Rostock. Math. Kolloq. **54**, 39–50 (2000)

Subject Classification (AMS)
54A20, 54B30, 54E05, 54E15, 54E17

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A note on antitonic convergence

This paper is dedicated in honour to Professor Harry Poppe with congratulations to his 65th birthday and also to Professors Gerhard Maeß and Günther Wildenhain on the occasions of their 60th birthdays

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0 Introduction

In the joint paper [2], Bentley, Herrlich and Lowen-Colebunders noted that $Conv_S$, the category of symmetric convergence spaces, and Chy, the category of Cauchy spaces, can be fully embedded into the Katětov's category Fil of filter-merotopic spaces [9]. Fil is a bicoreflective subcategory of Mer, the category of merotopic spaces, which is closely related to the concept of nearness introduced by Herrlich [8] who basically uses notions of set systems which are near. Katětov proved that Fil is cartesian closed and that the corresponding function space structure is the one of continuous convergence. As pointed out above, not all convergence spaces can be described by Fil, only the symmetric ones can be. Supertopological spaces in the sense of Doitchinov [5] or, more generally, the so called neighborhood spaces of Tozzi and Wyler [17], are a common generalization of all topological spaces and proximity spaces but fail to form a cartesian closed category.

Preuß [15] introduced semiuniform convergence spaces as a common generalization of (symmetric) limit spaces (and thus of symmetric topological spaces) as well as of uniform limit spaces (and thus of uniform spaces) with many convenient properties like cartesian closedness, hereditariness, and the fact that products of quotients are quotients again, which together build a strong topological universe. The latter mentioned two concepts lead us to define a notion of convergence in a more general setting. Moreover, it looks like as a first step in finding a more general concept which contains all the above mentioned spaces and, additionally, is being more set-like enough to allow constructions such as formation of functions spaces or universal one-point extensions. As a basic concept we use the convergence of filters to bounded subsets defined similarly as in the Wyler's paper [18]. But as for construction of function spaces with respect to evaluation maps our definition seems to be more natural because in some special cases (such as limit spaces or merotopic spaces) we obtain the usual function space structures.

1 Superfilterformities and related structures

Definition 1.1 *For a set X, a subset $\mathcal{B}^X \subseteq \mathcal{P}X$ (where $\mathcal{P}X$ denotes the set of all subsets of X) is called a* prebornology *or shortly a* B-structure *on X and the elements of $\mathcal{B}^X$ are called* bounded sets *if the following axioms are satisfied:*

(B1) $B' \subseteq B \in \mathcal{B}^X$ *implies* $B' \in \mathcal{B}^X$,

(B2) $\emptyset \in \mathcal{B}^X$,

(B3) $x \in X$ *implies* $\{x\} \in \mathcal{B}^X$.

Given a pair of B-structures $\mathcal{B}^X$ and $\mathcal{B}^Y$ on sets X and Y respectively, a map $f : X \to Y$ is called bounded *iff* $\{f[B]; B \in \mathcal{B}^X\} \in \mathcal{B}^Y$.

Remark 1.2 The category $BOUND$ whose objects are pairs $(X, \mathcal{B}^X)$ where X is a set and $\mathcal{B}^X$ is a B-structure on X and whose morphisms are bounded maps is topological and cartesian closed and has universal one-point extensions, which means that $BOUND$ is a *topological universe*.

Definition 1.3 *Given a B-structure $\mathcal{B}^X$ on a set X, a map $C : \mathcal{B}^X \to \mathcal{P}(\mathcal{P}(\mathcal{P}X))$ is called an* antitonic convergence operator *provided that the following axioms are satisfied:*

(ac1) $\mathcal{H}_1 \in C(B) \wedge \mathcal{H}_1 \ll \mathcal{H}_2$ *imply* $\mathcal{H}_2 \in C(B)$, *where* $\mathcal{H}_1 \ll \mathcal{H}_2$ *iff for each* $F_1 \in \mathcal{H}_1$ *there exists* $F_2 \in \mathcal{H}_2$ *such that* $F_2 \subseteq F_1$,

(ac2) $\emptyset \notin C(\emptyset)$ *and* $\{\emptyset\} \in C(X)$,

(ac3) $B_1 \subseteq B_2 \in \mathcal{B}^X$ *implies* $C(B_2) \subseteq C(B_1)$.

Examples 1.4 (i) For a preuniform convergence space $(X, \mathcal{J}_X)$ and $A \in \mathcal{P}X$ we put $C_{\mathcal{J}_X}(A) := \{\mathcal{F}; \mathcal{F} \times \dot{A} \in \mathcal{J}_X\}$ where $\mathcal{F} \times \dot{A}$ denotes the set generated by $\{F \times B; F \in \mathcal{F}, B \supseteq A\}$.

(ii) For a syntopogenous structure $\mathcal{S}$ on a set X in the sense of Császár [3] we put $C_{\mathcal{S}}(A) := \{\mathcal{F}; \exists <\in \mathcal{S}\ \forall x \in A :< (x) \subseteq \mathcal{F}\}$ where $< (x) := \{T \subseteq X; \{x\} < T\}$.

(iii) For a symmetrical semi-topogenous order $<$ on a set X in the sense of Császár [3] we put $C_<(A) := \{\mathcal{H}; <(B) \ll \mathcal{H}\}$ where $<(B) := \{F \subseteq X; X \setminus B < X \setminus F\}$.

(iv) For a convergence structure q on a set X we put $C_q(A) := \{\mathcal{F}^*; \forall x \in A\ \exists \mathcal{F} \in q(x) : \mathcal{F} \ll \mathcal{F}^*\}$.

(v) For a merotopy Γ on a set X we put $C_\Gamma(A) := \Gamma$.

At last we will mention an example with respect to generalized supertopological spaces in the sense of Doitchinov [5], [16]:

(vi) For a neighborhood space $(\mathcal{B}^X, \Theta)$ where Θ is a function from $\mathcal{B}^X$ to $\mathcal{P}(\mathcal{P}(\mathcal{P}X)))$ satisfying certain axioms we put $C_\Theta(B) := \{\mathcal{F}; \forall x \in B : \Theta(\{x\}) \ll \mathcal{F}\}$.

Remark 1.5 In all the above examples, if we reduce set systems to filters, we obtain the notion of *filter convergence*. Moreover, it is interesting to note that in the case $\mathcal{B}^X = \mathcal{P}X$ we can easily construct a bijection between the set of all *antitonic convergence operators* and the set of all *isotonic operators* as follows: For an antitonic convergence operator C and $A \in \mathcal{P}X$ we put $N_C(A) := \{\mathcal{H}; sec\mathcal{H} \in C(X \setminus A)\}$ where $sec\mathcal{H} := \{T \subseteq X; \forall F \in \mathcal{H} : F \cap T \neq \emptyset\}$. Then N_C fulfils the following axioms:

(in1) $\mathcal{H}_1 \in N_C(A) \wedge \mathcal{H}_2 \ll \mathcal{H}_1$ imply $\mathcal{H}_2 \in N_C(A)$,

(in2) $\emptyset \in N_C(\emptyset)$ and $\{\emptyset\} \notin N_C(X)$,

(in3) $B_1 \subseteq B_2$ implies $N_C(B_1) \subseteq N_C(B_2)$.

By imposing some next axioms we obtain the notion of a *near operator* and, more generally, if $\mathcal{B}^X$ is not specified, we get a *supernearness*, both introduced by myself [12], [13].

Now, after giving the previous background, we will define *superfilterformities* and consider some their relationships to other structures and also construct for them natural function space structures with respect to evaluation maps.

Definition 1.6 *For a set X, $FIL(X)$ denotes the set of all filters on X. A function $F : \mathcal{B}^X \to \mathcal{P}(FIL(X))$, where $\mathcal{B}^X$ is a B-structure on X, is called a* superfilterformity *and the pair $(\mathcal{B}^X, F)$ is called a* superfilterformic space *iff the following axioms are satisfied:*

(SF1) *If $B \in \mathcal{B}^X$, $\mathcal{F}_1 \in F(B)$ and $\mathcal{F}_2 \in FIL(X)$, then $\mathcal{F}_1 \subseteq \mathcal{F}_2$ implies $\mathcal{F}_2 \in F(B)$,*

(SF2) *$x \in X$ implies $\dot{x} \in F(x)$ where $\dot{x} := \{T \subseteq X; x \in T\}$,* (SF3) $B_1 \subseteq B_2 \in \mathcal{B}^X$ *implies* $F(B_2) \subseteq F(B_1)$, *which means that* F *is an* antitonic *function.*

Elements of $F(B)$ *are called* B-convergent *filters.*

Given a pair of superfilterformic spaces $(\mathcal{B}^X, F_X)$ *and* $(\mathcal{B}^Y, F_Y)$, *a bounded map* $f : X \to Y$ *is called an* sf-map *iff*

(sf) $B \in \mathcal{B}^X$ *and* $\mathcal{F}_1(B)$ *imply* $f(\mathcal{F}) \in F_2(B)$, *where* $f(\mathcal{F}) := \{G \subseteq Y; f^{-1}[G] \in \mathcal{F}\}$.

We also refer as to an sf-map f *by saying it preserves convergent filters in the above mentioned sense.*

We denote by $SUPFILFORM$ the category whose objects are the superfilterformic spaces and whose morphisms are the bounded maps which preserve convergent filters.

Examples 1.7 (i) Let $(\mathcal{B}^X, \Theta)$ be a subadditive neighborhood space, which means Θ is a function from $\mathcal{B}^X$ to $FIL(X)$ satisfying certain axioms including the following one:

(SA) $B \in \mathcal{B}^X$ implies $\bigcap\limits_{x \in B} \Theta(\{x\}) = \Theta(B)$.

Note that each subadditive neighborhood space $(\mathcal{B}^X, \Theta)$ is additive, which means Θ has the property:

(A) $\Theta(B' \cup B) = \Theta(B') \cap \Theta(B)$ whenever $B' \cup B \in \mathcal{B}^X$.

Further note that every neighborhood structure Θ on $\mathcal{D}^X := \{\emptyset\} \cup \{\{x\}; x \in X\}$, i.e. each pretopology is subadditive. Define $F_\Theta(B) := \{\mathcal{F} \in FIL(X); \forall x \in B : \Theta(\{x\}) \subseteq \mathcal{F}\}$.

(ii) For a limit space (X, q), we consider $\mathcal{P}X$ equipped with the superfilterformity F_q defined as follows:

$F_q(B) := \{\mathcal{F}' \in FIL(X); \forall x \in B\ \exists \mathcal{F} \in q(x) : \mathcal{F} \subseteq \mathcal{F}'\}$.

(iii) For a filtermerotopic space (X, Γ), we consider $\mathcal{P}X$ equipped with the superfilterformity F_Γ defined as follows:

$F_\Gamma(B) := \Gamma$ for each set $B \in \mathcal{P}X$.

(iv) Lastly, let $(X, \mathcal{J}_X)$ be a preuniform convergence space and let $\mathcal{B}^X$ be a prebornology. We consider $\mathcal{B}^X$ equipped with the superfilterformity $F_{\mathcal{J}_X}$ defined as follows:

$F_{\mathcal{J}_X}(B) := \{\mathcal{F} \in FIL(X); \mathcal{F} \times \dot{B} \in \mathcal{J}_X\}$ where $\mathcal{F} \times \dot{B}$ denotes the filter generated by $\{F \times B'; F \in \mathcal{F}, B' \supseteq B\}$.

In this context it seems to be of interest to search the question whether each superfilterformity is induced by a preuniform convergence space or, more precisely, whether we can construct to each superfilterformity F' a preuniform convergence structure $\mathcal{J}$ such that $F_{\mathcal{J}} = F'$. But this is not the aim of this paper. Nevertheless, an affirmative answer to this question is given at the end of the paper.

Theorem 1.8 *The category $SASNBD$ of subadditive neighborhood spaces and usual morphisms is isomorphic to a full subcatefory of $SUPFILFORM$.*

Proof: With respect to 1.7(i), let $(\mathcal{B}^X, F^*)$ be a superfilterformic space. Define an underlying neighborhood structure Θ_{F^*} by setting $\Theta_{F^*}(B) := \{U \subseteq X; \forall x \in B \forall \mathcal{F} \in F^*(\{x\}) : U \in \mathcal{F}\}$ for each $B \in \mathcal{B}^X$. Then we get an isomorphism between the category $SASNBD$ and a corresponding full subcategory of $SUPFILFORM$.

Theorem 1.9 *The category LIM of limit spaces and convergence preserving maps is isomorphic to a full subcatergory of $SUPFILFORM$.*

Proof: With respect to 1.7(ii), let $(\mathcal{P}X, F)$ be a superfilterformic space. Define an underlying convergence space [6] (X, q_F) by setting $q_F(X) := F(\{x\})$ for each $x \in X$. Imposing some aditional axiom on F, q_F is a limit structure. Therefore we obtain an isomorphism between the category LIM and a corresponding full subcategory of $SUPFILFORM$.

Theorem 1.10 *The category $FMER$ of filter-merotopic spaces and convergence preserving maps is isomorphic to a full subcategory of $SUPFILFORM$.*

Proof: With respect to 1.7(iii), let $(\mathcal{P}X, F)$ be a superfilterformic space. Define an underlying filter merotopy Φ_F by setting $\Phi_F := F(\emptyset)$. We obtain an isomorphism between the catergory $FMER$ and a corresponding full subcategory of $SUPFILFORM$.

2 Categorical properties of $SUPFILFORM$

First we note that superfilterformities on a B-structure $\mathcal{B}^X$ can be naturally ordered by putting $F_1 \leq F_2 :\Leftrightarrow \forall B \in \mathcal{B}^X : F_1(B) \subseteq F_2(B)$. The $SUPFILFORM$ fiber of $\mathcal{B}^X$ is a set and it will be denoted by $SUPFILFORM(\mathcal{B}^X)$ (i.e. $SUPFILFORM(\mathcal{B}^X)$ is the set of all superfilterformities on $\mathcal{B}^X$). In addition we observe that from the categorical point of view $SUPFILFORM(\mathcal{B}^X)$ has the terminal separator property, which means that $SUPFILFORM(\mathcal{B}^X)$ is a singleton whenever X is a singleton. We state that $\mathcal{B}^X$ equals to $\{\emptyset, X\}$ in this case. Moreover, the null filter generated by the empty set is allowed to be an element of $FIL(X)$. Thus, to prove that $SUPFILFORM$ is a topological category it is sufficient to show that it has initial structures.

Theorem 2.1 *For any set X, any family $(\mathcal{B}^{X_i}, F_i)_{i\in I}$ of superfilterformic spaces and any family $(f_i : X \to X_i)_{i\in I}$ of functions there exists a unique prebornology $\mathcal{B}^X_{\{f_i^{-1}\}}$ and a unique superfilterformity $F_{\{f_i^{-1}\}}$ on $\mathcal{B}^X_{\{f_i^{-1}\}}$ which is initial with respect to the given data $(\mathcal{B}^X_{\{f_i^{-1}\}}, f_i, (\mathcal{B}^{X_i}, F_i), I)$, i.e. such that for any superfilterformic space $(\mathcal{B}^Y, F)$ a map $g : (\mathcal{B}^Y, F) \to (\mathcal{B}^X_{\{f_i^{-1}\}}, F_{\{f_i^{-1}\}})$ is an sf-map iff for every $i \in I$ the composite map $f_i \circ g : (\mathcal{B}^Y, F) \to (\mathcal{B}^{X_i}, F_i)$ is an sf-map.*

Proof: We define the prebornology $\mathcal{B}^X_{\{f_i^{-1}\}}$ as follows:

$$\mathcal{B}^X_{\{f_i^{-1}\}} := \{B \subseteq X; \forall i \in I : f_i[B] \in \mathcal{B}^{X_i}\}.$$

Note that $\mathcal{B}^X_{\{f_i^{-1}\}}$ is the initial B-structure on X with respect to the above mentioned data. Moreover, we define $F_{\{f_i^{-1}\}}$ analogously in such a natural manner, namely:

$$F_{\{f_i^{-1}\}}(B) := \{\mathcal{F} \in FIL(X); \forall i \in I : f_i(\mathcal{F}) \in F_i(f_i[B])\} \text{ for each } B \in \mathcal{B}^X_{\{f_i^{-1}\}}).$$

Then $(\mathcal{B}^X_{\{f_i^{-1}\}}, F_{\{f_i^{-1}\}}$ is a superfilterformic space, and if $(\mathcal{B}^Y, F)$ is a superfilterformic space such that $g : (\mathcal{B}^Y, F) \to (\mathcal{B}^X_{\{f_i^{-1}\}}, F_{\{f_i^{-1}\}})$ is an sf-map, for every $i \in I$ the composite map $f_i \circ g : (\mathcal{B}^Y, F) \to (\mathcal{B}^{X_i}, F_i)$ is also an sf-map. Conversely, let latter situation be given. We are to show that $g : (\mathcal{B}^Y, F) \to (\mathcal{B}^X_{\{f_i^{-1}\}}, F_{\{f_i^{-1}\}})$ is an sf-map. So, let $\mathcal{F}$ be a B-convergent filter in Y. Our goal is to verify that $g(\mathcal{F})$ is a $g[B]$-convergent filter in X. To this end, let $i \in I$. We have $(f_i \circ g)(\mathcal{F}) \in F_i((f_i \circ g)[B]) = F_i(f_i[g[B]])$. Hence $g(\mathcal{F}) \in F_{\{f_i^{-1}\}}(g[B])$. Note also that $F_{\{f_i^{-1}\}}$ is uniquely determined.

Corollary 2.2 *SUPFILFORM is a topological category.*

Remark 2.3 The existence of initial structures in *SUPFILFORM* implies, by using purely categorical arguments, the existence of final structures in *SUPFILFORM*. So *SUPFILFORM* is complete and its limits (subspaces or products) are formed by supplying the corresponding limits in *SET* (the category of sets and functions) with the initial structures. Moreover, it is cocomplete and colimits (quotients or sums) are formed by supplying the corresponding colimits in *SET* with the final structures. Finally, we mention the fact that *SUPFILFORM* is also wellpowered and cowellpowered in consequence of the above details.

3 Function space structures with respect to evaluation maps

Theorem 3.1 *For any pair $(\mathcal{B}^X, F_1)$, $(\mathcal{B}^Y, F_2)$ of superfilterformic spaces, the set $Y^X := \{f; f : (\mathcal{B}^X, F_1) \to (\mathcal{B}^Y, F_2)$ is an sf-map$\}$ can be supplied in a natural way with a prebornology $\mathcal{B}^{Y^X}$ and this prebornology (which is a B-structure) can be then supplied with a* *superfilterformity F_{Y^X} such that the evaluation map $e : (\mathcal{B}^X, F_1) \times (\mathcal{B}^{Y^X}, F_{Y^X}) \to (\mathcal{B}^Y, F_2)$, $(x, f) \mapsto f(x)$, preserves convergent filters.*

Proof: We define $\mathcal{B}^{Y^X}$ by setting $\mathcal{B}^{Y^X} := \{B^* \subseteq Y^X; \forall B \in \mathcal{B}^X : B^*(B) \in \mathcal{B}^Y\}$, where $B^*(B) := \{f(b); f \in B^*, b \in B\}$. Now we define a superfilterformity F_{Y^X} on $\mathcal{B}^{Y^X}$ by setting for $B^* \in \mathcal{B}^{Y^X}$: $F_{Y^X}(B^*) := \{\mathcal{F}^* \in FIL(Y^X); \forall B \in \mathcal{B}^X \forall \mathcal{F} \in F_1(B) : e(\mathcal{F} \times \mathcal{F}^*) \in F_2(B^*(B))\}$, where $e(\mathcal{F} \times \mathcal{F}^*)$ denotes the filter generated by $\{e[G \times G^*]; G \in \mathcal{F}, G^* \in \mathcal{F}^*\}$. It is easy to verify that F_{Y^X} fulfils the axioms (SF1) and (SF3) in the definition of a superfilterformity (Definition 1.6). We will prove that F_{Y^X} fulfils also the axiom (SF2). To this end, let $f : (\mathcal{B}^X, F_1) \to (\mathcal{B}^Y, F_2)$ be an sf-map. We will show that $\dot{f} \in F_{Y^X}(\{f\})$. To this account, let B be an element of $\mathcal{B}^X$ and let $\mathcal{F}$ be a B-convergent filter. Our goal is to verify that $e(\mathcal{F} \times f) \in F_2(f[B])$. By the supposition it remains to prove the inclusion $f(\mathcal{F}) \subseteq e(\mathcal{F} \times f)$. As $C \in f(\mathcal{F})$ implies $f^{-1}[C] \in \mathcal{F}$ (see Definition 1.6), we have $f^{-1}[C] \times \{f\} \in \mathcal{F} \times \dot{f}$. Further, $C \supseteq e(f^{-1}[C] \times \{f\})$ because $z \in e(f^{-1}[C] \times \{f\})$ implies $z = e(x, f)$ for some $x \in f^{-1}[C]$ and hence $z \in C$. So the above mentioned inclusion is proved.

In the following step we show that the evaluation map is an sf-map. Since $BOUND$ is a topological universe (see Remark 1.2), the evaluation map $e : (\mathcal{B}^X, F_1) \times (\mathcal{B}^{Y^X}, F_{Y^X}) \to (\mathcal{B}^Y, F_2)$ is bounded with respect to the product prebornology $\mathcal{B}^X \times \mathcal{B}^{Y^X}$ on $X \times Y^X$. So, let R be an element of $\mathcal{B}^X \times \mathcal{B}^{Y^X}$ and let $\mathcal{F}$ be an R-convergent filter. We denote by $F_1 \times F^{Y^X}$ the product superfilterformity on $\mathcal{B}^X \times \mathcal{B}^{Y^X}$, i.e. the superfilterformity which is initial with respect to the data $(\mathcal{B}^X \times B^{Y^X}, P_X, P_{Y^X}, ((X, F_1), (Y^X, F_{Y^X}))$ (see Theorem 2.1), where P_X denotes the projection from $X \times Y^X$ to X, and P_{Y^X} denotes the projection from $X \times Y^X$ to Y^X. We are to show that $e(\mathcal{F}) \in F_2(e[R])$. By the supposition we have $P_X(\mathcal{F}) \in F_1(P_X[R])$ and $P_{Y^X}(\mathcal{F}) \in F^{Y^X}(P_{Y^X}[R])$. By definition of F^{Y^X} we get $e(P_X(\mathcal{F}) \times P_{Y^X}(\mathcal{F})) \in F_2(P_{Y^X}[R](P_X[R]))$. Now, according to Definition 1.6, it is sufficient to show the validity of the following two conditions:

(i) $P_{Y^X}[R](P_X[R]) \supseteq e[R]$,

(ii) $e(P_X(\mathcal{F}) \times P_{Y^X}(\mathcal{F})$.

Proof of (i): From $y \in e[R]$ it follows that $y = e(r)$ for some $r \in R$ which means $r = (x, f)$ for some $x \in X$ and $f \in Y^X$, hence $y = f(x)$. We have $f = P_{Y^X}(x, f) = P_{Y^X}(r)$ and $x = P_X(x, f) = P_X(r)$, thus $y \in P_{Y^X}[R](P_X[R])$.

Proof of (ii): From $S \in e(P_X(\mathcal{F}) \times P_{Y^X}(\mathcal{F}))$ it follows that $S \supseteq e[F \times F^*]$ for some $F \in P_X(\mathcal{F})$ and $F^* \in P_{Y^X}(\mathcal{F})$. We have $F \supseteq P_X[F_1]$ for some $F_1 \in \mathcal{F}$ and $F^* \supseteq P_{Y^X}[F_2]$ for some $F_2 \in \mathcal{F}$. Hence $F_1 \cap F_2 \in \mathcal{F}$ because $\mathcal{F}$ is a filter. Now, our goal is to show that $e[F_1 \cap F_2] \subseteq S$. We have $e[F_1 \cap F_2] \subseteq e[P_X[F_1 \cap F_2] \times P_{Y^X}[F_1 \cap F_2]]$, and the following two inclusions hold:

(1) $P_X[F_1] \supseteq P_X[F_1 \cap F_2]$,

(2) $P_{Y^X}[F_2] \supseteq P_{Y^X}[F_1 \cap F_2]$.

Consequently, $P_X[F_1] \times P_{Y^X}[F_2] \supseteq P_X[F_1 \cap F_2] \times P_{Y^X}[F_1 \cap F_2]$ and therefore $e[P_X[F_1 \cap F_2] \times P_{Y^X}[F_1 \cap F_2]]$ is a subset of the set $e[F \times F^*]$. But now we have $e[F_1 \cap F_2] \subseteq S$, which concludes the proof.

Now, on the other hand, we shall prove that for given superfilterformic spaces $(\mathcal{B}^X, F_1)$,$(\mathcal{B}^Y, F_2)$ ans $(\mathcal{B}^Z, F_3)$ the superfilterformity F^{Y^X} is weak enough to ensure that for any sf-map $f : (\mathcal{B}^X \times \mathcal{B}^Z, F_1 \times F_2) \to (\mathcal{B}^Y, F_2)$ the associated function $\bar{f} : (\mathcal{B}^Z, F_3) \to (\mathcal{B}^{Y^X}, F^{Y^X})$ defined by $\bar{f}(z)(x) := f(x, z)$ for each $z \in Z$ and each $x \in X$ is an sf-map.

Theorem 3.2 *For a triple of superfilterformic spaces $(\mathcal{B}^X, F_1), (\mathcal{B}^Y, F_2), (\mathcal{B}^Z, F_3)$ let $f : (\mathcal{B}^X \times \mathcal{B}^Z, F_1 \times F_3) \to (\mathcal{B}^Y, F_2)$ be an sf-map. Then the function $\bar{f} : (\mathcal{B}^Z, F_3) \to (\mathcal{B}^{Y^X}, F^{Y^X})$ defined by $\bar{f}(z)(x) := f(x, z)$ for each $z \in Z$ and each $x \in X$ is an sf-map.*

Proof: With respect to Remark 1.2 it suffices to show that for a given bounded set $\bar{B} \in \mathcal{B}^Z$ and a $\bar{B}$-convergent filter $\bar{\mathcal{F}}$ the image $\bar{f}(\bar{\mathcal{F}})$ is an $\bar{f}[\bar{B}]$-convergent filter, i.e. $\bar{f}(\bar{\mathcal{F}}) \in F^{Y^X}(\bar{f}[\bar{B}])$. We apply the definition of F^{Y^X}. Let $B \in \mathcal{B}^X$ and $\mathcal{F}$ be a B-convergent filter. We have to show that $e(\mathcal{F} \times \bar{f}(\bar{\mathcal{F}})) \in F_2(\bar{f}[\bar{B}](B))$. To show this we will verify that $\mathcal{F} \times \bar{\mathcal{F}} \in (F_1 \times F_3)(B \times \bar{B})$.

Since F is an sf-map, by supposition we have $f(\mathcal{F} \times \bar{\mathcal{F}}) \in F_2(f[B \times \bar{B}])$. Moreover, the inclusion $F_2(f[B \times \bar{B}]) \subseteq F_2(\bar{f}[\bar{B}](B))$ - note that $\bar{f}[\bar{B}](B) \subseteq f[B \times \bar{B}]$ is valid - results in $f(\mathcal{F} \times \bar{\mathcal{F}}] \in F_2(\bar{f}[\bar{B}])$. The desired result will be obtained as a consequence of the fact that $f(\mathcal{F} \times \bar{\mathcal{F}})$ is coarser then the filter $e(\mathcal{F} \times \bar{f}(\bar{\mathcal{F}}))$. To prove this fact, let $\bar{E}$ be an element of $f(\mathcal{F} \times \bar{\mathcal{F}}]$, i.e. $\bar{E} \supseteq f[F \times \bar{F}]$ for some $F \in \mathcal{F}$ and some $\bar{F} \in \bar{\mathcal{F}}$. We have $F \times \bar{f}[\bar{F}] \in \mathcal{F} \times \bar{f}(\bar{\mathcal{F}})$. It remains to show that $f[F \times \bar{F}] \supseteq e[F \times \bar{f}[\bar{F}]]$. From $y \in e[F \times \bar{f}[\bar{F}]]$ it follows that $y = e(x, f)$ for some $x \in F$ and some $f \in \bar{f}[\bar{F}]$. Now $f = \bar{f}(z)$ for some $z \in \bar{F}$, and we have $y = e(x, \bar{f}(z)) = \bar{f}(z)(x) = f(x, y)$ where $(x, y) \in F \times \bar{F}$.

It remains to prove the still open statement that $\mathcal{F} \times \bar{\mathcal{F}} \in (F_1 \times F_3)(B \times \bar{B})$. In other words, we have to check that

(i) $P_X(\mathcal{F} \times \bar{\mathcal{F}}) \in F_1(P_X[B \times \bar{B}])$,

(ii) $P_Z(\mathcal{F} \times \bar{\mathcal{F}}) \in F_3(P_Z[B \times \bar{B}])$

are valid. But the above two conditions are fulfilled whenever the following four inclusions hold:

(1) $P_X[B \times \bar{B}] \subseteq B$,

(2) $P_X(\mathcal{F} \times \bar{\mathcal{F}}) \supseteq \mathcal{F}$,

(3) $P_Z[B \times \bar{B}] \subseteq \bar{B}$,

(4) $P_Z(\mathcal{F} \times \bar{\mathcal{F}}) \supseteq \bar{\mathcal{F}}$.

Indeed, by supposition we firstly get $\mathcal{F} \in F_1(B) \subseteq F_1(P_X[B \times \bar{B}])$ and $\bar{\mathcal{F}} \in F_3(\bar{B}] \subseteq F_3(P_Z[B \times \bar{B}])$, and secondly we have $P_X(\mathcal{F} \times \bar{\mathcal{F}}) \in F_1(P_X[B \times \bar{B}])$ and $P_Z(\mathcal{F} \times \bar{\mathcal{F}}) \in F_3(P_Z[B \times \bar{B}])$. Thus, it remains to prove (1)-(4). But it is trivial to verify (1) and (3). To prove (2), let F be an element of $\mathcal{F}$. Since $\bar{\mathcal{F}} \neq \emptyset$, we can choose $\bar{F} \in \bar{\mathcal{F}}$ such that $F \times \bar{F} \in \mathcal{F} \times \bar{\mathcal{F}}$. Now, with respect to (1), we get $P_X[F \times \bar{F}] \subseteq F$, hence $F \in P_X(\mathcal{F} \times \bar{\mathcal{F}})$. The inclusion (4) can be shown in an analogous way.

4 $SUPFILFORM$ and exponential laws

To give a short summary we note that $SUPFILFORM$ is a topological category with well-behaved function space structures, i.e. it is cartesian closed. With respect to Theorem 3.1, Example 1.7(ii) and Theorem 1.9 or, respectively, Example 1.7(iii) and Theorem 1.10, the corresponding „reduced“ function space structures are the ones of continuous convergence. Moreover, $FMER$, $FCONV$ and some interesting neighborhood spaces can be fully embedded up to isomorphism into $SUPFILFORM$.

By purely categorical arguments (see [7]) the following three exponential laws hold in $SUPFILFORM$:

(1) First exponential law: $X^{Y \times Z}$ is isomorphic to $(X^Y)^Z$,

(2) Second exponential law: $(\prod\limits_{i \in I} X_i)^Y$ is isomorphic to $\prod\limits_{i \in I}(X_i^Y)$,

(3) Third exponential law: $X^{\coprod\limits_{i \in I} Y_i}$ is isomorphic to $\prod\limits_{i \in I}(X^{Y_i})$.

At least we mention that in $SUPFILFORM$ there also holds the interesting distributive law:

$X \times \coprod\limits_{i \in I} Y_i$ is isomorphic to $\coprod\limits_{i \in I}(X \times Y_i)$.

5 Filterunitopic spaces

In Example 1.7(iv) the question raises whether each superfilterformity is induced by a pre-uniform convergence. We will give an affirmative answer by constructing a more general category which includes both the superfilterformic spaces and the preuniform convergence spaces. The construction is very natural and seems to have some nice properties as discussed in the precedent paragraphs.

Definition 5.1 *For a set X, $FIL(X \times X)$ denotes the set of all filters on $X \times X$, i.e. on the cartesian product of X with itself. A function $\mu : \mathcal{B}^X \to \mathcal{P}(FIL(X \times X))$ from a B-structure $\mathcal{B}^X$ on X to subsets of $FIL(X \times X)$ is called a* filterunitopy *(or a* filterunitopic operator*), and the pair $(\mathcal{B}^X, \mu)$ is called a* filterunitopic space *iff the following axioms are satisfied:*

(fut1) $B \in \mathcal{B}^X$ *and* $\mathcal{W} \in \mu(B)$ *with* $\mathcal{W} \subseteq \mathcal{V} \in FIL(X \times X)$ *imply* $\mathcal{V} \in \mu(B)$,

(fut2) $x \in X$ *implies* $\dot{x} \times \dot{x} \in \mu(\{x\})$,

(fut3) $B_1 \subseteq B_2 \in \mathcal{B}^X$ *implies* $\mu(B_2) \subseteq \mu(B_1)$.

$\mathcal{W}$ is called a B-uniform filter iff $\mathcal{W} \in \mu(B)$.

Now, let $(\mathcal{B}^X, \mu_X)$ and $(\mathcal{B}^Y, \mu_Y)$ be filterunitopic spaces. A bounded map f is called an fut-map iff

(fut) $B \in \mathcal{B}^X$ *and* $\mathcal{W} \in \mu_X(B)$ *imply* $(f \times f)(\mathcal{W}) \in \mu_Y(f[B])$, *where* $(f \times f)(\mathcal{W}) := \{R \subseteq Y \times Y; (f \times f)^{-1}[R] \in \mathcal{W}\}$.

We also refer as to an fut-map by saying that it preserves uniform filters in the above mentioned sense.

We denote by F-$UNITOP$ the category whose objects are the filterunitopic spaces and whose morphisms are the bounded maps which preserve uniform filters.

Examples 5.2 (i) For a preuniform convergence space $(X, \mathcal{J}_X)$, we consider $\mathcal{P}X$ equipped with the filterunitopy $\mu_{\mathcal{J}_X}$ defined as follows:

$\mu_{\mathcal{J}_X} := \mathcal{J}_X$ for each $B \in \mathcal{P}X$.

(ii) For a superfilterformic space $(\mathcal{B}^X, F)$ we define a filterunitopic operator μ_F by setting

$\mu_F(B) := \{\mathcal{W} \in FIL(X \times X); \exists \mathcal{F} \in F(B) : \mathcal{F} \times \mathcal{F} \subseteq \mathcal{W}\}$.

Theorem 5.3 *The category PUConv of preuniform convergence spaces and uniformly continuous maps is isomorphic to a full subcategory of F-$UNITOP$.*

Proof: With respect to Example 5.2(i), let $(\mathcal{P}X, \mu)$ be a filterunitopic space. Define an underlying preuniform convergence structure $\mathcal{J}_\mu$ by setting $\mathcal{J}_\mu := \mu(\emptyset)$. By this way we obtain an isomorphism between the category $PUConv$ and a corresponding full subcategory of F-$UNITOP$.

Theorem 5.4 *The category SUPFILFORM is isomorphic to a full subcategory of F-UNITOP.*

Proof: With respect to Example 5.2(ii), let $(\mathcal{B}^X, \mu)$ be a filterunitopic space. Define an underlying superfilterformity F_μ by setting $F_\mu(B) := \{\mathcal{F} \in FIL(X); \mathcal{F} \times \mathcal{F} \in \mu(B)\}$ for each $B \in \mathcal{B}^X$. By this way we obtain an isomorphism between the category *SUPFILFORM* and a corresponding full subcategory of *F-UNITOP*.

Remark 5.5 Now it seems to be of interest to study categorical properties of *F-UNITOP* and, moreover, to deal with the possibility of constructing function space structures with respect to evaluation maps. But it is not the aim of this paper.

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received: September 30, 1997

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