn}\Theta\,, \text{ then } \mathrm{e}^{-1}(\Theta g) = \Theta^\wedge g^\sim .$$

Denote, for the moment, by $\mathcal{A}(E, I)$ the set of all I-ary partial operations in E and by $\mathcal{B}(\mathfrak{E}, I)$ the set of all I-ary operations Λ in $\mathfrak{E}$ such that for all $\phi \in \mathfrak{E}^I$

$$\text{if } \Lambda\phi \neq \emptyset \text{ then } \phi(i) \neq \emptyset \text{ for all } i \in I .$$

Then, the mapping $\Theta \longmapsto \Theta^\wedge(\Theta \in \mathcal{A}(E, I))$ is one-to-one onto $\mathcal{B}(\mathfrak{E}, I)$; its inverse mapping $(\cdot)^\vee$ is defined by letting for each $\Lambda \in \mathcal{B}(\mathfrak{E}, I)$

$$(\psi, x) \in \Lambda^\vee \text{ if and only if } \{x\} = \Lambda\psi$$

for all $\psi \in E^I$ and all $x \in E$. For each $f \in \mathfrak{E}^I$, we write Θf or $\underset{i \in I}{\Theta} f(i)$ instead of $\Theta^\wedge f$ without any danger of confusion. *For the remainder of this paper, let* $(E, \|\cdot\|)$ *be a Banach space.* If I is countable, the unconditional summation Σ_I in E (see [11],p.119 and p.128) is an I-ary partial operation in E.

d) *Convergence.* Let τ be the norm topology of $(E, \|\cdot\|)$. Let ΦM denote, for each set M, the class of all filtered families $(g, K, \mathfrak{b})$ in M, i.e. with $g(K) \subseteq M$. Denote by $\lim_\tau$ or just by lim the limit operation induced by τ; lim is a mapping from ΦE into E. Let $\mathcal{M}$ be the class of all filtered sets $(K, \mathfrak{b})$. Then lim is an $\mathcal{M}$-ary partial operation in E (see [11], p.120), and for the $\mathcal{M}$-ary operation $\lim^\wedge$ in $\mathfrak{E}$ the following holds for all $(g, K, \mathfrak{b}) \in \Phi\mathfrak{E}$:

$$\lim{}^\wedge(g, K, \mathfrak{b}) = \{x \in E \mid \exists B \in \mathfrak{b}\ (\exists\phi \in \underset{k \in B}{\mathrm{P}}\ g(k) : x = \lim(\phi, B, \mathfrak{b}_B))\} ,$$

where $\mathfrak{b}_B$ denotes the *trace* $\{B \cap C \mid C \in \mathfrak{b}\}$ *of the filter* $\mathfrak{b}$ *in* the set B.– The mapping e^{-1} is an isomorphism on the $\mathcal{M}$-ary partial algebra $(E, \lim)$ into the $\mathcal{M}$-ary algebra $(\mathfrak{E}, \lim^\wedge)$, namely onto the $\mathcal{M}$-ary partial algebra

$$(F,\, \lim{}^\wedge \cap (\Phi(F) \times F)) \text{ with } F = \mathfrak{E}\backslash\{\emptyset\}\,,$$

being called the *trace of* $(\mathfrak{E}, \lim^\wedge)$ *in* the set F. This means that, for each $x \in E$ and all $(g, K, \mathfrak{b}) \in \Phi(E)$, (1) the statement $(g, K, \mathfrak{b}) \in \mathrm{Dmn}(\lim)$ holds (i.e., $(g, K, \mathfrak{b})$ converges in E w. r. to the norm topology τ) if and only if $\lim^\wedge(g^\sim, K, \mathfrak{b}) \neq \emptyset$, and (2) $x = \lim(g, K, \mathfrak{b})$ holds if and only if $\{x\} = \lim^\wedge(g^\sim, K, \mathfrak{b})$.– In the sequel, we write, for each $(g, K, \mathfrak{b}) \in \Phi\mathfrak{E}$, also $\lim(g, K, \mathfrak{b})$ or ${}^{\mathfrak{b}}\lim\limits_{k \in K} g(k)$ instead of $\lim^\wedge(g, K, \mathfrak{b})$. In the special case that $K = \mathbb{N}$ and $\mathfrak{b} = \mathcal{F}\mathbb{N}$ (see b)), we replace ${}^{\mathfrak{b}}\lim\limits_{k \in K} g(k)$ by $\lim\limits_{k \in \mathbb{N}} g(k)$. (Of course, these considerations under d) hold in every Hausdorff space E.)

e) *Banach spaces.* The scalar domain of $(E, \|\cdot\|)$ is denoted by $\mathbb{K}$. The scalar multiplication in E, $(x, \alpha) \longmapsto x\alpha((x, \alpha) \in E \times \mathbb{K})$, is written in a right-hand notation. If $X, Y \in \mathfrak{E}$ and $\alpha \in \mathbb{K}$, then we define $X\alpha = \{x\alpha \mid x \in X\}$, $\|X\| = \{\|x\| \mid x \in X\}$, and (in accordance with the agreements in a) above for $\Theta = +$ or $\Theta = -$) $X \pm Y = \{x \pm y \mid x \in X \text{ and } y \in Y\}$. For each mapping h on a set T into E, we define $\|h\|$ to be the mapping $t \longmapsto \|h(t)\|$ $(t \in T)$; in the special case $E = \mathbb{R}$, $h^\sim$ is defined as under a) with T instead of I.

f) *Integration.* In this paper, we assume that a non-empty measure space (T, μ) with a *finite measure* μ is given. Each countable partition $\mathfrak{x}$ of T such that $\mathfrak{x} \subseteq \mathrm{Dmn}\,\mu$ is called a *μ-partition.* We recall that $\Omega(\mu)$ or Ω denotes the class of all μ-partitions of T. Let $\mathfrak{X} \subseteq \Omega$. Given $\mathfrak{x}, y \in \mathfrak{X}$, we agree that $\mathfrak{x} \preceq y$ holds, if, for each $Y \in \mathfrak{y}$, there exists an $X \in \mathfrak{x}$ such that $Y \subseteq X$. $\mathfrak{X}$ is called a *μ-partition system*, if $(\mathfrak{X}, \preceq)$ is a directed set. Define the binary operation $\vee$ in Ω by letting, for each $\mathfrak{x}, y \in \Omega$, $\mathfrak{x} \vee y = \{X \cap Y \mid X \in \mathfrak{x} \text{ and } Y \in \mathfrak{y}\}\backslash\{\emptyset\}$. We say that $\mathfrak{X}$ is $\vee$-*closed*, if $\vee(\mathfrak{X} \times X) \subseteq X$. If $\mathfrak{X}$ is $\vee$-closed, then $\mathfrak{X}$ is a μ-partition system; e.g., Ω is a μ-partition system, and the set $\mathfrak{X}_{fi}$ of all finite μ-partitions is a μ-partition system. Let $\mathfrak{X}$ be a μ-partition system of T. Then, we denote by $\mathfrak{X}^\#$ the set $\{(\mathfrak{x}, \phi) \mid x \in \mathfrak{X} \text{ and } \phi \in \mathcal{P}\mathfrak{x}\}$. We define $\preceq^\#$ to be the relation such that, for all $(\mathfrak{x}, \phi), (\mathfrak{y}, \psi) \in \mathfrak{X}^\#$, $(\mathfrak{x}, \phi) \preceq^\# (\mathfrak{y}, \psi)$, if $\mathfrak{x} \preceq \mathfrak{y}$. $(\mathfrak{X}^\#, \preceq^\#)$ is a directed set. By the agreement in b), $\mathcal{F}\mathfrak{X}$ and $\mathcal{F}\mathfrak{X}^\#$ denote the filters of perfinality on $\mathfrak{X}$ and $\mathfrak{X}^\#$. Given $f : T \longrightarrow \mathfrak{E}$, the *Riemann sum* $\mathrm{R}(f, \mathfrak{x}, \phi)$ of f belonging to $(\mathfrak{x}, \phi) \in \mathfrak{X}^\#$ is defined to be the set $\sum\limits_{X \in \mathfrak{x}} f(\phi(X))\mu X$. The *$(\mu, \mathfrak{X})$-integral* ${}^{\mathfrak{X}}\!\!\int f\, d\mu$ of f is defined to be the set

$$ {}^{\mathcal{F}\mathfrak{X}^\#}\lim_{(\mathfrak{x}, \phi) \in \mathfrak{X}^\#} \mathrm{R}(f, \mathfrak{x}, \phi)\,. $$

If $h : T \longrightarrow \mathbb{R}$ (where $\mathbb{R}$ denotes the real line), then, tacitly, ${}^{\mathfrak{X}}\!\!\int h^\sim d\mu$ denotes the $(\mu, \mathfrak{X})$-integral of $h^\sim$ for the Banach space $(\mathbb{R}, |\cdot|)$ instead of $(E, \|\cdot\|)$. – We call the $(\mu, \mathfrak{X})$-integral also $(\mu, \mathfrak{X})$-*subdivision integral.*

We add some results being proved in [6] and [7], in which f is a mapping on T into $\mathfrak{E}$ and $\mathfrak{X}, \mathfrak{Y}$ are μ-partition systems of T :

Lemma 1.1 ([6],Theorem 4). $\mathfrak{X} \subseteq \mathfrak{Y}$ *implies* ${}^{\mathfrak{X}}\!\!\int f\, d\mu \subseteq {}^{\mathfrak{Y}}\!\!\int f\, d\mu$.

We note a simple consequence of the preceding lemma applied in the Banach space $(\mathbb{R}, |\cdot|)$:

If $g \in E^T$ *and* ${}^{\mathfrak{X}}\!\!\int \|g\|^{\sim} d\mu = \{0\}$ *and the function* $\|g\|$ *is* μ*-measurable, then* $g(t) = 0$ *holds for* μ*-a.a.* $t \in T$.

[**Proof:** By Lemma 1, one has $0 \in {}^{\Omega}\!\!\int \|g\|^{\sim} d\mu$, thus (by Theorem 12 in [7]) $0 = {}^{\mathrm{H}}\!\!\int \|g\|\, d\mu$. Now apply Satz 8 on p.100 in [13]. $\square$]

f is called an $\mathfrak{X}-$*step function*, if there is an $\mathfrak{x} \in \mathfrak{X}$ and a mapping $\chi : \mathfrak{x} \longrightarrow \mathfrak{E}$ such that $f(x) = \chi(X)$ for all (X, x) with $X \in \mathfrak{x}$ and $x \in X$. If f is in such a relationship with $\mathfrak{x}$ and χ, we say that f is the $\mathfrak{X}$-step function determined by $(\mathfrak{x}, \chi)$. The class of all $\mathfrak{X}$-step functions on T into $\mathfrak{E}$ is denoted by $\mathcal{ST}(\mathfrak{X}, \mathfrak{E})$. Similarly, one defines the class $\mathcal{ST}(\mathfrak{X}, E)$ of all $\mathfrak{X}$-step funtions on T into E. Partially, the next lemma follows from Lemma 1.1:

Lemma 1.2 ([6], Theorem 5). *Let* $\mathfrak{x} \in \mathfrak{X}$, $\chi : \mathfrak{x} \longrightarrow \mathfrak{E}$, *and* f *be the* $\mathfrak{X}$*-step function determined by* $(\mathfrak{x}, \chi)$. *Then, one has*

$$ {}^{\mathfrak{X}}\!\!\int f\, d\mu = \sum_{X \in \mathfrak{x}} \chi(X)\mu X. $$

Given a μ-partition system $\mathfrak{X}$ of T and $\emptyset \neq R \in \operatorname{Dmn}\mu$, let $\mathfrak{X}_R = \{\mathfrak{x}_R \mid \mathfrak{x} \in \mathfrak{X}\}$, where $\mathfrak{x}_R = \{R \cap X \mid X \in \mathfrak{x}\} \backslash \{\emptyset\}$. Then, $\mathfrak{X}_R$ is a μ_R-partition system of R, where $\mu_R = \mu \mid (\operatorname{Dmn}\mu \cap \mathfrak{P}R)$. Then, one has the

Lemma 1.3 ([7], Theorem 9). *Assume* $\mathfrak{X}$ *to be* $\vee$*-closed and* $\mathfrak{y} \in \mathfrak{X}$. *If* $\mathfrak{y}$ *is finite, then, one has* ${}^{\mathfrak{X}}\!\!\int f\, d\mu = \sum_{Y \in \mathfrak{y}} {}^{\mathfrak{X}_Y}\!\!\int (f \mid Y) d\mu_Y$.

Corollary 1.1 *Let* $g : T \longrightarrow E$ *and* $h : T \longrightarrow E$. *Assume that* $\mathfrak{X}$ *is* $\vee$*-closed and that there is a set* $M \in \operatorname{Dmn}\mu$ *with* $\mu M = 0$ *such that* $\{M, T\backslash M\} \in \mathfrak{X}$ *and* $g(t) = h(t)$ *holds for all* $t \in T\backslash M$. *Then* ${}^{\mathfrak{X}}\!\!\int g^{\sim} d\mu = {}^{\mathfrak{X}}\!\!\int h^{\sim} d\mu$.

In particular, if $\mathfrak{X} = \{\mathfrak{x} \in \Omega \mid \mathfrak{x}$ *is finite* $\}$ or $\mathfrak{X} = \Omega$ and $g(t) = h(t)$ holds for μ-almost all $t \in T$, then ${}^{\mathfrak{X}}\!\!\int g^{\sim} d\mu = {}^{\mathfrak{X}}\!\!\int h^{\sim} d\mu$ (cf. Proposition 30 in [7], p.18).

For the remainder of this paper, *we reserve the symbol* $\mathfrak{X}$ *for the system of all finite* μ*-partitions of* T and the symbol $\mathfrak{Y}$ *for some (given)* μ*-partition system of* T.

We shall make use of the following link to the integration of bounded and measurable real-valued functions:

Lemma 1.4 *Let* $(E, \|\cdot\|)$ *be the Banach space* $(\mathbb{R}, |\cdot|)$, *and assume* $g : T \longrightarrow E$ *to be* μ*-measurable. Then,* (a)$\Longrightarrow$(c) *and* (b)$\Longrightarrow$(c) *hold true:*

(a) *g is bounded;*

(b) *μ is complete and g is bounded μ-a.e.;*

(c) ${}^{\mathfrak{Y}}\!\!\int g^{\sim} d\mu \neq \emptyset$, *namely* $\mathrm{e}({}^{\mathfrak{Y}}\!\!\int g^{\sim} d\mu) = {}^{\mathrm{H}}\!\!\int g\, d\mu$, *if* $\mathfrak{X} \subseteq \mathfrak{Y}$. (For the terminology, see [7],p.15.)

Proof: (a)$\Longrightarrow$(c). For abbreviation, put $s(\mathfrak{x}) = \sum\limits_{X \in \mathfrak{x}} \inf\limits_{t \in X} g(t)\mu X$ and $S(\mathfrak{x}) = \sum\limits_{X \in \mathfrak{x}} \sup\limits_{t \in X} g(t)\mu X$ for all $\mathfrak{x} \in \mathfrak{X}$, furthermore $j(\mathfrak{x}, \phi) = s(\mathfrak{x})$ and $J(\mathfrak{x}, \phi) = S(\mathfrak{x})$ for all $(\mathfrak{x}, \phi) \in \mathfrak{X}^{\#}$. Then,

$$s(\mathfrak{x}) \leq \mathrm{e}(\mathrm{R}(g^{\sim}, \mathfrak{x}, \phi) \leq S(\mathfrak{x}) \text{ holds for all } (\mathfrak{x}, \phi) \in \mathfrak{X}^{\#}\,. \tag{1}$$

Since g is bounded and μ-measurable and $\mu T < +\infty$, there is (cf. the proof of Satz 1a, p.95 in [13]) an $r \in \mathbb{R}$ such that

$$r = \sup_{\mathfrak{x} \in \mathfrak{X}} s(\mathfrak{x}) = {}^{\mathcal{F}\mathfrak{X}^{\#}}\lim_{(x, \phi) \in \mathfrak{X}^{\#}} j(\mathfrak{x}, \phi) \text{ and}$$

$$r = \inf_{\mathfrak{x} \in \mathfrak{X}} S(\mathfrak{x}) = {}^{\mathcal{F}\mathfrak{X}^{\#}}\lim_{(\mathfrak{x}, \phi) \in \mathfrak{X}^{\#})} J(\mathfrak{x}, \phi)$$

and therefore, by (1), $r \in {}^{\mathfrak{X}}\!\!\int g^{\sim} d\mu$. Furthermore, use Lemma 1.1 here and Theorem 12 in [7].

(b)$\Longrightarrow$(c). There is a set $M \in \mathrm{Dmn}\,\mu$ such that $\mu M = \emptyset$ and g bounded on $T \backslash M$. Consider the non-trivial case $M \neq \emptyset$. Define the function $f : T \longrightarrow E$ by letting $f(t) = g(t)$ on $T \backslash M$ and $f(t) = 0$ on M. Then f is bounded and (μ being complete) μ-measurable. Now, one applies (a)$\Longrightarrow$(c) to f instead of g occuring there. In view of Corollary 1.1, one has ${}^{\mathfrak{X}}\!\!\int f^{\sim} d\mu = {}^{\mathfrak{X}}\!\!\int g^{\sim} d\mu$. Thus (c) holds true by Lemma 1.1. □

In this paper, a function $g : T \longrightarrow \overline{\mathbb{R}}$ will be called *μ-summable* if $g \in \mathrm{Dmn}({}^{\mathrm{H}}\!\!\int \cdot\, d\mu)$ and $| {}^{\mathrm{H}}\!\!\int g\, d\mu | < +\infty$.

2 More on the relationship between Bochner integral and subdivision integral

For each set M and each mapping $g : M \longrightarrow E^T$, we write in this paper, for each $m \in M$, also g_m instead of $g(m)$. We use this agreement especially, if g is a sequence in E^T, i.e. if $M = \mathbb{N}$.

In the next considerations, E^T is considered as a linear space w.r. to the pointwise summation $+ : E^T \times E^T \longrightarrow E^T$ and the pointwise scalar multiplication $\cdot : E^T \times \mathbb{K} \longrightarrow E^T$.

Definition 2.1 (a) *A filtered family $(g, K, \mathfrak{b})$ in E^T is called a $(\mu, \mathfrak{Y})$-*Cauchy family*, if $0 \in {}^{\mathfrak{b}\times^\star\mathfrak{b}}\lim\limits_{(k,l)\,\in\, K\times K} {}^{\mathfrak{Y}}\!\!\int \|g_k - g_l\|^\sim d\mu$, where $\mathfrak{b}\times^\star\mathfrak{b}$ denotes the cardinal product of the filter $\mathfrak{b}$ with itself. (Of course, we will speak of a $(\mu, \mathfrak{Y})$-*Cauchy sequence* in the case $K = \mathbb{N}$ and $\mathfrak{b} = \mathcal{F}\mathbb{N}$.)*

(b) *A linear subspace L of E^T is called to be $(\mu, \mathfrak{Y})$-*absolutely integrable*, if $f \in L$ implies ${}^{\mathfrak{Y}}\!\!\int f^\sim d\mu \neq \emptyset$ and ${}^{\mathfrak{Y}}\!\!\int \|f\|^\sim d\mu \neq \emptyset$.*

Obviously, a filtered family $(g, K, \mathfrak{b})$ in E^T is a $(\mu, \mathfrak{Y})$-Cauchy family if and only if, for all $\epsilon > 0$, there is a $B \in \mathfrak{b}$ such that $\emptyset \neq {}^{\mathfrak{Y}}\!\!\int \|g_k - g_l\|^\sim d\mu < \epsilon$ holds for all $k,\ l \in B$.

Definition 2.2 *Let $S \subseteq T$, $f : T \longrightarrow E$, and $(h, K, \mathfrak{b})$ be a filtered family in E^T.*

(a) $(h, K, \mathfrak{b})$ converges uniformly on S *to f, if for each $\epsilon > 0$ there is a $B \in \mathfrak{b}$ such that $\|f(t) - h_k(t)\| < \epsilon$ holds for all $k \in B$ and all $t \in S$. If $S = T$, we omit mostly the reference "on S" to S.*

(b) $(h, K, \mathfrak{b})$ converges uniformly μ-a.e. *(on T) to f, if there is a set $M \in \operatorname{Dmn}\mu$ with $\mu M = 0$ such that $(h, K, \mathfrak{b})$ converges uniformly on $T \backslash M$ to f.*

(c) $(h, K, \mathfrak{b})$ $(\mu, \mathfrak{Y})$-converges in mean to f, *if* $0 \in {}^{\mathfrak{b}}\lim\limits_{k\,\in\,K} {}^{\mathfrak{Y}}\!\!\int \|f - h_k\|^\sim d\mu$.

Note a possible terminological conflict between the relation "converges uniformly μ-a.e." occuring here and the relation "converges μ-almost uniformly" appearing in connection with Egoroff's theorem (see, e.g., [12], p.88) but not occuring in this paper.

Definition 2.1/b is illustrated and the relationship of the notions introduced in the Definitions 2.1/a and 2.2/c to the corresponding classical notions is discussed in the following remark, in which we refer to results and a definition formulated later:

Remark 2.1 A) The linear subspaces L_1, L_2, $\cdots$, L_5 of E^T defined in (a)–(d) are $(\mu, \mathfrak{Y})$-absolutely integrable (for a linear subspace of E^T which is not $(\mu, \mathfrak{Y})$-absolutely integrable see the remarks after Theorem 3.3):

(a) Let $\mathfrak{Y} = \mathfrak{X}$ and $L_1 = \mathcal{ST}(\mathfrak{X}, E)$. (See Lemma 1.2.)

(b) Let $\mathfrak{Y} = \mathfrak{X}$ and $L_2 = \{f \in \operatorname{Dmn}\,({}^{\mathrm{D}}\!\!\int \cdot\, d\mu) \mid f \text{ is bounded } \mu\text{-a.e.}\}$. (See Definition 2.3, Theorem 2.2/a, and Proposition 2.10.)

(c) Let $\mathfrak{Y} = \Omega$ and let L_3 be the set of all $f \in \mathcal{ST}(\mathfrak{Y}, E)$ such that ${}^{\Omega}\!\!\int \|f\|^{\sim} d\mu \neq \emptyset$. (In view of Lemma 1.2, the last condition for f is just a short form for a condition about the absolute convergence of a certain sum. – See Theorem 14 in [11], p.128.)

(d) Let $\mathfrak{Y} = \Omega$, let $L_4 = \mathrm{Dmn}({}^{\mathrm{D}}\!\!\int \cdot\, d\mu)$ and $L_5 = \mathrm{Dmn}({}^{\mathrm{B}}\!\!\int \cdot\, d\mu)$. (See Definition 2.3, Proposition 2.8, Definition 8 in [7], p.20, Theorem 2.1, and Remark 2.3.)

B) Let $\mathfrak{Z}$ be a μ-partition system of T with $\mathfrak{Y} \subseteq \mathfrak{Z}$. Then, one has by Lemma 1.1: If a linear subspace L of E^T is $(\mu, \mathfrak{Y})$-absolutely integrable, it is also $(\mu, \mathfrak{Z})$-absolutely integrable.

C) If L is $(\mu, \mathfrak{Y})$-absolutely integrable, then the mapping $f \longmapsto \mathrm{e}({}^{\mathfrak{Y}}\!\!\int \|f\|^{\sim} d\mu)$ $(f \in L)$ is a semi-norm (denoted by $\mathrm{sn}(\mu, \mathfrak{Y})$) on L. (For the terminology, see Section 1/a.)

D) Let (L, sn) be a semi-normed linear space. Denote by σ the topology on L induced in the usual way by sn. Let $(h, K, \mathfrak{b}) \in \Phi(L)$ and $x \in L$. $(h, K, \mathfrak{b})$ is called a *Cauchy family in* the topological vector space (L, σ), if $h(\mathfrak{b})$ is a Cauchy filter (see, e.g., Definition 1 in [18], p.128) in (L, σ). Then, $(h, K, \mathfrak{b})$ is a Cauchy family in (L, σ) if and only if, for each $\epsilon > 0$, there is a $B \in \mathfrak{b}$ such that $\mathrm{sn}(h(k) - h(l)) < \epsilon$ for all $k, l \in B$. Denote by $\mathrm{Lim}_{\sigma} : \Phi(L) \longrightarrow \mathfrak{P}L$ the limit operator belonging to σ (see [10], p.183). Then $x \in \mathrm{Lim}_{\sigma}(h, K, \mathfrak{b})$ holds if and only if, for all $\epsilon > 0$, there is a $B \in \mathfrak{b}$ such that $\mathrm{sn}(h(k) - x) < \epsilon$ for all $k \in B$. If $\emptyset \neq \mathrm{Lim}_{\sigma}(h, K, \mathfrak{b})$, then $(h, K, \mathfrak{b})$ is a Cauchy family in (L, σ). – Now, let especially L be a linear subspace of E^T being $(\mu, \mathfrak{Y})$-absolutely integrable and let sn be the semi-norm $\mathrm{sn}(\mu, \mathfrak{Y})$ on L defined under C. Then $(h, K, \mathfrak{b})$ is a $(\mu, \mathfrak{Y})$-Cauchy family if and only if $(h, K, \mathfrak{b})$ is a Cauchy family in (L, σ). Furthermore, $(h, K, \mathfrak{b})$ $(\mu, \mathfrak{Y})$-converges in mean to x if and only if $x \in \mathrm{Lim}_{\sigma}(h, K, \mathfrak{b})$.

Proposition 2.1 *Let L be a linear subspace of E^T being $(\mu, \mathfrak{Y})$-absolutely integrable. Let $(g, K, \mathfrak{b})$ be a filtered family in L. Then, one has* (a), (b), (c), *and* (d)*:*

(a) *If $(g, K, \mathfrak{b})$ is a $(\mu, \mathfrak{Y})$-Cauchy family, then* ${}^{\mathfrak{b}}\lim\limits_{k \in K} {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu \neq \emptyset$.

(b) *If there is an $f \in E^T$ such that $(g, K, \mathfrak{b})$ converges uniformly to f, then $(g, K, \mathfrak{b})$ is a $(\mu, \mathfrak{Y})$-Cauchy family.*

(c) *If $\mathfrak{Y}$ is $\vee$-closed and, for each set $M \in \mathrm{Dmn}\,\mu$ having the properties $\mu M = 0$, $\emptyset \neq M \neq E$, the set $\{M, T \backslash M\}$ is an element of $\mathfrak{Y}$, then* (b) *holds with "uniformly μ-a.e." instead of "uniformly".*

(d) *If there is an $f \in E^T$ such that $(g, K, \mathfrak{b})$ $(\mu, \mathfrak{Y})$-converges in mean to f, then $(g, K, \mathfrak{b})$ is a $(\mu, \mathfrak{Y})$-Cauchy family.*

(Remark: Of course, the premise within the statement (c) is satisfied, e.g., if $\mathfrak{Y} = \mathfrak{X}$.)

Proof: Ad (a) Let $h = (\mathrm{e}(\,{}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu))_{k\in K}$. Then, by the premise within (a), $(h, K, \mathfrak{b})$ is a Cauchy family in the topological vector space (E, τ) (for the terminology, see Remark 2.1/D), therefore (since (E, τ) is complete) the conclusion within (a) is true.

Ad (b) and (c). We skip the trivial case $\mu T = 0$. Assume that $(g, K, \mathfrak{b})$ converges uniformly μ-a.e. to f $(\in E^T)$. Then, there a set $M \in \mathrm{Dmn}\,\mu$ with $\mu M = 0$ such that $(g, K, \mathfrak{b})$ converges uniformly on $T\backslash M$ to f. Let $\epsilon > 0$. Then, there is a $B \in \mathfrak{b}$ such that $\|f(t) - g_k(t)\| < \frac{\epsilon}{3(\mu T)}$ for all $k \in B$ and all $t \in T\backslash M$. Therefore, with the abbreviation $N = T\backslash M$, the chain

$$\begin{aligned} \emptyset \neq {}^{\mathfrak{Y}}\!\!\int \|g_k - g_l\|^{\sim} d\mu \leq \sup_{t\,\in\, N} \,(\|g_k(t) - g_l(t)\|)\mu_N(N) \leq \\ \leq \sup_{t\,\in\, N}\,(\|f(t) - g_k(t)\| + \|f(t) - g_l(t)\|)\mu T < \epsilon \end{aligned}$$

holds for all $k, l \in B$ in the case $M = \emptyset$ without any further assumptions (thus (b) is true), while, in the case $M \neq \emptyset$, it holds for all $k, l \in B$ under the assumption made in (c) (use of Lemma 1.3).

Ad (d). Use Proposition 26 in [7], p.17, the elementary fact that, for each pair of mappings $f, j : T \longrightarrow \mathbb{R}$, $f(t) \leq j(t)$ for all $t \in T$ implies ${}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu \leq {}^{\mathfrak{Y}}\!\!\int j^{\sim} d\mu$, furthermore the linearity of L and the supposition for L. $\square$

For the remainder of this paper, let $f : T \longrightarrow E$.

By Theorem 12 in [7], p.16, for a sequence g in E^T the following is evident: g (μ, Ω)-converges in mean to f if and only if, for some $p \in \mathbb{N}$, the function $\|f - g_n\|$ is μ-summable for all $n \geq p$, and $\lim\limits_{n\,\geq\, p} {}^{\mathrm{H}}\!\!\int \|f - g_n\|\, d\mu = 0$ holds.

In the general case of $(\mu, \mathfrak{Y})$-convergence in mean of filtered families one has the following: If $(g, K, \mathfrak{b})(\in \Phi(E^T))$ $(\mu, \mathfrak{Y})$-converges in mean to f, then (by Lemma 1.1) there is a $B \in \mathfrak{b}$ such that $\emptyset \neq {}^{\mathfrak{Y}}\!\!\int \|f - g_k\|^{\sim} d\mu = {}^{\Omega}\!\!\int \|f - g_k\|^{\sim} d\mu$, thus (by Theorem 12 in [7]) ${}^{\mathrm{H}}\!\!\int \|f - g_k\|\, d\mu = \mathrm{e}(\,{}^{\Omega}\!\!\int \|f - g_k\|^{\sim} d\mu)$ holds for all $k \in B$, and $0 = {}^{\mathfrak{c}}\lim\limits_{k\,\in\, B}\ {}^{\mathrm{H}}\!\!\int \|f - g_k\|\, d\mu$ with $\mathfrak{c} = \mathfrak{b}_B$ (= trace of the filter $\mathfrak{b}$ in the set B). – Let L be a linear subspace of E^T such that $j \in L$ implies that $\|j\|$ is μ-measurable. (If the measure μ is complete, the linear space $L = \mathrm{Dmn}\,(\,{}^{\mathrm{D}}\!\!\int \cdot\, d\mu)$ (see Definition 2.3) has this property (see Proposition 2.8/b).) Then, the intersection of the relation "$(\mu, \mathfrak{Y})$-converges in mean to" (between the class $\Phi(E^T)$ and the set E^T) with the class $\Phi(L) \times L$ is "μ-almost" a function, i.e., if $(g, K, \mathfrak{b}) \in \Phi(L)$, $f, j \in L$, and $(g, K, \mathfrak{b})$ $(\mu, \mathfrak{Y})$-converges in mean to f and j, then $f(t) = j(t)$ holds for μ-a.a. $t \in T$. (Use of the preceding lines, of [13], p.102, 4.3.4, Satz 1, and of the note following to Lemma 1.1.)

The assertion (a)$\Longrightarrow$(b) in the following proposition (cf. [5], p.130, Proposition 14) generalizes a well-known relationship between the classical mean convergence and pointwise convergence (see, e.g., [21], p.219, within a proof):

Proposition 2.2 *Assume the measure μ to be complete. Let g be a sequence in E^T. Then* (a) *implies* (b)*:*

(a) *g $(\mu, \mathfrak{Y})$-converges in mean to f.*

(b) *There is a subsequence of g converging pointwise μ-a.e. to f.*

If, additionally, for almost all (i.e. eventually all) $n \in \mathbb{N}$, $f - g_n$ is bounded μ-a.e., then (a) *implies* (c)*:*

(c) *g $(\mu, \mathfrak{X})$-converges in mean to f.*

Proof: Assume (a). Then, by Lemma 1.1, $g\,(\mu, \Omega)$-converges in mean to f, thus (by Theorem 12 in 3.Satz in [13], p.96), for some $p \in \mathbb{N}$, the function $\|f - g_n\|$ is μ-summable, therefore (by 3.Satz in [13], p.96) μ-measurable (since μ is complete) for all $n \geq p$, and one has $\lim\limits_{n \geq p} {}^{\mathrm{H}}\!\!\int \|f - g_n\|\, d\mu = 0$. Now, *formally*, we have the same situation as in the proof of Proposition 14 in [5], p.130, and we can continue *formally* almost word by word like there: For each $k \in \mathbb{N}$ there exists an $n_k \in \mathbb{N}$ with $n_k \geq p$ such that

$$ {}^{\mathrm{H}}\!\!\int \|f - g_{n_k}\|\, d\mu < (1/(2^{2k})\,, \tag{1} $$

and, of course, one can reach that $n_k < n_{k+1}$ holds for each $k \in \mathbb{N}$. The sequence (g_{n_k}) converges pointwise μ-a.e. to f.

In order to show this, one observes that, for each $k \in \mathbb{N}$, the set E_k defined by

$$ E_k = \{t \in T \mid 1/(2^k) \leq \|f(t) - g_{n_k}(t)\|\} \tag{2} $$

is a member of $\operatorname{Dmn} \mu$ for all $k \in \mathbb{N}$, because $\|f - g_{n_k}\|$ is μ-measurable. Using 4.Satz in [13], p.100, one obtains (by means of (1) and (2)) that

$$ \mu(E_k) \leq 1/(2^k) \text{ holds for all } k \in \mathbb{N}\,. \tag{3} $$

Let, now, $F = \bigcap_{k \in \mathbb{N}} \bigcup_{i=k}^{\infty} E_i$. Then, one has (in view of (3) and of the completeness of μ) $\mu F = 0$. Using (2), one obtains that on $T \backslash F$ the sequence (g_{n_k}) converges pointwise to f. We have showed (b).

Now assume $f - g_n$ to be bounded μ-a.e. for almost all $n \in \mathbb{N}$, say for all $n \geq q$ for some $q \geq p$. Let $n \geq q$. The function $\|f - g_n\|$ being bounded μ-a.e. and μ-measurable, one obtains (by Lemma 1.4) ${}^{\mathrm{H}}\!\!\int \|f - g_n\|\, d\mu \in {}^{\mathfrak{X}}\!\!\int \|f - g_n\|^{\sim} d\mu = {}^{\Omega}\!\!\int \|f - g_n\|^{\sim} d\mu$. Therefore, since g (μ, Ω)-converges in mean to f, one has (c). $\square$

(Note: For the relationship between the relation "(μ, Ω)-converges in mean to" and the strict specialization of the relation "converges in mean to" in the space $\mathcal{L}^1_E(m)$ defined in [5],p.127, to the present situation of a positive scalar measure, see Remark 2.4 below.)

Lemma 2.1 (Proposition 32 in [7]). *Assume that* (a) *E is finite-dimensional or* (b) $\mathfrak{Y} \subseteq \mathfrak{X}$. *Let $(h, K, \mathfrak{b})$ be a filtered family in E^T. Then, one has: If $(h, K, \mathfrak{b})$ $(\mu, \mathfrak{Y})$-converges in mean to f, then* $^{\mathfrak{b}}\lim\limits_{k \in K} \;{}^{\mathfrak{Y}}\!\!\int h_k{}^{\sim} d\mu \subseteq {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$.

Proposition 2.3 *Let μ be complete and f be bounded μ-a.e. Let g be a sequence in $\mathcal{ST}(\mathfrak{X}, E)$.*

Then (a) *implies* (b):

(a) *g (μ, Ω)-converges in mean to f.*

(b) $\lim\limits_{n \in \mathbb{N}} \;{}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu \subseteq {}^{\mathfrak{X}}\!\!\int f^{\sim} d\mu$.

Proof: By Proposition 2.2, (a) implies that g $(\mu, \mathfrak{X})$-converges in mean to f, thus, by Lemma 2.1, one has (b). □

Remark 2.2 As an immediate consequence of Proposition 31 in [7], one obtains the following statement: If ${}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu \neq \emptyset$ and the filtered family $(h, K, \mathfrak{b})$ in $E^T(\mu, \mathfrak{Y})$-converges in mean to f, then $^{\mathfrak{b}}\lim\limits_{k \in K} {}^{\mathfrak{Y}}\!\!\int h_k{}^{\sim} d\mu \subseteq {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$.

The next considerations assure that our setting of convergence in mean (using the subdivision integral, in particular the Birkhoff integral ${}^{\Omega}\!\!\int \cdot \, d\mu$, in its definition) fits in the usual approach to convergence in mean occuring in a natural way in the theory of completion of certain seminormed spaces (see, e.g., [13], p.137, [21], p.218–220, [5], p.127–133).

Lemma 2.2 *Let g be a $(\mu, \mathfrak{Y})$-Cauchy sequence in $\mathcal{ST}(\mathfrak{Y}, E)$. Then there are a $(\mu, \mathfrak{Y})$-Cauchy sequence h in $\mathcal{ST}(\mathfrak{Y}, E)$, a strictly increasing mapping $\phi : \mathbb{N} \longrightarrow \mathbb{N}$, and a mapping $j : T \longrightarrow E$ such that the statements* (1)–(7) *defined next hold true:*

(1) *$h_n(t) = g_n(t)$ for all $n \in \mathbb{N}$ and for μ-a.a. $t \in T$.*

(2) *$j(t) = \lim\limits_{k \in \mathbb{N}} h_{\phi(k)}(t)$ for all $t \in T$.*

(3) *$(j - g_n)(t) = (j - h_n)(t)$ for all $n \in \mathbb{N}$ and for μ-a.a. $t \in T$.*

(4) *$\|j\|$ and all functions $\|j - h_n\|$ (with $n \in \mathbb{N}$) are μ-measurable.*

(5) *g and h (μ, Ω)-converge in mean to j.*

(6) *If μ is complete, then $\|j - g_n\|$ is μ-measurable for all $n \in \mathbb{N}$.*

(7) *If $j - g_n$ is bounded μ-a.e. for almost all (i.e. for eventually all) $n \in \mathbb{N}$, then g and h $(\mu, \mathfrak{X})$-converge in mean to j.*

(Remark: Within the scope of the lemma, the four implications

$$(1)\Longrightarrow(3);\ (2)\Longrightarrow(4);\ (1)\wedge(2)\Longrightarrow(6);\ (3)\wedge(4)\wedge(5)\Longrightarrow(7)$$

hold true.)

Proof: (We adapt the main part of the proof of Theorem 1 in [21], p.218–219, where the Daniell integration for measurable real-valued functions instead of ${}^{\mathrm{H}}\!\!\int \cdot\, d\mu$ is used and, up to the different terminology, our case $\mathfrak{Y} = \mathfrak{X}$ for complete measure μ is covered; cf. also the proof of Satz 1 in [13], p.137.) Without any danger of confusion, the sign ${}^{\mathrm{c}}\Sigma$ denotes the conditional summation either in $\overline{\mathbb{R}}$ or in E, depending on its application (for the terminology, see [10], p.187, or [11], p.119, respectively). Tacitly, we will use of Theorem 12 in [7].

Since g ist a $(\mu, \mathfrak{Y})$-Cauchy sequence (thus, by Lemma 1.1, a (μ, Ω)-Cauchy sequence), there is a stricly increasing mapping $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ such that

$$\underset{k\,\in\,\mathbb{N}}{{}^{\mathrm{c}}\Sigma}\ {}^{\mathrm{H}}\!\!\int \|g_{\phi(k+1)} - g_{\phi(k)}\|\, d\mu < +\infty\,. \tag{a}$$

For remainder of the proof, we put $\phi(k) = n_k$ for all $k \in \mathbb{N}$. Since, for each $k \in \mathbb{N}$, the function $\|g_{n_{k+1}} - g_{n_k}\|$ is μ-measurable, one has by (a) and Levi's theorem on monotone convergence (see 3.Satz, p.105 in [13])

$${}^{\mathrm{H}}\!\!\int \Big(\underset{k\,\in\,\mathbb{N}}{{}^{\mathrm{c}}\Sigma}\ \|g_{n_{k+1}}(t) - g_{n_k}(t)\|\Big)_{t\in T}\, d\mu < +\infty\,. \tag{b}$$

Define the function $G : T \longrightarrow \overline{\mathbb{R}}$ by

$$G(t) = \|g_{n_1}(t)\| + \underset{k\,\in\,\mathbb{N}}{{}^{\mathrm{c}}\Sigma}\ \|g_{n_{k+1}}(t) - g_{n_k}(t)\|$$

for all $t \in T$. By (b) (cf. [13], p.96, 2.Satz), there is a set $M \in \mathrm{Dmn}\,\mu$ such that $\mu M = 0$ and $G(t) < +\infty$ for all $t \in T\backslash M$. Define the sequence h in $\mathcal{ST}(\mathfrak{Y}, E)$ by letting, for each $n \in \mathbb{N}$, $h_n(t) = g_n(t)$ for all $t \in T\backslash M$ and $h_n(t) = 0$ for all $t \in M$ (where 0 denotes the zero element of E). (Thus, one has (1). Since g is a $(\mu, \mathfrak{Y})$-Cauchy sequence in $\mathcal{ST}(\mathfrak{Y}, E)$, so also h.) Then, for each $t \in T$, the set $L(t)$, defined by

$$L(t) = \{h_{n_1}(t)\} + \underset{k\,\in\,\mathbb{N}}{{}^{\mathrm{c}}\Sigma}\ \{h_{n_{k+1}}(t) - h_{n_k}(t)\}\,,$$

is non-empty, since absolute convergence of a series in E implies its conditional convergence (for the notation, see Theorem 14 in [11], p.128). Define the mapping $j : T \longrightarrow E$ by letting $j(t) = \mathrm{e}L(t)$ for all $t \in T$. Then, (2) and, therefore, also (4) holds, since one has, e.g., $\|j(t) - h_n(t)\| = \lim\limits_{k \in \mathbb{N}} \|h_{n_k}(t) - h_n(t)\|$ for all $t \in T$ and all $n \in \mathbb{N}$. Furthermore, one has, for each $q \in \mathbb{N}$ and each $t \in T$, $j(t) - h_{n_q}(t) = {}^{\mathrm{c}}\!\sum\limits_{k \geq q} (h_{n_{k+1}}(t) - h_{n_k}(t))$, thus

$$\lim_{q \in \mathbb{N}} {}^{\mathrm{H}}\!\!\int \|j - h_{n_q}\| \, d\mu = 0 \,. \tag{c}$$

[We have used the following: By the definition of h, the statement (a) and therefore (b) holds for h instead of g. Consequently, for each $q \in \mathbb{N}$, the function H_q, defined by $H_q(t) = {}^{\mathrm{c}}\!\sum\limits_{k \geq q} \|h_{n_{k+1}}(t) - h_{n_{k(t)}}\|$ for all $t \in T$, is μ-summable. Now apply Fatou's lemma (see 2.Satz, p.105 in [13]) to the sequence (H_q) and use (4).] – Since

$${}^{\mathrm{H}}\!\!\int \|j - h_n\| \, d\mu \leq {}^{\mathrm{H}}\!\!\int \|h_n - h_{n_q}\| \, d\mu + {}^{\mathrm{H}}\!\!\int \|h_{n_q} - j\| \, d\mu$$

holds for all n, $q \in \mathbb{N}$, one obtains (using (c) and the fact that h is a $(\mu, \mathfrak{Y})$-Cauchy sequence) that h (μ, Ω)-converges in mean to j, therefore, using (3) and Corollary 1.1, the statement (5). Assume the premise within the statement (7) and choose a $p \in \mathbb{N}$ such that the function $j - g_n$ is bounded μ-a.e. for all $n \geq p$. Let $n \geq p$. Then, in view of (3), (4), Corollary 1.1, and Lemma 1.4, the statement (d) formulated next holds true:

$$\emptyset \neq {}^{\mathfrak{X}}\!\!\int \|j - h_n\|^{\sim} d\mu = {}^{\mathfrak{X}}\!\!\int \|j - g_n\|^{\sim} d\mu = {}^{\Omega}\!\!\int \|j - g_n\|^{\sim} d\mu \,. \tag{d}$$

Combining (d) with (5), one obtains the conclusion within (7). □

In the following remark, where we slightly generalize Proposition 33 and Theorem 13 in [7], p.21, the preceding lemma is applied for $\mathfrak{Y} = \Omega$:

Remark 2.3 In order to have the μ-partition system $\mathfrak{X}$ available, we have the measure μ assumed to be *finite* in this paper. For this remark, we replace this assumption by requiring μ to be (only) σ-finite. Then, e.g., Definitions 2.1 and 2.2 need not be changed and the parts (a) and (d) of Proposition 2.1 keep to be valid. Also, Lemma 2.2, reformulated with the words "and if μ is finite" added to the premise within (7), remains true. We assume these modifications, now, tacitly: In Definition 8 in [7], p.20, the statements (a) and (b) imply (by Proposition 2.1/d) that the sequence (g_n), occuring there, is a (μ, Ω)-Cauchy sequence. (Observe that, by (a), $g_n \in L_3$ (see Remark 2.1) holds for all $n \in \mathbb{N}$.) Therefore (by Lemma 2.2 and Corollary 1.1) the T-ary partial operation ${}^{\mathrm{B}}\!\!\int \cdot \, d\mu$ in E is the set of all $(j, x) \in E^T \times E$ such there exists a (μ, Ω)-Cauchy sequence h in $\mathcal{ST}(\Omega, E)$ having the properties (α), (β), and (γ) defined next:

(α) ${}^{\Omega}\!\!\int \|h_n\|^{\sim} d\mu \neq \emptyset$ for all $n \in \mathbb{N}$.

(β) h converges pointwise μ-a.e. to j.

(γ) $x \in \lim\limits_{n \in \mathbb{N}} {}^{\Omega}\!\!\int h_n{}^{\sim} d\mu$.

The elements of Dmn(${}^{\mathrm{B}}\!\!\int \cdot\, d\mu$) are now called to be B-*Bochner integrable*. We assert: *If f is* B-*Bochner integrable, then $\|f\|$ is* B-*Bochner integrable in the Banach space* $(\mathbb{R}, |\cdot|)$, *and one has* ${}^{\mathrm{B}}\!\!\int \|f\|\, d\mu \in {}^{\Omega}\!\!\int \|f\|^{\sim} d\mu$, *thus $\|f\|$ is μ-summable.* [*Proof.* Choose $x \in E$ and a (μ, Ω)-Cauchy sequence h in $\mathcal{ST}(\Omega, E)$ having the properties (α), (β) with f instead of j, and (γ). Then, the sequence $(\|h_n\|)$ is a (μ, Ω)-Cauchy sequence in $\mathcal{ST}(\Omega, \mathbb{R})$ (use of Lemma 1.2 for $E = \mathbb{R}$) with the property (α), thus (by Proposition 2.1/a, applied to $L = L_3$ (see Remark 2.1) with $E = \mathbb{R}$) $\emptyset \neq \lim\limits_{n \in \mathbb{N}} {}^{\Omega}\!\!\int \|h_n\|^{\sim} d\mu$. If one combines this with (α), the modified (β), and the continuity of $\|\cdot\|$, one obtains that $\|f\|$ is B-Bochner integrable in $\mathbb{R}$. Since $\mathbb{R}$ is finite-dimensional and $(\|h_n\|)$ (μ, Ω)-converges in mean to $\|f\|$ (use of Lemma 2.2 and Corollary 1.1, one has (by Lemma 2.1) $\lim\limits_{n \in \mathbb{N}} {}^{\Omega}\!\!\int \|h_n\|^{\sim} d\mu \subseteq {}^{\Omega}\!\!\int \|f\|^{\sim} d\mu$, thus the remainder of the assertion is proven (use of Theorem 12 in [7],p.16).$\square$] Therefore, in Proposition 33 in [7], p.21, consequently in Theorem 13 in [7], p.21,(which uses this proposition) the supposition of the completeness of μ is allowed to be deleted. (In the proof of the latter theorem, the second occurence of "$h^{\star}$" should be replaced by "h" (printing error).)

We apply Lemma 2.2 for $\mathfrak{Y} = \mathfrak{X}$:

Proposition 2.4 *Let g be a $(\mu, \mathfrak{X})$-Cauchy sequence in $\mathcal{ST}(\mathfrak{X}, E)$ converging pointwise μ-a.e. to f. Then, one has* (a) *and* (b)*:*

(a) *g (μ, Ω)-converges in mean to f and $\emptyset \neq \lim\limits_{n \in \mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu$. If the linear space E is finite-dimensional, then, additionally,* $\lim\limits_{n \in \mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu = {}^{\Omega}\!\!\int f^{\sim} d\mu$.

(b) *If $f - g_n$ is bounded μ-a.e. for almost all (i.e. for eventually all) $n \in \mathbb{N}$, then g $(\mu, \mathfrak{X})$-converges in mean to f, and $\emptyset \neq \lim\limits_{n \in \mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu = {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$ holds, provided that $\mathfrak{X} \subseteq \mathfrak{Y}$.*

Proof: Choose a $(\mu, \mathfrak{X})$-Cauchy sequence h in $\mathcal{ST}(\mathfrak{X}, E)$, a strictly increasing mapping $\phi : \mathbb{N} \longrightarrow \mathbb{N}$, and a mapping $j : T \longrightarrow E$ such that the statements (1)–(7) in Lemma 2.2 hold true. Then, by (1) and (2), $f(t) = j(t)$ holds for μ-a.a. $t \in T$, therefore, by the

Corollary 1.1 and by (5), the sequence g (μ,Ω)-converges in mean to f. By Proposition 2.1/a, one has $\emptyset \neq \lim\limits_{n\,\in\,\mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu$. Therefore, if the linear space E is finite-dimensional, then, by Lemma 2.1, $\lim\limits_{n\,\in\,\mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu = {}^{\Omega}\!\!\int f^{\sim} d\mu$. (Observe that in any case ${}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu = {}^{\Omega}\!\!\int g_n{}^{\sim} d\mu$ by Lemma 1.2.) This settles the proof of (a). Combining (a) with (7) and Lemma 1.1, one obtains (b). □

We adapt now the following definition of the Bochner integral using only finite μ-partitions of T:

Definition 2.3 (cf. Definition 1 in [5], p.120, there for vector measures). *We define the relation* ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu$ *between* E^T *and* E *to be the set of all* $(j,x) \in E^T \times E$ *such that there is a* $(\mu,\mathfrak{X})$*-Cauchy sequence* g *in* $\mathcal{ST}(\mathfrak{X},E)$ *converging pointwise* μ*-a.e. to* j *and having the property* $x \in \lim\limits_{n\,\in\,\mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu$. *We call* ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu$ *the* D-Bochner integral. *(Remark: The relation* ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu$ *is a* T*-ary partial operation in* E *(use, e.g., Proposition 2.7). The elements of* Dmn $({}^{\mathrm{D}}\!\!\int \cdot\, d\mu)$ *are called to be* D-Bochner integrable. *Instead of* $({}^{\mathrm{D}}\!\!\int \cdot\, d\mu)^{\wedge}$ *we write also* ${}^{\mathrm{D}}\!\!\int^{\wedge} \cdot\, d\mu$ *(see Section* 1/c*).*

Proposition 2.5 *If* f *is an* $\mathfrak{X}$*-step function determined by* $(\mathfrak{x},\chi)$ *with* $\mathfrak{x} \in \mathfrak{X}$ *and* $\chi : \mathfrak{x} \longrightarrow E$*, then* f *is D-Bochner integrable, and one has* ${}^{\mathrm{D}}\!\!\int f\, d\mu = \sum\limits_{X\,\in\,\mathfrak{x}} f(\chi(X))\mu X =$ e$({}^{\mathfrak{X}}\!\!\int f^{\sim} d\mu)$.

Proof: Ad (a). The sequence $(f,f,\cdots)$ is a $(\mu,\mathfrak{X})$-Cauchy sequence. Use, furthermore, Lemma 1.2 and the finiteness of $\mathfrak{x}$. □

We denote by ${}^{\mathrm{el}}\!\!\int \cdot\, d\mu$ the set of all $(f,x) \in \mathcal{ST}(\mathfrak{X},E) \times E$ such that $x = \sum\limits_{X\,\in\,\mathfrak{x}} \chi(X)\mu X$ holds true for some pair $(\mathfrak{x},\lambda)$ with $\mathfrak{x} \in \mathfrak{X}$ and $\lambda : \mathfrak{x} \longrightarrow E$ which determines f. The relation ${}^{\mathrm{el}}\!\!\int \cdot\, d\mu$ is a function on $\mathcal{ST}(\mathfrak{X},E)$ into E being called the *elementary integral* for (μ,E).

By Proposition 2.5, one has $({}^{\mathfrak{X}}\!\!\int \cdot\, d\mu)^{\vee} | \mathcal{ST}(\mathfrak{X},E) = {}^{\mathrm{el}}\!\!\int \cdot\, d\mu$.

Avoiding the use of ${}^{\mathfrak{X}}\!\!\int \cdot\, d\mu$, but using the elementary integral ${}^{\mathrm{el}}\!\!\int \cdot\, d\mu$ twice (first in the Banach space $\mathbb{R}$, secondly in E)), one can describe the mapping ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu$ in the following way:

Proposition 2.6 ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu$ *is the set of all* $(f,x) \in E^T \times E$ *such there exists a sequence* g *in* $\mathcal{ST}(\mathfrak{X},E)$ *such that* (a), (b), *and* (c) *hold true:*

(a) $0 = \lim\limits_{\substack{n\,\in\,\mathbb{N}\\ m\,\in\,\mathbb{N}}} {}^{\mathrm{el}}\!\!\int \|g_n - g_m\|\, d\mu$;

(b) *g converges pointwise μ-a.e. to f;*

(c) $x = \lim\limits_{n \in \mathbb{N}} {}^{\mathrm{el}}\!\!\int g_n \, d\mu.$

Proof: Use of Lemma 1.2. □.

The following characterization of the D-Bochner integral provides its description via mean convergence, which becomes more familiar for measurable integrands j (see Proposition 3.2 below and [4]).

Proposition 2.7 *In every case (resp. in the case of complete μ) ${}^{\mathrm{D}}\!\!\int \cdot \, d\mu$ is the set of all $(j, x) \in E^T \times T$ such there exists a sequence g in $\mathcal{ST}(\mathfrak{X}, E)$ having the properties* (a), (b), *and* (c) *(resp. the properties* (b) *and* (c)*) defined next:*

(a) *g converges to j pointwise μ-a.e..*

(b) *g (μ, Ω)-converges in mean to j.*

(c) $x = \lim\limits_{n \in \mathbb{N}} {}^{\mathrm{el}}\!\!\int g_n \, d\mu.$

Proof: Observe first that, for each sequence g in $\mathcal{ST}(\mathfrak{X}, E)$, g is a $(\mu, \mathfrak{X})$-Cauchy sequence if and only if g is a (μ, Ω)-Cauchy sequence. Now, apply the Propositions 2.4 and 2.1/d in the general case, and, additionally, Proposition 2.2 in the special case of a complete μ. □

Recall that the T-ary partial operation $({}^{\Omega}\!\!\int \cdot \, d\mu)^{\vee}$ in E (being also denoted by ${}^{\Omega}\!\!\int^{\vee} \cdot \, d\mu$) is defined to be the function $\{(g, x) \in E^T \times E \mid x \in {}^{\Omega}\!\!\int g^{\sim} d\mu\}$ (see Corollary to Proposition 18 and Section 0/a in [6]).

The Bochner integral ${}^{\mathrm{B}}\!\!\int \cdot \, d\mu$, as it has been described in [7], Definition 8 there, is related to ${}^{\mathrm{D}}\!\!\int \cdot \, d\mu$ and to ${}^{\Omega}\!\!\int \cdot \, d\mu$ in the following way:

Theorem 2.1 *The following chain holds true:*

$$ {}^{\mathrm{D}}\!\!\int \cdot \, d\mu \subseteq {}^{\mathrm{B}}\!\!\int \cdot \, d\mu \subseteq ({}^{\Omega}\!\!\int \cdot \, d\mu)^{\vee} . $$

Proof: For the first inclusion, use Proposition 2.7, for the second one use Theorem 13 in [7], where the completeness of μ need not be supposed (see Remark 2.3). □

As a consequence, one has: If f is D-Bochner integrable, then ${}^{\mathrm{D}}\!\!\int f \, d\mu$ is the limit of a filtered family of Riemann sums which are constructed by means of f. In Theorem 2.2, it will be assured that, if f is bounded μ- a.e., those Riemann sums can be chosen as *finite* sums. (This remark justifies the title of the paper.)

Proposition 2.8 (For the first the assertion in (a), see Proposition 4 in [5], p.122.) *Let f be* D*-Bochner integrable. Then, one has* (a) *and* (b)*:*

(a) $\|f\|$ *is* D*-Bochner integrable in the Banach space* $(\mathbb{R}, |\cdot|)$ *and* μ*-summable; more precisely, one has the chain*

$$^{\mathrm{D}}\!\!\int \|f\|\, d\mu = \, ^{\Omega}\!\!\int^{\vee} \|f\|\, d\mu = \, ^{\mathrm{H}}\!\!\int \|f\|\, d\mu < +\infty.$$

(b) *If μ is complete, then $\|f\|$ is μ-measurable.*

Proof: Proving the first assertion in (a), use Proposition 2.5, the continuity of $\|\cdot\|$, and the fact that, for each $(\mu, \mathfrak{X})$-Cauchy sequence g in $\mathcal{ST}(\mathfrak{X}, E)$, the sequence $\|g_n\|_{n \in \mathbb{N}}$ is a $(\mu, \mathfrak{X})$-Cauchy sequence in $\mathcal{ST}(\mathfrak{X}, \mathbb{R})$. Proving the μ-summability of f, combine Theorem 12 in [7] with Theorem 2.1. In order to prove (b), apply either additionally 3.Satz, p.96 in [13] or combine the continuity of $\|\cdot\|$ with the μ-measurability of each function $\|g_n\|$. $\square$

Remark 2.4 Denote, for this remark, the linear space Dmn$(\,^{\mathrm{D}}\!\!\int \cdot\, d\mu)$ by L and define the relation "D-*converges in mean to*" ($= R$ for abbreviation) to be the set of all $(h, g) \in L^{\mathbb{N}} \times L$ such that $0 = \lim\limits_{n \in \mathbb{N}} \,^{\mathrm{D}}\!\!\int \|g - h_n\|\, d\mu$ holds (use of Proposition 2.8). For abbreviation, let S denote the relation "(μ, Ω)-converges in mean to". Then, by Proposition 2.8, one has the equation $R = S \cap (L^{\mathbb{N}} \times L)$ (where, of course, each sequence j in E^T is identified with the filtered family $(j, \mathbb{N}, \mathcal{F}\mathbb{N})$). The notion of the "D-convergence in mean" is a strict specialization of the notion of "mean convergence" in $\mathcal{L}^1_E(m)$ defined in [5], p.127. Now, reconsider the proof of Proposition 2.2: If we would have supposed f and the functions g_n to be D-Bochner integrable, the greatest part of the proof would have become a strict special case of the cited proof in [5].

Lemma 2.3 (cf. [5], p.128, Proposition 12). *If g is a $(\mu, \mathfrak{X})$-Cauchy sequence in $\mathcal{ST}(\mathfrak{X}, E)$ converging pointwise μ-a.e to f, then*

$$0 = \lim_{n \in \mathbb{N}} \,^{\mathrm{D}}\!\!\int \|f - g_n\|\, d\mu$$

holds in the Banach space $(\mathbb{R}, |\cdot|)$*, i.e. " g* D*-converges in mean to f "* (see Remark 2.4).

Proof: Formulating the lemma, we have tacitly used Propositions 2.4/a and 2.8. In order to prove the assertion, just reformulate the proof of the cited Proposition in [5] in our terminology. $\square$

Within the preceding lemma, $^{\mathrm{D}}\!\!\int \|f - g_n\|\, d\mu \in \, ^{\Omega}\!\!\int \|f - g_n\|^{\sim} d\mu$ holds for all $n \in \mathbb{N}$ by Proposition 2.8. Thus, the lemma implies the first assertion of Proposition 2.4/a. [But, in view of the implicit dependence of Proposition 2.8 on Proposition 2.4/a, this implication is not another proof of that assertion.] On the other hand, one can use it in the proof of the following description of the D-Bochner integral (for complete μ) related to $(E, \|\cdot\|)$ in terms of the D-Bochner integral related to the Banach space $(\mathbb{R}, |\cdot|)$ and the $(\mu, \mathfrak{X})$-subdivision integral of $\mathfrak{X}$-step functions:

Proposition 2.9 *Let M be the set of all $(j, x) \in E^T \times E$ such that there exists a sequence g in $\mathcal{ST}(\mathfrak{X}, E)$ having the properties*

$$0 \in \lim_{n \in \mathbb{N}} {}^{\mathrm{D}}\!\!\int^{\wedge} \|j - g_n\|^{\sim} \, d\mu \text{ and } x \in \lim_{n \in \mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu \, .$$

Then ${}^{\mathrm{D}}\!\!\int \cdot \, d\mu \subseteq M$ holds in any case. If μ is complete, then $M = {}^{\mathrm{D}}\!\!\int \cdot \, d\mu$.

Proof: Proving ${}^{\mathrm{D}}\!\!\int \cdot \, d\mu \subseteq M$, one uses Definition 2.3, Lemma 2.3, and Proposition 2.7. Proving $M \subseteq {}^{\mathrm{D}}\!\!\int \cdot \, d\mu$, one uses Proposition 2.8/b (in the Banach space $(\mathbb{R}, |\cdot|)$), the completeness of μ, and Proposition 2.7. $\square$

Now, we raise the question, whether or under which conditions in the statement ${}^{\mathrm{D}}\!\!\int^{\wedge} f^{\sim} d\mu \subseteq {}^{\Omega}\!\!\int f^{\sim} d\mu$ (in Theorem 2.1) the μ-partition system Ω is allowed to be replaced by $\mathfrak{X}$. Here is a partial answer:

Theorem 2.2 (a) *If f is bounded μ-a.e., then ${}^{\mathrm{D}}\!\!\int^{\wedge} f^{\sim} d\mu \subseteq {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$ holds, provided that $\mathfrak{X} \subseteq \mathfrak{Y}$.*

(b) *If the sequence g in $\mathcal{ST}(\mathfrak{X}, E)$ converges uniformly μ-a.e. to f, then $\emptyset \neq \lim\limits_{n \in \mathbb{N}} {}^{\mathfrak{Y}}\!\!\int g_n{}^{\sim} d\mu =$*

$= {}^{\mathrm{D}}\!\!\int^{\wedge} f^{\sim} d\mu = {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$ holds, provided that $\mathfrak{X} \subseteq \mathfrak{Y}$.

Proof: Ad (a). Note: If μ is complete, then, one has by Proposition 2.7, Proposition 2.3, and Lemma 1.1 the validity of (a). In the general case we conclude the following way:

Assume the premise of (a) and the inclusion $\mathfrak{X} \subseteq \mathfrak{Y}$. Let $x \in {}^{\mathrm{D}}\!\!\int^{\wedge} f^{\sim} d\mu$. Then, there exists a $(\mu, \mathfrak{X})$-Cauchy sequence g in $\mathcal{ST}(\mathfrak{X}, E)$ converging pointwise μ-a.e. to f such that $x \in \lim\limits_{n \in \mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu$. Choose such a g. Then, by the premise, there is an $M \in \operatorname{Dmn} \mu$ such that $\mu M = 0$ and, for each $n \in \mathbb{N}$, the function $(f - g_n)|T \backslash M$ is bounded. By Proposition 2.4/b, one obtains $x \in {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$.

Ad (b). Assume the premise of (b) and the inclusion $\mathfrak{X} \subseteq \mathfrak{Y}$. Then g is a $(\mu, \mathfrak{X})$-Cauchy sequence (by Proposition 2.1/b) converging pointwise μ-a.e. to f, thus (by Proposition 2.1/a and Definition 2.3) $\emptyset \neq \lim\limits_{n \in \mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu = {}^{\mathrm{D}}\!\!\int^{\wedge} f^{\sim} d\mu$. Because g converges even uniformly μ-a.e. to f, there is a $p \in \mathbb{N}$ and an $M \in \operatorname{Dmn} \mu$ such that $\mu M = 0$ and for each $n \geq p$ the function $(f - g_n)|T\backslash M$ is bounded. By Proposition 2.4/b, one obtains $\emptyset \neq \lim\limits_{n \in \mathbb{N}} {}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu = {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$. Observe, furthermore, that ${}^{\mathfrak{X}}\!\!\int g_n{}^{\sim} d\mu = {}^{\mathfrak{Y}}\!\!\int g_n{}^{\sim} d\mu$ holds for all $n \in \mathbb{N}$ by Lemma 1.2, since $\mathfrak{X} \subseteq \mathfrak{Y}$. □

Next, we note a simple consequence of the preceding theorem applied in the Banach space $(\mathbb{R}, |\cdot|)$:

Proposition 2.10 *If f is* D*-Bochner integrable and bounded μ-a.e., then* ${}^{\mathrm{D}}\!\!\int \|f\| \, d\mu \in {}^{\mathfrak{Y}}\!\!\int \|f\|^{\sim} d\mu$*, provided that* $\mathfrak{X} \subseteq \mathfrak{Y}$.

Proof: By Proposition 2.8/b, $\|f\|$ is D-Bochner integrable in the Banach space $(\mathbb{R}, |\cdot|)$; thus, the assertion holds by Theorem 2.2/a and Lemma 1.1. □

Next, we give another proof of Theorem 2.2/b not using implicitly Lemma 2.2 (via Proposition 2.4) but based on the following elementary lemma the validity of which is not restricted to classical sequences as it is the case for Lemma 2.2.

Lemma 2.4 *Let $(h, K, \mathfrak{b})$ be a filtered family in E^T. If $(h, K, \mathfrak{b})$ converges uniformly μ-a.e. to f and ${}^{\mathfrak{X}}\!\!\int \|f - h_k\|^{\sim} d\mu \neq \emptyset$ holds for $\mathfrak{b}$-almost all $k \in K$* (for the terminology, see [11], p.116)*, then $(h, K, \mathfrak{b})$ $(\mu, \mathfrak{X})$-converges in mean to f and one has*

$$ {}^{\mathfrak{b}}\lim_{k \in K} {}^{\mathfrak{X}}\!\!\int h_k{}^{\sim} d\mu \subseteq {}^{\mathfrak{X}}\!\!\int f^{\sim} d\mu. $$

(Remark. By the following proof, the lemma is logically equivalent with the statement one gets from it by canceling the quantifier "μ-a.e.")

Proof: We skip the trivial case $\mu T = 0$.

1. By the supposition, there is a set $M \in \operatorname{Dmn} \mu$ such that $\mu M = 0$ and $(h, K, \mathfrak{b})$ converges uniformly on $T\backslash M$ to f. Define a mapping $g : K \longrightarrow E^T$ by letting, for each $k \in K$, $g_k(t) = h_k(t)$ for all $t \in T\backslash M$ and $g_k(t) = f(t)$ for all $t \in M$. Then, the filtered family $(g, K, \mathfrak{b})$ converges uniformly to f, and (by Corollary 1.1) the equations ${}^{\mathfrak{X}}\!\!\int \|f - h_k\|^{\sim} d\mu = {}^{\mathfrak{X}}\!\!\int \|f - g_k\|^{\sim} d\mu$, ${}^{\mathfrak{X}}\!\!\int h_k{}^{\sim} d\mu = {}^{\mathfrak{X}}\!\!\int g_k{}^{\sim} d\mu$ hold for all $k \in K$. Therefore, it suffices to prove the lemma under the supposition that $(\star)$ $(h, K, \mathfrak{b})$ converges uniformly to f.

2. Assume $(\star)$. By the premise, there is a $B \in \mathfrak{b}$ such that

$$ \emptyset \neq {}^{\mathfrak{X}}\!\!\int \|f - h_k\|^{\sim} d\mu \text{ for all } k \in B \,. \tag{1} $$

Let $\epsilon > 0$. By the assumption $(\star)$, there is a $C \in \mathfrak{b}$ such that

$$ \|f(t) - h_k(t)\| < \frac{\epsilon}{\mu T} \text{ for all } t \in T \text{ and all } k \in C \,. $$

Therefore, we have for each $(\mathfrak{x}, \phi) \in \mathfrak{X}^{\#}$

$$\mathrm{R}(\|f - h_k\|, \mathfrak{x}, \phi) < \epsilon \text{ for all } k \in C\,,$$

thus, by (1), $\emptyset \neq {}^{\mathfrak{X}}\!\!\int \|f - h_k\|^{\sim} d\mu \leq \epsilon$ for all $k \in B \cap C$, and therefore, by the choice of ϵ, $(h, K, \mathfrak{b})$ $(\mu, \mathfrak{X})$-converges in mean to f. In view of Lemma 2.1, one obtains ${}^{\mathfrak{b}}\lim\limits_{k \in K} {}^{\mathfrak{X}}\!\!\int h_k{}^{\sim} d\mu \subseteq {}^{\mathfrak{X}}\!\!\int f^{\sim} d\mu$. □

Second proof of Theorem 2.2/b: We skip the trivial case $\mu T = 0$. By the premise within Theorem 2.2/b, there is a set $M \in \operatorname{Dmn} \mu$ such that $\mu M = 0$ and

$(\star)$ the sequence $(g_n|(T \backslash M))_{n \in \mathbb{N}}$ converges uniformly to $f|(T \backslash M)$.

For abbreviation, we put $Q = T \backslash M$, $\mathfrak{Z} = \mathfrak{X}_Q$, $\nu = \mu_Q$, $j = f|Q$, and $h_n = g_n|Q$ for all $n \in \mathbb{N}$. Observe that $Q \neq \emptyset$, since $\mu T > 0$. Let $m \in \mathbb{N}$. Since $\|h_n - h_m\|$ is ν-measurable for all $n \in \mathbb{N}$ and $\|j - h_m\|(t) = \| \lim\limits_{n \in \mathbb{N}} h_n(t) - h_m(t)\| = \lim\limits_{n \in \mathbb{N}} \|h_n(t) - h_m(t)\|$ holds for all $t \in Q$, the function $\|j - h_m\|$ is ν-measurable. Because of $(\star)$, there is an $n_0 \in \mathbb{N}$ such that $\|j - h_n\|$ is bounded for all $n \geq n_0$. Therefore, by Lemma 1.4, ${}^{\mathfrak{Z}}\!\!\int \|j - h_n\|^{\sim} d\nu$ is non-empty for all those n. Let $n \geq n_0$. If $M = \emptyset$, the inequality ${}^{\mathfrak{X}}\!\!\int \|f - g_n\|^{\sim} d\mu \neq \emptyset$ is proved; otherwise, one has $\{M, Q\} \in \mathfrak{X}$, and, therefore, this inequality holds by Lemma 1.3. Now use Lemma 2.4, Proposition 2.1, Definition 2.3, and Lemma 1.1. □

Theorem 2.2/b is generalized by the next theorem clearing up the (expected) behavior of the subdivision integral w.r. to the uniform convergence of filtered families in E^T. Observe that its proof does not take the detour via mean convergence we have taken implicitly in the special case (by using Lemma 2.2 or Lemma 2.4).

Theorem 2.3 *Let $(g, K, \mathfrak{b})$ be a filtered family in E^T converging uniformly μ-a.e. to f. If (α) f is $\mathfrak{Y}$-integrable or (β) $\mathfrak{Y} = \mathfrak{X}$, then* (a) *and* (b) *hold true:*

(a) ${}^{\mathfrak{b}}\lim\limits_{k \in K} {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu \subseteq {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$.

(b) *If there exists a $\mathfrak{Y}$-absolutely integrable linear subspace L of E^T such that $g(K) \subseteq L$, then $\emptyset \neq {}^{\mathfrak{b}}\lim\limits_{k \in K} {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu = {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$.*

Note: If the quantifier "μ-a.e." is canceled, then "$\mathfrak{Y} = \mathfrak{X}$" is allowed to be replaced by "$\mathfrak{Y} \subseteq \mathfrak{X}$".

Proof: (b) follows from (a) by means of Proposition 2.1.

Ad (a). We consider the non-trivial case $\mu T > 0$.

1. Assume first that $(g, K, \mathfrak{b})$ converges uniformly to f. We assume (α) or (γ) $\mathfrak{Y} \subseteq \mathfrak{X}$. ($(\gamma)$ replaces (β) above.) Let $x \in {}^{\mathfrak{b}}\lim\limits_{k \in K} {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu$ and $\epsilon > 0$. Then there is a $B \in \mathfrak{b}$ such that

(1) $\emptyset \neq \| {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu - \{x\}\| < \frac{\epsilon}{2}$ for all $k \in B$

and

(2) $\|f(t) - g_k(t)\| < \frac{\epsilon}{4\mu T}$ for all $k \in B$ and all $t \in T$.

Since $\mathfrak{b}$ is a filter, B is non-empty; choose a $k \in B$. In view of (1), one has $\emptyset \neq {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu$. Thus, there is an $\mathfrak{x}_0 \in \mathfrak{Y}$ such that

(3) $\emptyset \neq \|\mathrm{R}(g_k^{\sim}, \mathfrak{x}, \phi) - {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu\| < \frac{\epsilon}{4}$ for all $\mathfrak{x} \in \mathfrak{Y}$ with $\mathfrak{x}_0 \preceq \mathfrak{x}$ and all $\phi \in \mathcal{P}\mathfrak{x}$.

Now the proof branches: Case 1. If (α) is valid, then there is an $\mathfrak{x}_1 \in \mathfrak{Y}$ such that $\mathrm{R}(f^{\sim}, \mathfrak{x}, \phi) \neq \emptyset$ for all $\mathfrak{x} \in \mathfrak{Y}$ with $\mathfrak{x}_1 \preceq \mathfrak{x}$ and all $\phi \in \mathcal{P}\mathfrak{x}$; in this case $\mathfrak{x}_2 = \mathfrak{x}_1 \vee \mathfrak{x}_0$. Case 2. If (γ) but not (α) is valid then $\mathrm{R}(f^{\sim}, \mathfrak{x}, \phi) \neq \emptyset$ holds true for all $\mathfrak{x} \in \mathfrak{Y}$; in this case put $\mathfrak{x}_2 = \mathfrak{x}_0$.

In both branches we continue in the following way: Let $\mathfrak{x} \in \mathfrak{Y}$ with $\mathfrak{x}_2 \preceq \mathfrak{x}$ and $\phi \in \mathcal{P}\mathfrak{x}$. Then, in view of $\mathrm{R}(g_k^{\sim}, \mathfrak{x}, \phi) \neq \emptyset$ (use of "$\neq$" in (3)) and of (2), one obtains the chain

(4) $\emptyset \neq \|\mathrm{R}(f^{\sim}, \mathfrak{x}, \phi) - \mathrm{R}(g_k^{\sim}, \mathfrak{x}, \phi)\| \leq \mathrm{R}(\|f - g_k\|^{\sim}, \mathfrak{x}, \phi) < \frac{\epsilon}{4}$,

where the third Riemann sum (which is non-empty) is related to the Banach space $(\mathbb{R}, |\cdot|)$. (Cf. Proposition 10 and (0.12) in [6], p.490 and p.487.) The combination of (3) and (4) provides

(5) $\emptyset \neq \|\mathrm{R}(f^{\sim}, \mathfrak{x}, \phi) - {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu\| < \frac{\epsilon}{2}$.

Combining (5) with (1), one gets $\emptyset \neq \|\mathrm{R}(f^{\sim}, \mathfrak{x}, \phi) - \{x\}\| < \epsilon$. In view of the choice of $(\mathfrak{x}, \phi)$ we have proved $x \in {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$.

2. Since, by the supposition, $(g, K, \mathfrak{b})$ converges uniformly μ-a.e. to f, there is a set $M \in \operatorname{Dmn}\mu$ such $(g, K, \mathfrak{b})$ converges uniformly on $T\backslash M$ to f. Define a mapping $h : K \longrightarrow E^T$ by letting, for each $k \in K$, $h_k(t) = g_k(t)$ for all $t \in T\backslash M$ and $h_k(t) = f(t)$ for all $t \in M$. Then, the filtered family $(h, K, \mathfrak{b})$ converges uniformly to f. Under the supposition (α), one obtains by Lemma 1.1, Corollary 1.1, and Part **1** of this proof

$$ {}^{\mathfrak{b}}\lim_{k \in K} {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu \subseteq {}^{\mathfrak{b}}\lim_{k \in K} {}^{\Omega}\!\!\int g_k{}^{\sim} d\mu = {}^{\mathfrak{b}}\lim_{k \in K} {}^{\Omega}\!\!\int h_k{}^{\sim} d\mu \subseteq {}^{\Omega}\!\!\int f^{\sim} d\mu = {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu. $$

Under supposition (β), one has by Corollary 1.1 and Part **1** of this proof

$$ {}^{\mathfrak{b}}\lim_{k \in K} {}^{\mathfrak{Y}}\!\!\int g_k{}^{\sim} d\mu = {}^{\mathfrak{b}}\lim_{k \in K} {}^{\mathfrak{Y}}\!\!\int h_k{}^{\sim} d\mu \subseteq {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu\,. \qquad \square $$

Note: By Definition 2.3, the statement one gets from Theorem 2.3/b by putting there $K = \mathbb{N}$, $\mathfrak{b} = \mathcal{F}\mathbb{N}$, $\mathfrak{Y} = \mathfrak{X}$, and $L = \mathcal{ST}(\mathfrak{X}, E)$ coincides logically with Theorem 2.2/b.

Remark 2.5 The translation of G. Birkhoff´s Theorem 22 in [2], p.372, into our language is following nice assertion: "If g is a sequence in E^T converging uniformly to f and having the property that ${}^{\Omega}\!\!\int g_n{}^{\sim} d\mu \neq \emptyset$ for all $n \in \mathbb{N}$, then $\emptyset \neq \lim\limits_{n\,\in\,\mathbb{N}} {}^{\Omega}\!\!\int g_n{}^{\sim} d\mu = {}^{\Omega}\!\!\int f^{\sim} d\mu$." Unfortunately, we cannot follow the arguments of Birkhoff´s proof. On the other hand, we do not have an example contradicting the assertion. For $\mathfrak{Y} = \Omega$ and $(K, \mathfrak{b}) = (\mathbb{N}, \mathcal{F}\mathbb{N})$ the preceding theorem is contained in Birkhoff´s theorem. For arbitrary $\mathfrak{Y}$ it contains a (possibly well-known) analogue for the D- and B-Bochner integral formulated next.

Corollary 2.1 *Read the superscript* "C" *as* "D" *or as* "B"*. Let* $f \in \mathrm{Dmn}\ {}^{\mathrm{C}}\!\!\int \cdot\, d\mu$*. Let* $(g, K, \mathfrak{b})$ *be a filtered family in* $\mathrm{Dmn}\ {}^{\mathrm{C}}\!\!\int \cdot\, d\mu$ *converging uniformly to* f*. Then, one has*

$$^{\mathfrak{b}}\lim_{k\,\in\,K}\ {}^{\mathrm{C}}\!\!\int g_k^{\sim}\, d\mu = {}^{\mathrm{C}}\!\!\int f^{\sim}\, d\mu.$$

Proof: Since $f \in \mathrm{Dmn}\ {}^{\mathrm{C}}\!\!\int \cdot\, d\mu$ and $g_k \in \mathrm{Dmn}\ {}^{\mathrm{C}}\!\!\int \cdot\, d\mu$ for all $k \in K$, one has $\emptyset \neq {}^{\mathrm{C}}\!\!\int f^{\sim}\, d\mu = {}^{\Omega}\!\!\int f^{\sim} d\mu$ and $\emptyset \neq {}^{\mathrm{C}}\!\!\int g_k^{\sim}\, d\mu = {}^{\mathrm{B}}\!\!\int g_k^{\sim}\, d\mu = {}^{\Omega}\!\!\int g_k{}^{\sim} d\mu$ for all $k \in K$ by Theorem 2.1. Now apply Theorem 2.3/b for $\mathfrak{Y} = \Omega$ and the linear subspace $L = \mathrm{Dmn}\ {}^{\mathrm{B}}\!\!\int \cdot\, d\mu$ of E^T (which is Ω-absolutely integrable (see Remark 2.3). □

We conclude this section by reconsidering Proposition 2.4/a. For this purpose, we define the statement (A) by the following lines:

(A) *If* $h : \mathbb{N} \longrightarrow \mathcal{ST}(\mathfrak{X}, \mathbb{R})$ *is a* $(\mu, \mathfrak{X})$*-Cauchy sequence and* $j : T \longrightarrow \mathbb{R}$ *a function such that* h *converges pointwise* μ*-a.e. to* j*, then* j *is* μ*-summable, and one has*

$$\lim_{n\,\in\,\mathbb{N}}\ {}^{\mathrm{H}}\!\!\int \mid j - h_n \mid\ d\mu = 0 \ \text{and}\ {}^{\mathrm{H}}\!\!\int j\, d\mu = \lim_{n\,\in\,\mathbb{N}}\ {}^{\mathrm{H}}\!\!\int h_n\, d\mu.$$

We define the statement **(B)** to be the statement one obtains from (A) by canceling there " $\lim\limits_{n\,\in\,\mathbb{N}}\ {}^{\mathrm{H}}\!\!\int \mid j - h_n \mid\ d\mu = 0$ and ".

Then, we obtain

Proposition 2.11 *Each of the two statements* (A), (B) *is logically equivalent to* Proposition 2.4/a.

Proof: **1.** By Theorem 12 in [7], p.16, Proposition 2.4/a implies (A).

2. Trivially, (A) implies (B).

3. Assume (B). Assume the premise within Proposition 2.4/a. Define $k : \mathbb{N} \longrightarrow (\mathcal{ST}(\mathfrak{X}, \mathbb{R}))^{\mathbb{N}}$ by $k(n) = (\|g_n - g_m\|)_{m \in \mathbb{N}}$ for all $n \in \mathbb{N}$. By the choice of g and f in the premise of Proposition 2.4/a, for each $n \in \mathbb{N}$, $k(n)$ turns out to be a $(\mu, \mathfrak{X})$-Cauchy sequence in $\mathcal{ST}(\mathfrak{X}, \mathbb{R})$ converging pointwise μ-a.e. to the function $\|f - g_n\|$. By (B), one obtains therefore, for all $n \in \mathbb{N}$,

$$^{\mathrm{H}}\!\!\int \|f - g_n\| \, d\mu = \lim_{m \in \mathbb{N}} {}^{\mathrm{H}}\!\!\int (k(n))(m) \, d\mu < +\infty. \tag{1}$$

Let $\epsilon > 0$. By the choice of g, there is a $p \in \mathbb{N}$ such that $^{\mathrm{H}}\!\!\int \|g_n - g_m\| \, d\mu \in {}^{\mathfrak{X}}\!\!\int \|g_n - g_m\|^{\sim} d\mu < \epsilon$ holds for all $n, m \geq p$, thus, by (1), $^{\mathrm{H}}\!\!\int \|f - g_n\| \, d\mu \leq \epsilon$ holds for all $n \geq p$. (So far, we have just translated the proof of Proposition 12 in Dinculeanu [5], p.128, into our language.) Thus, by Theorem 12 in [7], p.16, g (μ, Ω)-converges in mean to f. The remainder of the assertion of Proposition 2.4/a follows by means of Proposition 2.1/a and Lemma 2.1. □

Remark 2.6 Let $(E, \|\cdot\|) = (\mathbb{R}, |\cdot|)$. Then, one has by Theorem 2.1 here and Theorem 12 in [7], p.16, the chain $^{\mathrm{D}}\!\!\int \cdot \, d\mu \subseteq {}^{\mathrm{B}}\!\!\int \cdot \, d\mu \subseteq ({}^{\mathrm{H}}\!\!\int \cdot \, d\mu)|M$, where M denotes the set of all μ-summable functions $j : T \longrightarrow \mathbb{R}$. Of course, the inclusion $^{\mathrm{D}}\!\!\int \cdot \, d\mu \subseteq ({}^{\mathrm{H}}\!\!\int \cdot \, d\mu)|M$ follows also from the statement (A) which is true by Proposition 2.11.

3 Summability of real-valued functions and integrability of vector-valued functions

It is usual (see, e.g., [4], p.41) to call f *μ-measurable* – for clarity we say also *$(\mu, \|\cdot\|)$-measurable* – if there exists a sequence g in $\mathcal{ST}(\mathfrak{X}, E)$ converging pointwise μ-a.e. to f. [For a generalization of this notion, see Definition 4 in [5], p.89.] If μ is complete, f is μ-measurable in the sense of Dinculeanu, loc.cit., if and only if f is $(\mu, \|\cdot\|)$-measurable (use of Theorem 1 in [5], p.94, and Theorem 2 in [5], p.99).] For later use we note that:

(3.1) *Every element of $\mathcal{ST}(\Omega, E)$ is $(\mu, \|\cdot\|)$-measurable.* (Use of $\mu T < +\infty$.)

For the remainder of this section, we assume μ to be complete.

If $(E, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, then f is $(\mu, \|\cdot\|)$-measurable if and only if f is μ-measurable. Therefore, there cannot arise any confusion if we just say "μ-measurable function" instead of "$(\mu, \|\cdot\|)$-measurable function", as we do in the following. We denote by $\mathcal{M}(\mu, T, E)$ the set of all μ-measurable functions on T into E.

In the remainder of this section, we rediscuss the relationships between the integrals occuring in Theorems 2.1 and 2.2. In the sequel, we say that f be B-*Bochner integrable* resp. Ω-*integrable* if $f \in \mathrm{Dmn}\, {}^{\mathrm{B}}\!\!\int \cdot \, d\mu$ resp. $^{\Omega}\!\!\int f^{\sim} d\mu \neq 0$.

For later use, we note a well-known fact:

(3.2) *If g is a sequence of μ-measurable functions $g_n : T \longrightarrow E$ converging pointwise μ-a.e. to f, then f is μ-measurable.*

Preparing the general case of integration of μ-measurable functions on T into E, we consider the integration of functions on T into $\mathbb{R}$:

For the next considerations, the elementary integral ${}^{\mathrm{el}}\!\!\int \cdot\, d\mu$ (defined in Section 2) is related to the Banach space $\mathbb{R}$.

Definition 3.1 *A linear mapping J* from $\mathbb{R}^T$ *(i.e. defined on a subset of $\mathbb{R}^T$) into $\mathbb{R}$ is called a μ-*summation *if* (J 1) *through* (J 4) *hold true:*

(J 1) $\mathcal{ST}(\mathfrak{X},\mathbb{R}) \subseteq \mathrm{Dmn}\, J \subseteq \mathcal{M}(\mu, T, \mathbb{R})$ *and* $J|\mathcal{ST}(\mathfrak{X},\mathbb{R}) = {}^{\mathrm{el}}\!\!\int \cdot\, d\mu$.

(J 2) $f \in \mathrm{Dmn}\, J$ *and* $A \in \mathrm{Dmn}\, \mu$ *implies* $f \cdot 1_A \in \mathrm{Dmn}\, J$.

(J 3) *If* $f, g \in \mathrm{Dmn}\, J$ *then* $f \leq g$ *implies* $J(f) \leq J(g)$.

(J 4) *If* (g_n) *is an increasing sequence in* $\mathrm{Dmn}\, J$ *converging pointwise to a function* $f (\in \mathbb{R}^T)$ *and having the property that the sequence* $(J(g_n))$ *is bounded, then* $f \in \mathrm{Dmn}\, J$ *and* $J(f) = \lim\limits_{n \in \mathbb{N}} J(g_n)$.

We recall that in the Banach space $E = \mathbb{R}$ the relationship between ${}^{\Omega}\!\!\int^{\vee} \cdot\, d\mu$ and the integral ${}^{\mathrm{H}}\!\!\int \cdot\, d\mu$ is described by

(1) ${}^{\Omega}\!\!\int^{\vee} \cdot\, d\mu = {}^{\mathrm{H}}\!\!\int \cdot\, d\mu \cap (\mathbb{R}^T \times \mathbb{R})$

(see Theorem 12 in [7], p.16). Let ${}^{\mathrm{Ba}}\!\!\int \cdot\, d\mu$ denote the "(μ)-Integral" defined in [1], p.64, Definition 12.1 (being considered here as a mapping *from* $\overline{\mathbb{R}}^T$ into $\overline{\mathbb{R}}$, defined on the set of all "(μ)-integrierbaren Funktionen" $g : T \longrightarrow \overline{\mathbb{R}}$). In analogy to (1) we define now the mapping ${}^{\mathrm{su}}\!\!\int \cdot\, d\mu$ by

${}^{\mathrm{su}}\!\!\int \cdot\, d\mu = {}^{\mathrm{Ba}}\!\!\int \cdot\, d\mu \cap (\mathbb{R}^T \times \mathbb{R})$. (We agree in the definitions $f^+ = \max\{f, 0\}$ and $f^- = \min\{f, 0\}$ for $f : T \longrightarrow \mathbb{R}$.)

In this terminology, we obtain

Theorem 3.1 *Let $E = \mathbb{R}$. There is exactly one μ-summation J, which we denote by $J(\mu)$, and one has $J(\mu) = {}^{\mathrm{su}}\!\!\int \cdot\, d\mu = {}^{\mathrm{D}}\!\!\int \cdot\, d\mu = {}^{\mathrm{B}}\!\!\int \cdot\, d\mu = {}^{\Omega}\!\!\int^{\vee} \cdot\, d\mu$. The function f is μ-summable if and only if $f \in J(\mu)$.*

Proof: 1. Clearly, there is a μ-summation, for instance the mapping ${}^{\Omega}\!\!\int^{\vee} \cdot\, d\mu$ (cf. Theorem 12 in [7], p.16, Lemma 1.2, [13], p.93, Satz 3, p.105, Satz 3, Theorems 7 and 8 in [7], p.12, and [13], p.105, Satz 3).

2. There is at most one μ-summation. Let J be a μ-summation. We show that $J = {}^{\mathrm{su}}\!\!\int \cdot\, d\mu$. By the way, we show $J \subseteq {}^{\mathrm{D}}\!\!\int \cdot\, d\mu$.

2a. Let $(f, x) \in J$. By (J 1), f is μ-measurable, thus (by (J 2)) $f^+, f^- \in \mathrm{Dmn}\, J$, thus $x = J(f) = J(f^+) + J(f^-)$, since $f = f^+ + f^-$ and J is linear.

2aα**.** Since f^+ is μ-measurable and non-negative, there exists an increasing sequence (g_n) in $\mathcal{ST}(\mathfrak{X}, \mathbb{R})$ with $g_n \geq 0$ for all $n \in \mathbb{N}$ converging pointwise to f^+. By (J 1) through (J 4) one has $f^+ \in \mathrm{Dmn}\,({}^{\mathrm{su}}\!\!\int \cdot\, d\mu)$ and

(2) $J(f^+) = \lim\limits_{n \in \mathbb{N}} {}^{\mathrm{el}}\!\!\int g_n \, d\mu = {}^{\mathrm{su}}\!\!\int f^+ \, d\mu.$

Furthermore, (g_n) being increasing, one has

$$ {}^{\mathrm{el}}\!\!\int |g_n - g_m| \, d\mu = |\, {}^{\mathrm{el}}\!\!\int g_n \, d\mu - {}^{\mathrm{el}}\!\!\int g_m \, d\mu| \longrightarrow 0 \text{ as } n, m \longrightarrow \infty. $$

Using this and the first sign = in (2), one obtains (by Definition 2.3 and Proposition 2.8/a) $J(f^+) = {}^{\mathrm{D}}\!\!\int f^+ \, d\mu$.

2aβ**.** Since mapping J is linear, one has $-f^- \in \mathrm{Dmn}\, J$. Paraphrasing the preceding paragraph **2a**α with $-f^-$ replacing f^+, one gets $J(-f^-) = {}^{\mathrm{su}}\!\!\int -f^- \, d\mu = {}^{\mathrm{D}}\!\!\int -f^- \, d\mu$.

Combining paragraphs **2a**α and **2a**β and the linearity of the mappings J, ${}^{\mathrm{su}}\!\!\int \cdot\, d\mu$, and ${}^{\mathrm{D}}\!\!\int \cdot\, d\mu$, one obtains $(f, x) \in {}^{\mathrm{su}}\!\!\int \cdot\, d\mu \cap {}^{\mathrm{D}}\!\!\int \cdot\, d\mu$.

2b. We show ${}^{\mathrm{su}}\!\!\int \cdot\, d\mu \subseteq J$. Let $(f, x) \in {}^{\mathrm{su}}\!\!\int \cdot\, d\mu$. Then, there exists an increasing sequence (g_n) in $\mathcal{ST}(\mathfrak{X}, \mathbb{R})$ converging pointwise to f^+ and having the property ${}^{\mathrm{su}}\!\!\int f^+ \, d\mu = \lim\limits_{n \in \mathbb{N}} {}^{\mathrm{el}}\!\!\int g_n \, d\mu$. Using (J 1), the boundedness of the sequence $({}^{\mathrm{el}}\!\!\int g_n \, d\mu)$, and (J 4), one gets $f^+ \in \mathrm{Dmn}\, J$ and ${}^{\mathrm{su}}\!\!\int f^+ \, d\mu = J(f^+)$. Analogously, one has ${}^{\mathrm{su}}\!\!\int (-f^-) \, d\mu = J(-f^-)$. Since ${}^{\mathrm{su}}\!\!\int \cdot\, d\mu$ and J are linear mappings, we have proved $(f, x) \in J$.

3. Combining Part 1 with Part 2 of this proof, we have proved the existence and the uniqueness of a μ-summation and the chain $J(\mu) = {}^{\mathrm{su}}\!\!\int \cdot\, d\mu = {}^{\Omega}\!\!\int^{\vee} \cdot\, d\mu \subseteq {}^{\mathrm{D}}\!\!\int \cdot\, d\mu$. The remainder of the assertion follows by means of Theorem 2.1 and the definition of " μ-summable". □

The conditions (J 2) and (J 4) in Definition 3.1 are allowed to be replaced by weaker ones:

Proposition 3.1 *A linear mapping J from $\mathbb{R}^T$ into $\mathbb{R}$ is a μ-summation in $\mathbb{R}$ if and only if the above statements* (J 1) *and* (J 3) *and the next statements* (J 2′) *and* (J 4′) *hold true:*

(J 2′) *If $f \in \operatorname{Dmn} J$ then $f^+ \in \operatorname{Dmn} J$.*

(J 4′) *If (g_n) is an increasing sequence in $\mathcal{ST}(\mathfrak{X}, \mathbb{R})$ converging pointwise to an element f of E^T and having the property that the sequence $(J(g_n))$ is bounded, then $f \in \operatorname{Dmn} J$ and $\lim_{n \in \mathbb{N}} J(g_n) = J(f)$.*

Proof: If a linear mapping J has the properties (J 1), (J 2′), (J 3), and (J 4′), then (paraphrasing Part 2 of the proof of Theorem 3.1, one gets) $J = {}^{\mathrm{su}}\!\!\int \cdot \, d\mu$, thus, by Theorem 3.1, $J = J(\mu)$. $\square$

We return to the general case of integration in the Banach space E.

Theorem 2.1 and Proposition 2.8/b are supplemented by the next two theorems. [Note: For the statements (a) $\Longleftrightarrow$ (b) and (a) $\Longleftrightarrow$ (c), occuring in the following theorem, cf. Proposition 20 in [5], p.136, *or* Theorem 2 in [4], p.45, and Theorem 3.74 in [17], p.80, respectively.]

Theorem 3.2 *The statements* (a), (b), (c), (d) *defined next are logically equivalent:*

(a) *f is μ-measurable and $\|f\|$ is μ-summable;*

(b) *f is* D*-Bochner integrable.*

(c) *f is* B*-Bochner integrable;*

(d) $\emptyset \neq {}^{\mathrm{D}}\!\!\int^{\wedge} f^{\sim} d\mu = {}^{\mathrm{B}}\!\!\int^{\wedge} f^{\sim} d\mu = {}^{\Omega}\!\!\int f^{\sim} d\mu.$

In particular, (a) *implies* (e)*, defined next:*

(e) *f is Ω-integrable.*

In the case $(E, \|\cdot\|) = (\mathbb{R}, |\cdot|)$, each of the statements (a) *through* (d) *is equivalent to the statement* (e)*.*

Proof: Trivially, (d) implies (b). By Theorem 2.2, (b) implies (d). For the conclusion from (a) to (b) we refer to [4], loc. cit. (which we are allowed to in view of Proposition 3.2 below). (b) implies (c) by Theorem 2.1, while (c) implies (a) by Proposition 33 in [7], p.21, (or by Remark 2.3) and (2.2). The assertion for $E = \mathbb{R}$ follows from Theorem 3.1 by means of (a) $\Longleftrightarrow$ (b). $\square$

Theorem 3.3 *One has* (a) *and, if there is an infinite* $\mathfrak{x} \in \Omega$ *such that* $\mu X > 0$ *for all* $X \in \mathfrak{x}$, (b)*:*

(a) ${}^{\mathrm{D}}\!\!\int \cdot \, d\mu = {}^{\mathrm{B}}\!\!\int \cdot \, d\mu$;

(b) *if* ${}^{\mathrm{B}}\!\!\int \cdot \, d\mu = ({}^{\Omega}\!\!\int \cdot \, d\mu)^{\vee}$ *then the Banach space* E *is finite-dimensional.*

Proof: Ad (a). Use Theorem 3.2.

Ad (b). Choose an $\mathfrak{x} \in \Omega$ as required to exist above; let $\mathfrak{x} = \{X_1, X_2, \cdots\}$. Assume the Banach space E to be infinite-dimensional. Then, by the Dvoretzky-Rogers theorem (see, e.g., [19], p.27,Theorem 6), there is a sequence $(a_n)_{n\in\mathbb{N}}$ in E such that $\underset{n \in \mathbb{N}}{\Sigma} \{a_n\} \neq \emptyset$ and $\underset{n \in \mathbb{N}}{\Sigma} \{\|a_n\|\} = \emptyset$. Define $\chi : \mathfrak{x} \longrightarrow E$ by $\chi(X_n) = a_n \cdot \frac{1}{\mu X_n}$ for all $n \in \mathbb{N}$. Let f be the Ω-step function determined by $(\mathfrak{x}, \chi)$. Then one obtains, by Lemma 1.2, ${}^{\Omega}\!\!\int f^{\sim} d\mu = \underset{n \in \mathbb{N}}{\Sigma} \{a_n\} \neq \emptyset$ and ${}^{\Omega}\!\!\int \|f\|^{\sim} d\mu = \underset{n \in \mathbb{N}}{\Sigma} \{\|a_n\|\} = \emptyset$, therefore, by Theorem 3.2 or Proposition 33 in [7],p.21 (use of the completeness of μ), ${}^{\mathrm{B}}\!\!\int^{\wedge} f^{\sim} d\mu = \emptyset$. Thus ${}^{\mathrm{B}}\!\!\int \cdot \, d\mu \neq ({}^{\Omega}\!\!\int \cdot \, d\mu)^{\vee}$. □

As a consequence of Theorem 3.3 one has: If the Banach space E is infinite-dimensional and if there is an μ-partition as described in Theorem 3.3, then (by Theorem 3.2) the linear space $L = \mathrm{Dmn}({}^{\Omega}\!\!\int \cdot \, d\mu)^{\vee})$ (see Corollary to Proposition 18 in [6], p.495) is not a (μ, Ω)-absolutely integrable subspace of E^T (cf. Definition 2.1 /b).

Theorem 2.2 is supplemented by the following theorem, where the superscript "D" is allowed to be replaced by "B" (in view of Theorem 3.3).

Theorem 3.4 *Assume* f *to be bounded* μ*-a.e. Then the following statements* (a), (b), *and* (c) *are equivalent:*

(a) f *is* μ*-measurable;*

(b) f *is* D*-Bochner integrable;*

(c) $\emptyset \neq {}^{\mathrm{D}}\!\!\int^{\wedge} f^{\sim} d\mu = {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$, *provided that* $\mathfrak{X} \subseteq \mathfrak{Y}$.

Proof: By Theorem 3.2, (b) implies (a). Assume (a). Then, since μ is complete, the function $\|f\|$ is μ-measurable and (by supposition) bounded μ-a.e., therefore (by Lemma 1.4) it is μ-summable. Thus, the combination of Theorems 3.2 and 2.2 provides the statements (b) and (b) $\Longleftrightarrow$ (c). □

Next, we make sure that in the present situation the D-Bochner integral coincides with the Bochner integral used in [4].

Proposition 3.2 $^{\mathrm{D}}\!\!\int \cdot\, d\mu$ *is the set of all* $(f, x) \in \mathcal{M}(\mu, T, E) \times E$ *such that, for some sequence* g *in* $\mathcal{ST}(\mathfrak{X}, E)$*,* (a) *and* (b) *defined next hold true:*

(a) $0 = \lim\limits_{n \in \mathbb{N}} {}^{\mathrm{H}}\!\!\int \|f - g_n\|\, d\mu$;

(b) $x = \lim\limits_{n \in \mathbb{N}} {}^{\mathrm{el}}\!\!\int g_n\, d\mu$.

Proof: 1. By Theorem 3.2, one has Dmn $^{\mathrm{D}}\!\!\int \cdot\, d\mu \subseteq \mathcal{M}(\mu, T, E)$.

2. If g is a sequence in $\mathcal{ST}(\mathfrak{X}, E)$ Ω-converging in mean to an element f of $\mathcal{M}(\mu, T, E)$, then $^{\Omega}\!\!\int \|f - g_n\|^{\sim} d\mu \neq \emptyset$ for eventually all $n \in \mathbb{N}$, say for all $n \geq n_0$, and for those n $^{\mathrm{H}}\!\!\int \|f - g_n\|\, d\mu \in {}^{\Omega}\!\!\int \|f - g_n\|^{\sim} d\mu$. Thus, then (a) holds true.

3. If (a) holds true for some $f \in \mathcal{M}(\mu, T, E)$ and some sequence g in $\mathcal{ST}(\mathfrak{X}, E)$, then eventually all functions $\|f - g_n\|$ are μ-summable, thus g Ω-converges in mean to f.

4. Furthermore, use Proposition 2.7. □

4 Integration w. r. to an arbitrarily fine μ-partition system

Up to now, on the set T no topological structure was given. For the remainder of this paper, we assume that (T, d) be a metric space. For each $X \subseteq T$, we denote by diam(X) the diameter of X w. r. to d; we define the mapping Diam: $\mathfrak{P}\mathfrak{P}T \longrightarrow \overline{\mathbb{R}}$ by letting $\mathrm{Diam}(\mathfrak{x}) = \sup\limits_{X \in \mathfrak{x}} \mathrm{diam}(X)$ for all $\mathfrak{x} \in \mathfrak{P}\mathfrak{P}T$.

Definition 4.1 *We call a set* $\mathfrak{N}$ *of partitions of* T *to be* arbitrarily fine w. r. to d (*or* d-arbitrarily fine)*, if, for each* $\epsilon > 0$*, there is an* $\mathfrak{x} \in \mathfrak{N}$ *such that Diam*$(\mathfrak{x}) < \epsilon$.

Obviously, $\mathfrak{N}$ is d-arbitrarily fine if and only if there exists a sequence $(\mathfrak{x}_n)_{n \in \mathbb{N}}$ in $\mathfrak{N}$ such that $\lim\limits_{n \in \mathbb{N}}{}_{\sigma}\, \mathrm{Diam}(\mathfrak{x}_n) = 0$, where $\lim_\sigma$ denotes the limit operation in the extended real line w.r. to its usual topology σ. If $\mathfrak{R}$ and $\mathfrak{S}$ are sets of partitions of T with $\mathfrak{R} \subseteq \mathfrak{S}$ then the statement "$\mathfrak{R}$ is d-arbitrarily fine" implies "$\mathfrak{S}$ is d-arbitrarily fine".

Example 4.1 Let $n \in \mathbb{N}$ and ν be the Lebesgue-Borel measure (for the terminology, see [1], p.37) or the Lebesgue measure on $\mathbb{R}^n$; let T be a non-empty compact subset of $\mathbb{R}^n$. Define μ by $\mu X = \nu X$ for all $X \in$Dmn$\,\nu$ with $X \subseteq T$. Then, $\mathfrak{X}$ is arbitrarily fine w. r. to the Euclidian metric of $\mathbb{R}^n$ restricted to $T \times T$.

Remark 4.1 The comparison of the notion of an "arbitrarily fine μ-partition system" with proposition 8.3.3.1 in [13], p.184, on a measure being "eng adaptiert" to a topology suggests to generalize the preceding considerations by replacing there the metric space (T, d) by a topological space (T, ρ) or a uniform space $(T, \mathfrak{U})$ and modifying the notion "d-arbitrarily fine" in a suitable way. Given $(T, \mathfrak{U})$, a set $\mathfrak{N}$ of partitions of T should be called $\mathfrak{U}$-*arbitrarily fine*, if, for each $U \in \mathfrak{U}$, there is an $\mathfrak{x} \in \mathfrak{N}$ such that $X \times X \subseteq U$ holds for all $X \in \mathfrak{x}$. Of a special interest, of course, would be, in this context, the topology $\rho(\mathfrak{U})$ induced by $\mathfrak{U}$. Is there a natural relationship between the property of a μ-partition system to be "$\mathfrak{U}$-arbitrarily fine" and and the property of a measure to be "eng adaptiert" to $\rho(\mathfrak{U})$?

For the remainder of this paper, let $\mathfrak{Z}$ be a third *μ-partition system* (beside $\mathfrak{X}$ and $\mathfrak{Y}$).

Proposition 4.1 *Let (T, d) be compact and assume that $\mathfrak{Z}$ is* d*-arbitrarily fine. If f is continuous* (*w. r. to* d *and the norm of E*)*, then one has* (a) *and* (b)*:*

(a) *There exists a sequence $g : \mathbb{N} \longrightarrow \mathcal{ST}(\mathfrak{Z}, E)$ converging uniformly to f.*

(b) *If $\mathfrak{Z} \subseteq \mathfrak{X} \subseteq \mathfrak{Y}$, then ${}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu$ is non-empty.*

Proof: Since $\mathfrak{Z}$ is d-arbitrarily fine, there exists a sequence $(\mathfrak{z}_n)_{n \in \mathbb{N}}$ in $\mathfrak{Z}$ with the property $\lim\limits_{n \in \mathbb{N}}{}^{\sigma} \mathrm{Diam}(\mathfrak{z}_n) = 0$. Choose $\phi \in \underset{n \in \mathbb{N}}{\mathrm{P}} \mathcal{P}_{\mathfrak{z}_n}$ and put $\phi(n) = \phi_n$ for each $n \in \mathbb{N}$. Define, for each $n \in \mathbb{N}$, a $\mathfrak{Z}$-step function $g_n : \mathfrak{z}_n \longrightarrow E$ by letting, for each $t \in T$,

$$g_n(t) = f(\phi_n(Z)), \text{ if } t \in Z \in \mathfrak{z}_n \,.$$

Let $\epsilon > 0$. Since f is continuous and T is compact, there is a $\delta > 0$ such that for all $t_1, t_2 \in T$

$$\|f(t_1) - f(t_2)\| < \epsilon, \text{ if } \mathrm{d}(t_1, t_2) < \delta \,. \tag{1}$$

There is an $n_0 \in \mathbb{N}$ such that

$$\mathrm{Diam}(\mathfrak{z}_n) < \delta \text{ for all } n \geq n_0 \,. \tag{2}$$

Let $n \geq n_0$ and $t \in T$. Then, there is a Z with $t \in Z \in \mathfrak{z}_n$. One has $\mathrm{diam}(Z) < \delta$ by (2), therefore, by (1), $\|g_n(t) - f(t)\| < \epsilon$, since $g_n(t) = f(\phi_n(Z))$. Thus (a) is proven. (b) follows from (a) by means of Theorem 2.2/b. □

We conclude this paper with a short view to the Riemann integration:

Definition 4.2 (a) *For each $\mathfrak{x}, y \in \mathfrak{Y}$, let $\mathfrak{x} \preceq_{\mathrm{R}} \mathfrak{y}$, if Diam$(\mathfrak{y}) \leq$ Diam$(\mathfrak{x})$.* (*Remark: Since $(\mathfrak{Y}, \preceq)$ is a directed set, also $(\mathfrak{Y}, \preceq_{\mathrm{R}})$ is a directed set.*)

(b) *For each $(\mathfrak{x}, \phi), (\mathfrak{y}, \psi) \in \mathfrak{Y}^{\#}$, let $(\mathfrak{x}, \phi)(\preceq_{\mathrm{R}})^{\#}(\mathfrak{y}, \psi)$, if $\mathfrak{x} \preceq_{\mathrm{R}} \mathfrak{y}$.* (*Remark: $(\mathfrak{Y}^{\#}, (\preceq_{\mathrm{R}})^{\#})$ is a directed set. We denote by $\mathcal{F}(\mathfrak{X}^{\#}, (\preceq_{\mathrm{R}})^{\#})$ the filter of perfinality on this directed set* (*see Section 1/b*)*.*)

(c) *The $(\mu, \mathfrak{Y})$*-Riemann integral *${}^{\mathrm{R}\mathfrak{Y}}\!\!\int \cdot \, d\mu$ is the mapping on $\mathfrak{E}^T$ into $\mathfrak{E}$ being defined by letting, for each $g \in \mathfrak{E}^T$,*

$$^{\mathrm{R}\mathfrak{Y}}\!\!\int g\,d\mu \;=\; {}^{\mathcal{F}(\mathcal{Y}^{\#},\,(\preceq_{\mathrm{R}})^{\#})}\lim_{(\mathfrak{x},\,\phi)\,\in\,\mathfrak{Y}^{\#}} \mathrm{R}(f,\mathfrak{x},\phi)\,.$$

(*For the terminology, see Section* 1/d *and* f.)

The following example opens a wide range of investigation in the survey paper [8]:

Example 4.2 Let $T=[a,b]\subseteq\mathbb{R}$ with $a<b$. Take μ as in Example 4.1. Define $\mathfrak{Y}_1$ to be set of all finite partitions of $[a,b[$ into half-open intervals of the form $[c,d[$ with $a\le c<d\le b$ and let $\mathfrak{Y}=\{\mathfrak{y}\cup\{\{b\}\}\mid\mathfrak{y}\in\mathfrak{Y}_1\}$. Alternatively, define $\mathfrak{Y}$ to be set of all finite partitions of T into intervals, where also half-open intervals and singletons $[c,c]$ with $c\in T$ are admitted. In both cases, $({}^{\mathfrak{Y}}\!\!\int\cdot\,d\mu)^{\vee}$ (see Propositon 18 in [6]) coincides with the integral defined in [8], p.924, Definition 2(b) there, while $({}^{\mathrm{R}\mathfrak{Y}}\!\!\int\cdot\,d\mu)^{\vee}$ coincides with the integral defined in [8], p.924, Definition 2(a) there. (For the latter use of $^{\vee}$, see the terminology introduced in Section 0 in [6], which is applicable here by Proposition 18 in [6] and Proposition 4.2 below.) By Theorem 3 in [8], p.924, one has ${}^{\mathrm{R}\mathfrak{Y}}\!\!\int\cdot\,d\mu={}^{\mathfrak{Y}}\!\!\int\cdot\,d\mu$.

Since $(\preceq^{\#})\subseteq((\preceq_{\mathrm{R}})^{\#})$, one obtains

Proposition 4.2 *For each $g\in\mathfrak{E}^T$, one has the inclusion ${}^{\mathrm{R}\mathfrak{Y}}\!\!\int g\,d\mu\subseteq{}^{\mathfrak{Y}}\!\!\int g\,d\mu$.*

Proposition 4.3 *If ${}^{\mathrm{R}\mathfrak{Y}}\!\!\int f^{\sim}d\mu$ and ${}^{\mathrm{R}\mathfrak{Z}}\!\!\int f^{\sim}d\mu$ are non-empty, then*

$${}^{\mathrm{R}\mathfrak{Y}}\!\!\int f^{\sim}d\mu={}^{\mathrm{R}\mathfrak{Z}}\!\!\int f^{\sim}d\mu\,.$$

Proof: By Lemma 1.1 and Proposition 4.2, one has ${}^{\mathrm{R}\mathfrak{Y}}\!\!\int f^{\sim}d\mu\subseteq{}^{\Omega}\!\!\int f^{\sim}d\mu$ and ${}^{\mathrm{R}\mathfrak{Z}}\!\!\int f^{\sim}d\mu\subseteq{}^{\Omega}\!\!\int f^{\sim}d\mu$. Now, use the premise. □

Proposition 4.4 *Let (T,d) be compact; assume $\mathfrak{Y}\subseteq\mathfrak{X}$ and $\mathfrak{Y}$ to be* d*-arbitrarily fine. Then, one has: If f is continuous, then ${}^{\mathrm{R}\mathfrak{Y}}\!\!\int f^{\sim}d\mu$ is non-empty.*

Proof: We skip the trivial case $\mu T=0$ and assume $\mu T>0$. Let $\epsilon>0$. Since T is compact and f is continuous, there is a $\delta>0$ such that

$$\text{for all } s,t\in T,\ \mathrm{d}(s,t)<\delta \text{ implies } \|f(s)-f(t)\|<\frac{\epsilon}{\mu T}\,. \tag{1}$$

Because $\mathfrak{Y}$ is d-arbitrarily fine, there is an $\mathfrak{x}_0\in\mathfrak{Y}$ such that

$$\mathrm{Diam}(\mathfrak{x}_0)<\frac{\delta}{2}\,. \tag{2}$$

Let $\mathfrak{x},\mathfrak{y}\in\mathfrak{Y}$ such that $\mathfrak{x}_0\preceq_{\mathrm{R}}\mathfrak{x},\mathfrak{y}$. Furthermore, let $\phi\in\mathcal{P}\mathfrak{x}$ and $\psi\in\mathcal{P}\mathfrak{y}$. Since $(\mathfrak{Y},\preceq)$ is a directed set, there is a $\mathfrak{z}\in\mathfrak{Y}$ with $\mathfrak{x},\mathfrak{y}\preceq\mathfrak{z}$. Choose such a $\mathfrak{z}$. For all $Z\in\mathfrak{z}$, there are exactly one $\alpha(Z)\in\mathfrak{x}$ such that $Z\subseteq\alpha(Z)$ and exactly one $\beta(Z)\in\mathfrak{y}$ such that $Z\subseteq\beta(Z)$. Define the mappings $g,h:\mathfrak{z}\longrightarrow E$ by letting $g(Z)=f(\phi(\alpha(Z)))$ and $h(Z)=f(\psi(\beta(Z)))$ for all $Z\in\mathfrak{z}$. Using (2) and the relationship between $\mathfrak{x}_0$, $\mathfrak{x}$, $\mathfrak{y}$, and $\mathfrak{z}$, one obtains $\mathrm{d}(\phi(\alpha(Z)),\psi(\beta(Z)))<\delta$ for all $Z\in\mathfrak{z}$, thus, using (1) and the relationship between $\mathfrak{x}$, $\mathfrak{y}$, and $\mathfrak{z}$, the chain

$$\|\mathrm{R}(f^\sim, \mathfrak{x}, \phi) - \mathrm{R}(f^\sim, \mathfrak{y}, \psi)\| \leq \sum_{Z \in \mathfrak{z}} \|g^\sim(Z) - h^\sim(Z)\|\mu Z < \epsilon\,,$$

where the three occuring sums are singletons, since $\mathfrak{x}$, $\mathfrak{y}$, and $\mathfrak{z}$ are finite sets. (Observe that, e.g., $\sum_{Z \in \mathfrak{z}} g(Z)\mu Z \in \mathrm{R}(f^\sim, \mathfrak{x}, \phi)$ holds, since $\mathfrak{x} \preceq \mathfrak{z}$ (use of Proposition 20 and Lemma 1 in [6]).) We have shown that the net

$$(j, \mathfrak{Y}^\#, (\preceq_{\mathrm{R}})^\#) \text{ with } j(\mathfrak{u}, \kappa) = \mathrm{e}(\mathrm{R}(f^\sim, \mathfrak{u}, \kappa)) \text{ for all } (\mathfrak{u}, \kappa) \in \mathfrak{Y}^\#$$

is a Cauchy net in the Banach space $(E, \|\cdot\|)$, therefore (by Definition 4.2) the set ${}^{\mathrm{R}\mathfrak{Y}}\!\!\int f^\sim d\mu$ is non-empty. (For the definition of the mapping e, see Section 1/a.) □

Of course, Proposition 4.1/b (even if one supposes there $\mathfrak{Z} \subseteq \mathfrak{X} \cap \mathfrak{Y}$ instead of $\mathfrak{Z} \subseteq \mathfrak{X} \subseteq \mathfrak{Y}$) follows also from the Propositions 4.2 and 4.4 (use of Lemma 1.1); see also Proposition 4.6/b.

For the remainder of this paper, we assume $\mathfrak{Z}$ to be d-arbitrarily fine and T to be compact. We add now some remarks on the integration of continuous functions. – As in Remark 2.1, we denote by L_2 the linear space of all those D-Bochner integrable functions $g : T \longrightarrow E$, which are bounded μ-a.e. In the following, $\mathcal{C}(T, E)$ denotes the set of all (d, $\|\cdot\|$)-continuous functions on T into E.

Proposition 4.5 *If $f \in \mathcal{C}(T, E)$ and $\mathfrak{Z} \subseteq \mathfrak{X}$, then ${}^{\mathrm{R}\mathfrak{Z}}\!\!\int f^\sim d\mu = {}^{\mathrm{R}\mathfrak{X}}\!\!\int f^\sim d\mu$.*

Proof: Combine Propositions 4.4 and 4.3. □

In the sense of Proposition 4.5, for continuous f, the integral ${}^{\mathrm{R}\mathfrak{X}}\!\!\int f\,d\mu$ represents all integrals ${}^{\mathrm{R}\mathfrak{W}}\!\!\int f\,d\mu$, $\mathfrak{W}$ being a d-arbitrarily fine μ-partition system of T with $\mathfrak{W} \subseteq \mathfrak{X}$. So, it seems to be justified to reserve a special symbol, namely ${}^{\mathrm{R}}\!\!\int \cdot\, d\mu$, for ${}^{\mathrm{R}\mathfrak{X}}\!\!\int \cdot\, d\mu$, and to call ${}^{\mathrm{R}}\!\!\int \cdot\, d\mu$ the *μ-Riemann integral.* (This terminology deviates from that used in [8], p.925, definition of "Riemann integrable". See our Example 4.2: There, for continuous f (by Proposition 4.4 and Lemma 1.1) ${}^{\mathrm{R}\mathfrak{Y}}\!\!\int f^\sim d\mu = {}^{\mathrm{R}}\!\!\int f^\sim d\mu = {}^{\mathfrak{X}}\!\!\int f^\sim d\mu$.)

Proposition 4.6 *One has* (a) *and* (b):

(a) *If $\mathfrak{X}$ is d-arbitrarily fine, then $\mathcal{C}(T, E) \subseteq L_2$.*

(b) *If $f \in \mathcal{C}(T, E)$, then ${}^{\mathrm{R}\mathfrak{Z}}\!\!\int f^\sim d\mu = {}^{\mathrm{D}}\!\!\int^\wedge f^\sim d\mu = {}^{\mathfrak{Y}}\!\!\int f^\sim d\mu \neq \emptyset$, provided that $\mathfrak{Z} \subseteq \mathfrak{X} \cap \mathfrak{Y}$.*

Proof: Ad (a). Assume $\mathfrak{X}$ to be d-arbitrarily fine. Let $f \in \mathcal{C}(T, E)$. Then, by Proposition 4.1, there is a sequence $h : \mathbb{N} \longrightarrow \mathcal{ST}(\mathfrak{X}, E)$ converging uniformly to f. Thus, by

Theorem 2.2/b, one has $(\star)\ \emptyset \neq {}^{\mathrm{D}}\!\!\int^{\wedge} f^{\sim} d\mu = {}^{\mathfrak{X}}\!\!\int f^{\sim} d\mu$; therefore, the function f is D-Bochner integrable, while, being continuous on a compact set, it is bounded. Thus, one has $f \in L_2$.

Ad (b). Let $\mathfrak{Z} \subseteq \mathfrak{X} \cap \mathfrak{Y}$. Using the Propositions 4.2, 4.4, and Lemma 1.1, one gets ${}^{\mathrm{R}\mathfrak{Z}}\!\!\int f^{\sim} d\mu = {}^{\mathfrak{Y}}\!\!\int f^{\sim} d\mu = {}^{\mathfrak{X}}\!\!\int f^{\sim} d\mu$, therefore (using the statement $(\star)$ which is applicable, since $\mathfrak{X}$ containing $\mathfrak{Z}$ as a subset is d-arbitrarily fine) the assertion (b). $\square$

From Theorem 2.3, we regain a classical result on the Riemann integration being contained in

Proposition 4.7 *If $(g, K, \mathfrak{b})$ is a filtered family in $\mathcal{C}(T, E)$ converging uniformly to f, then one has:*

$$\emptyset \neq {}^{\mathfrak{b}}\lim_{k \in K} {}^{\mathrm{R}\mathfrak{Z}}\!\!\int g_k{}^{\sim} d\mu = {}^{\mathrm{R}\mathfrak{Z}}\!\!\int f^{\sim} d\mu, \textit{ provided that } \mathfrak{Z} \subseteq \mathfrak{X}\,.$$

Proof: By the premise, f is continuous. Now, apply Proposition 4.6 (to f and all g_k with $k \in K$) and then Corollary 2.1. $\square$

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[1] Printing errors: On p.489, line 23 from above, replace "0" by "$\emptyset$". On p.490, line 23 from above, move the subscript "$i \in I$" of the sign "$\sum$" as a subscript to the sign "P".

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[2]Printing errors: On p.20, lines 17–18 from below, move the subscript "$n \in \mathbf{N}$" of the integral sign as a subscript to the sign "lim". On p.21, lines 13 from below, replace "$h^\star$" by "h". On p.21, line 10 from below, replace the superscript "$^\wedge$" by the superscript "$^\vee$".

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