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Cauchy filters and strong completeness of quasi-uniform spaces

ABSTRACT. We introduce and study the notions of a strongly completable and of a strongly complete quasi-uniform space. A quasi-uniform space (X,U) is said to be strongly complete if every Cauchy filter (in the sense of Sieber and Pervin) clusters in the uniform space $(X,\mathcal{U}\vee\mathcal{U}^{-1})$. An interesting motivation for the study of this notion of completeness is the fact, proved here, that the quasi-uniformity induced by the complexity space is strongly complete but not Corson complete. We recall that the (quasi-metric) complexity space was introduced by Schellekens to study complexity analysis of programs. We characterize those quasi-uniform space that are strongly completable and show that a quasi-uniform space is strongly complete if and only if it is bicomplete and strongly completable. We observe that every T_0 strongly complete quasi-uniform space is Smyth complete. We also show that every T_1 strongly complete quasi-uniform space is small-set symmetric, so every T_1 strongly complete quasi-metric space is (completely) metrizable.

KEY WORDS. Cauchy filter, strongly complete, Corson complete, Smyth complete, bicomplete, small set-symmetric, complexity space.

1 Introduction

Throughout this paper the letters $\mathbb{R}$, $\mathbb{N}$ and ω will denote the sets of reals, positive integers and nonnegative integers, respectively.

Terms and undefined concepts may be found in [4] and [7].

Given a quasi-uniform space $(X,\mathcal{U})$ (we shall denote by $\mathcal{U}^s$ the coarsest uniformity finer than $\mathcal{U}$ and its conjugate $\mathcal{U}^{-1}$ (i.e. $\mathcal{U}^s=\mathcal{U}\vee\mathcal{U}^{-1}$). If $U\in\mathcal{U}$ we denote by U^s the entourage $U\cap U^{-1}$ of $\mathcal{U}^s$.

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Let us recall that every quasi-uniformity $\mathcal{U}$ on a set X induces a topology $T(\mathcal{U}) = \{A \subseteq X|$ for each $x \in A$ there is $U \in \mathcal{U}$ such that $U(x) \subseteq A\}$, where $U(x) = \{y \in X|(x, y) \in U\}$.

According to [4], a quasi-uniform space $(X,\mathcal{U})$ is called bicomplete if $(X,\mathcal{U}^s)$ is a complete uniform space. A bicompletion of (X, U) is a bicomplete quasi-uniform space $(Y, \mathcal{V})$ which has a $T(\mathcal{V}^s)$-dense subspace quasi-unimorphic to $(X,\mathcal{U})$. It was shown in [15] and in [4] that every quasi-uniform space $(X,\mathcal{U})$ admits a bicompletion $(\widetilde{X},\widetilde{\mathcal{U}})$ such that if $(X,\mathcal{U})$ is a T_0 quasi-unifom space then $(\widetilde{X},\widetilde{\mathcal{U}})$ is T_0 and it is the unique (up to quasi-unimorphism) bicompletion of $(X,\mathcal{U})$.

In the context of this paper, a quasi-metric on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x, y, z \in X$: (i) $d(x, y) = d(y, x) = 0 \Leftrightarrow x = y$ and (ii) $d(x, y) \leq d(x, z) + d(z, y)$.

If d is a quasi-metric on a set X and $x \in X$, the set $\{y \in X|d(x, y) < r\}$ is called the open r-sphere around x and is denoted by $S_d(x, r)$. The conjugate d^{-1} of the quasi-metric d is given by $d^{-1}(x, y) = d(y, x)$. Then we shall denote by d^s the metric defined on X by $d^s = d \vee d^{-1}$.

Every quasi-metric d on a set X induces a quasi-uniformity U_d on X which has as a base the family of sets of the form $\{(x, y) \in X \times X|d(x, y) < 2^{-n}\}$, for $n \in \mathbb{N}$ (see [4] p. 3]). The topology $T(U_d)$ will be denoted simply by $T(d)$.

In [16] M. Schellekens introduced the quasi-metric complexity space as a part of the development of a topological foundation for the complexity analysis of programs. Via the analysis of its dual it was proved in [13] that the complexity space is Smyth complete. In Section 3 of this paper we shall show that actually the (dual) complexity space admits a stronger form of completeness based on the use of Cauchy filters (in the sense of Sieber and Pervin) having a sup-cluster point. This kind of completeness will be called "strong completeness". Thus, in Section 2 we define the notions of a strongly completable and of a strongly complete quasi-uniform space and obtain some properties of such spaces. In particular, we characterize both strongly completable and strongly complete quasi-uniform spaces and deduce that every strongly completable quasi-uniform space is Smyth completable and that every T_0 strongly complete quasi-uniform space is Smtyh complete. We give examples which show that the converse implications do not hold. We also observe that a quasi-uniform space is totally bounded if and only if it is precompact and strongly completable. We show that every T_1 strongly complete quasi-uniform space is small-set symmetric, so every T_1 strongly complete quasi-metric space is completely metrizable. Finally, in Section 3 we show, in addition to the result cited above, that the (dual) complexity space is not Corson complete (in the sense of [11]) and give an example of a weightable Smyth complete quasi-metric space having a maximum which is not strongly complete.

2 Strongly complete quasi-uniform spaces

Let us recall that a filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is Cauchy [17] provided that for each $U \in \mathcal{U}$ there is $x \in X$ such that $U(x) \in \mathcal{F}$. $\mathcal{F}$ is left K-Cauchy [12] provided that for each $U \in \mathcal{U}$ there is an $F \in \mathcal{F}$ such that $U(x) \in \mathcal{F}$ for all $x \in F$.

$(X,\mathcal{U})$ is said to be (left K-) complete if every (left K-) Cauchy filter has a $T(\mathcal{U})$-cluster point. lt is clear that every left K-complete quasi-uniform space is complete and it is well-known that the converse is not true. On the other hand, although every left K-Cauchy filter converges to its cluster points, there exist complete quasi-uniform spaces having non $T(\mathcal{U})$-convergent Cauchy filters (see [4], [6]).

In [8] H.P.A. Künzi proved that a quasi-uniform space $(X,\mathcal{U})$ is Smyth complete if and only if each left K-Cauchy filter is $T(\mathcal{U}^s)$-convergent to a unique point of X and that $(X,\mathcal{U})$ is Smyth completable if and only if every left K-Cauchy filter is a Cauchy filter on the uniform space $(X,\mathcal{U}^s)$.

Definition 1 *A quasi-uniform space $(X,\mathcal{U})$ is called* strongly complete *if each Cauchy filter on $(X,\mathcal{U})$ has a $T(\mathcal{U}^s)$-cluster point.*

Definition 2 *Let $(X,\mathcal{U})$ be a quasi-uniform space. A* strong completion *of $(X,\mathcal{U})$ is a strongly complete quasi-unifom space $(Y,\mathcal{V})$ in which $(X,\mathcal{U})$ can be quasi-uniformly embedded as a $T(\mathcal{V}^s)$-dense subspace. In this case we say that $(X,\mathcal{U})$ is* strongly completable.

We will say that a quasi-metric space (X,d) is strongly completable *(resp.* strongly complete*) if the quasi-uniform space (X,U_d) is strongly completable (resp. strongly complete). Similarly, (X,d) is called Smyth complete if (X,U_d) is Smyth complete.*

Lemma 1 *Let $(Y,\mathcal{V})$ be a quasi-uniform space and let X be a $T(\mathcal{V}^s)$-dense subset of Y. If $\mathcal{F}$ is a Cauchy filter on $(Y,\mathcal{V})$, then*

$$\mathcal{G} = \{U^s(F) \cap X | F \in \mathcal{F}, U \in \mathcal{V}\}$$

is a Cauchy filter base on $(X,\mathcal{V}|X \times X)$.

Proof: Let $\mathcal{F}$ be a Cauchy filter on $(Y,\mathcal{V})$. Let $U \in \mathcal{V}$. We shall show that there is $x \in X$ such that $U(x) \in \mathcal{G}$. Choose a $V \in \mathcal{V}$ such that $V^3 \subseteq U$. Then there exists a $y \in Y$ such that $V(y) \in \mathcal{F}$. Since X is $T(\mathcal{V}^s)$-dense in Y, there exists an $x \in V^s(y) \cap X$. Since $V^s(V(y)) \cap X \in \mathcal{G}$, it will be sufficient to see that $(V^s(V(y)) \cap X) \subseteq U(x)$. Indeed, given $z \in V^s(V(y)) \cap X$, there is $a \in V(y)$ such that $z \in V^s(a)$. Thus $z \in V^2(y)$. Since $y \in V(x)$, we conclude that $z \in V^3(x) \subseteq U(x)$. The proof is complete. ■

Theorem 1 *A quasi-uniform space $(X,\mathcal{U})$ is strongly completable if and only if every Cauchy filter on $(X,\mathcal{U})$ is contained in a Cauchy filter on the uniform space $(X,\mathcal{U}^s)$.*

Proof: Suppose that $(X,\mathcal{U})$ is strongly completable. Then there is a quasi-unimorphism f from $(X,\mathcal{U})$ to a $T(\mathcal{V}^s)$-dense subspace of a strongly complete quasi-uniform space $(Y,\mathcal{V})$. Let $\mathcal{F}$ be a Cauchy filter on $(X,\mathcal{U})$. Clearly $\{f(F) : F \in \mathcal{F}\}$ is a Cauchy filter base on $(Y,\mathcal{V})$, so it has a $T(\mathcal{V}^s)$-cluster point $y \in Y$. Then, the filter generated by $\{f^{-1}(V^s(y)) \cap F | F \in \mathcal{F}, V \in \mathcal{V}\}$ is a Cauchy filter on $(X,\mathcal{U}^s)$ which contains $\mathcal{F}$.

Conversely, let $(\widetilde{X},\widetilde{\mathcal{U}})$ be a bicompletion of $(X,\mathcal{U})$. If $\mathcal{F}$ is a Cauchy filter on $(\widetilde{X},\widetilde{\mathcal{U}})$, then it follows from Lemma 1 that the filter base $\mathcal{G} = \{U^s(F) \cap X | F \in \mathcal{F}, U \in \widetilde{\mathcal{U}}\}$ is Cauchy on $(X,\mathcal{U})$. So $\mathcal{G} \subseteq \mathcal{H}$ for some Cauchy filter $\mathcal{H}$ on $(X,\mathcal{U}^s)$. Denote by $\widetilde{\mathcal{H}}$ the filter generated on $\widetilde{X}$ by $\mathcal{H}$. Then $\widetilde{\mathcal{H}}$ is $T(\widetilde{\mathcal{U}}^s)$-convergent to some point $y \in \widetilde{X}$. Hence y is a $T(\widetilde{\mathcal{U}}^s)$-cluster point of $\mathcal{F}$. We conclude that $(X,\mathcal{U})$ is strongly completable. ■

Remark 1 It follows from the preceding result that if $(X,\mathcal{U})$ is a T_o strongly completable quasi-uniform space, then its bicompletion is the unique strong completion of $(X,\mathcal{U})$.

Recall that a quasi-uniform space $(X,\mathcal{U})$ is precompact provided that for each $U \in \mathcal{U}$ there is a finite subset A of X such that $U(A) = X$. $(X,\mathcal{U})$ is said to be totally bounded if the uniform space $(X,\mathcal{U}^s)$ is precompact (see [4], [7]). Every totally bounded quasi-uniform space is precompact but the converse does not hold. On the other hand, a quasi-uniform space is precompact if and only if every ultrafilter is a Cauchy filter [4].

Corollary 1 *A quasi-uniform space is totally bounded if and only if its precompact and strongly completable.*

Proof: Let $(X,\mathcal{U})$ be a precompact strongly completable quasi-uniform space. Let $\mathcal{F}$ be an ultrafilter on X. By the precompactness of $(X,\mathcal{U})$, $\mathcal{F}$ is a Cauchy (ultra) filter. So, by Theorem 1, $\mathcal{F}$ is a Cauchy filter on the uniform space $(X,\mathcal{U}^s)$. Therefore $(X,\mathcal{U})$ is totally bounded. The converse follows from Theorem 1 and the preceding observations. ■

It is essentially known (see, for instance, the proof of [11] Corollary 3) that a left K-Cauchy filter on a quasi-uniform space $(X,\mathcal{U})$ is Cauchy on $(X,\mathcal{U}^s)$ if and only if it is contained in a Cauchy filter on $(X,\mathcal{U}^s)$. From this fact and Theorem 1 we deduce the following result.

Corollary 2 *Every strongly completable quasi-uniform space is Smyth completable.*

Theorem 2 *A quasi-uniform space $(X,\mathcal{U})$ is strongly complete if and only if it is bicomplete and strongly completable.*

Proof: Suppose that $(X,\mathcal{U})$ is bicomplete and strongly completable. Let $\mathcal{F}$ be a Cauchy filter on $(X,\mathcal{U})$. By Theorem 1, $F \in \mathcal{G}$ for some Cauchy filter $\mathcal{G}$ on $(X,\mathcal{U}^s)$. Hence $\mathcal{G}$ is $T(\mathcal{U}^s)$-convergent to a point $x \in X$. So x is a $T(\mathcal{U}^s)$-cluster point of $\mathcal{F}$. We conclude that $(X,\mathcal{U})$ is strongly complete. The converse is obvious. ■

Corollary 3 *A T_0 quasi-uniform space $(X,\mathcal{U})$ is strongly complete if and only if it is Smyth complete and strongly completable.*

Proof: Suppose that $(X,\mathcal{U})$ is a T_0 strongly complete quasi-uniform space. Let $\mathcal{F}$ be a left K-Cauchy filter on $(X,\mathcal{U})$. By Corollary 2, $\mathcal{F}$ is a Cauchy filter on $(X,\mathcal{U}^s)$. So it is $T(\mathcal{U}^s)$-convergent to a point of X, by Theorem 2. Hence $(X,\mathcal{U})$ is Smyth complete. The converse follows from Theorem 2. ■

The following is a simple example of a compact Hausdorff Smyth complete quasi-metric space which is not strongly complete.

Example 1 Let d be the quasi-metric defined on ω by $d(0,n) = 1/n$ for all $n \in \mathbb{N}$, $d(n,m) = 1$ for all $n \in \mathbb{N}$ and $m \in \omega$ with $n \neq m$, and $d(n,n) = 0$ for all $n \in \omega$. Clearly (ω, d) is a compact Hausdorff Smyth complete quasi-metric space. However, it is not strongly complete because the filter generated by $\{\{m \in \mathbb{N} : m \geq n\} : n \in \mathbb{N}\}$ is a Cauchy filter on (ω, U_d) without $T((U_d)^s)$-cluster points.

Proposition 1 *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then the uniform space $(X,\mathcal{U}^s)$ is compact if and only if $(X,\mathcal{U})$ is precompaet and strongly complete.*

Proof: Suppose that $(X,\mathcal{U})$ is precompact and strongly complete. Then $(X,\mathcal{U})$ is bicomplete. Moreover, it is totally bounded by Corollary 1. We conclude that $(X,\mathcal{U}^s)$ is a compact uniform space. The converse is obvious. ■

In [1] P. Fletcher and W. Hunsaker introduced the notion of a small-set symmetric quasi-uniform space. It was shown in [9] that a quasi-uniform space $(X,\mathcal{U})$ is small-set symmetric if and only if $T(U^{-1}) \subseteq T(\mathcal{U})$.

Proposition 2 *A T_1 quasi-uniform space is strongly complete if and only if it is complete and small-set symmetric.*

Proof: Let $(X,\mathcal{U})$ be a strongly complete T_1 quasi-uniform space. Obviously, it is complete. In order to prove that $(X,\mathcal{U})$ is also small-set symmetric suppose that there exists $x \in X$ and $U \in \mathcal{U}$ such that $V(x)\backslash U^{-1}(x) \neq \emptyset$ for all $V \in \mathcal{U}$. Thus, the filter generated by $\{V(x)\backslash U^{-1}(x) | V \in \mathcal{U}\}$ is a Cauchy filter on $(X,\mathcal{U})$. Let $y \in X$ be a $T(\mathcal{U}^s)$-cluster point of such a filter. Then $x = y$. Indeed, given $V \in \mathcal{U}$ there is $W \in \mathcal{U}$ such that $W^2 \subseteq V$. Since $W^s(y) \cap W(x) \neq \emptyset$, it follows that $y \in V(x)$. Consequently $y \in \cap_{V\in\mathcal{U}} V(x)$, so $y = x$. Therefore $U^s(x) \cap (U(x)\backslash U^{-1}(x)) \neq \emptyset$, a contradiction. We conclude that $T(\mathcal{U}^{-1}) \subseteq T(\mathcal{U})$. Hence $(X,\mathcal{U})$ is small-set symmetric. The converse is almost obvious, so its simple proof is omitted. ■

Corollary 4 *Every strongly complete T_1 quasi-metric space is completely metrizable.*

Proof: Let (X, d) be a strongly complete T_1 quasi-metric space. By Proposition 2 we obtain that $T(d^{-1}) \subseteq T(d)$, so $(X, T(d))$ is a metrizable space. Now the conclusion follows from the fast, proved in [2], that every metrizable space which admits a complete quasi-metric is completely metrizable. ■

3 Strong completeness of the complexity space

Let us recall [16] that the complexity space is the pair $(\mathcal{C}, d_{\mathcal{C}})$, where

$$\mathcal{C} = \left\{ f : \omega \to (0, +\infty] | \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < +\infty \right\}$$

and $d_{\mathcal{C}}$ is the quasi-metric on $\mathcal{C}$ defined by

$$d_{\mathcal{C}}(f, g) = \sum_{n=0}^{\infty} 2^{-n} \left[\left(\frac{1}{g(n)} - \frac{1}{f(n)} \right) \vee 0 \right]$$

whenever $f, g \in \mathcal{C}$.

The dual complexity space $(\mathcal{C}^*, d_{\mathcal{C}^*})$ is defined in [13] as follows:

$$\mathcal{C}^* = \left\{ f : \omega \to [0, +\infty) | \sum_{n=0}^{\infty} 2^{-n} f(n) < +\infty \right\}$$

and $d_{\mathcal{C}^*}$ is the quasi-metric on $\mathcal{C}^*$ given by

$$d_{\mathcal{C}^*}(f, g) = \sum_{n=0}^{\infty} 2^{-n} [(g(n) - f(n)) \vee 0]$$

whenever $f, g \in \mathcal{C}^*$.

It is observed in [13] that the inversion function $\Psi : \mathcal{C}^* \to \mathcal{C}$ is a quasi-isometry from $(\mathcal{C}^*, d_{\mathcal{C}^*})$ to $(\mathcal{C}, d_{\mathcal{C}})$, because $d_{\mathcal{C}}(\Psi(f), \Psi(g)) = d_{\mathcal{C}}(1/f, 1/g) = d_{\mathcal{C}^*}(f, g)$, whenever $f, g \in \mathcal{C}^*$.

The fact that the dual complexity space admits a structure of quasi-normed semilinear space (see [14]) provided a first motivation to the authors for the use of the dual complexity space rather than the original one in the study of the properties of completeness, compactness and total boundedness of the complexity space (see [13]). A second motivation for the use of the dual space is the fact that the definition of the dual is mathematically somewhat more appealing, since $d_{\mathcal{C}^*}$ is "derived" from the restriction to $[0, +\infty)$ of the standard quasi-metric u defined on $\mathbb{R} \times \mathbb{R}$, by $u(x, y) = (y - x) \vee 0$. Consequently, the presentation of the proofs becomes somewhat more elegant.

The quasi-metric of pointwise convergence of u is the quasi-metric u_P defined on $\mathbb{R}^\omega \times \mathbb{R}^\omega$ by $u_P(f,g) = \sum_{n=0}^{\infty} 2^{-n} \min\{u(f(n),g(n)),1\}$. Thus the metric $(u_P)^s$ induces the usual topology of pointwise convergence on $\mathbb{R}^\omega$.

A quasi-metric space (X,d) is called weightable [10] if there is a function $w : X \to [0,+\infty)$ such that for all $x,y \in X$:

$$d(x,y) + w(x) = d(y,x) + w(y).$$

In this case, we say that w is a weighting function for (X,d).

It was proved in [16] that the complexity space $(C,d_{\mathcal{C}})$ is weigthable with weighting function $w_{\mathcal{C}}$ defined by $w_{\mathcal{C}}(f) = \sum_{n=0}^{\infty} 2^{-n}(i/f(n))$, for all $f \in \mathcal{C}$. Similary, the dual complexity space $(\mathcal{C}^*, d_{\mathcal{C}^*})$ is weightable with weighting function $w_{\mathcal{C}^*}$ defined by $w_{\mathcal{C}^*}(f) = \sum_{n=0}^{\infty} 2^{-n} f(n)$, for all $f \in \mathcal{C}^*$.

A quasi-metric space (X,d) *has a maximum* provided there is $x_0 \in X$ such that $d(x,x_0) = 0$ for all $x \in X$. It is obvious that the (dual) complexity space has a maximum. Before we prove that the (dual) complexity space is strongly complete, it seems interesting to note that there exists a non strongly complete weightable Smyth complete quasi-metric space which has a maximum.

Example 2 Let $X = \omega \cup \{\infty\}$. Define a quasi-metric d on X by $d(0,x) = 0$ for all $x \in X$, $d(\infty,n) = 1$ for all $n \in \mathbb{N}$, $d(\infty,0) = 2$, $d(n,m) = 1$ for all $n,m \in \mathbb{N}$ with $n \neq m$, $d(n,\infty) = 0$ for all $n \in \mathbb{N}$, $d(n,0) = 1$ for all $n \in \mathbb{N}$, and $d(x,x) = 0$ for all $x \in X$. It is immediate to check that (X,d) is Smyth complete and that ∞ is a maximum for (X,d). Furthermore (X,d) is weightable with weighting function w given by $w(0) = 2$, $w(\infty) = 0$ and $w(n) = 1$ for all $n \in \mathbb{N}$.

Proposition 3 *The dual complexity space $(\mathcal{C}^*, d_{\mathcal{C}^*})$ is strongly complete.*

Proof: Let $\mathcal{F}$ be a Cauchy filter on $(\mathcal{C}^*, \mathcal{U}_{d_{\mathcal{C}*}}$. Then, for each $k \in \mathbb{N}$ there is an $f_k \in \mathcal{C}^*$ such that $S_{d_{\mathcal{C}^*}}(f_k, 2^{-3k}) \in \mathcal{F}$. Put $F_k = S_{d_{\mathcal{C}^*}}(f_k, 2^{-3k})$ for all $k \in \mathbb{N}$.

Furthermore, for each $f \in F_1$, $w_{\mathcal{C}^*}(f) \leq w_{\mathcal{C}^*}(f_1) + d_{\mathcal{C}^*}(f_1, f)$. Hence $\sum_{n=0}^{\infty} 2^{-n} f(n) < w_{\mathcal{C}^*}(f_1) + 2^{-3}$ and thus $f(n) < 2^n(w_{\mathcal{C}^*}(f_1) + 1)$ for all $f \in F_1$ and $n \in \omega$.

Denote by K the compact space $\prod_{n=0}^{\infty}[0, 2^n(w_{\mathcal{C}^*}(f_1)+1)]$, and by $\overline{F \cap K}$ the closure of $F \cap K$ in K for all $F \in \mathcal{F}$. (Note that for each $F \in \mathcal{F}$, $F \cap K \neq \emptyset$ because $F_1 \subseteq K$.)

Next we show that for each $F \in \mathcal{F}$, $(\overline{F \cap K}) \cap (\cap_{k=1}^{\infty} \overline{F_k \cap K}) \neq \emptyset$.

Indeed, fix $F \in \mathcal{F}$. For each $k \in \mathbb{N}$ there is $g_k \in F \cap F_k$, so $(g_k)_{k \in \mathbb{N}}$ is a sequence in K and, thus, it clusters to a function $g \in K$ with respect to $T((u_P)^s)$. Therefore $g \in (\overline{F \cap K}) \cap (\cap_{k=1}^{\infty} \overline{F_k \cap K})$.

In particular, it follows from the above observation that $\cap_{k=1}^{\infty}\overline{F_k \cap K}$ is a nonempty compact subset of K, so the filter base $\{(\overline{F \cap K}) \cap (\cap_{k=1}^{\infty}\overline{F_k \cap K}) | F \in \mathcal{F}\}$ clusters to some function $h \in \cap_{k=1}^{\infty}\overline{F_k \cap K}$ with respect to $T((u_P)^s)$.

Now we want to show that $h \in \mathcal{C}^*$ and $\mathcal{F}$ clusters to h with respect to $T((d_{\mathcal{C}^*})^s)$. Thus $(\mathcal{C}^*, d_{\mathcal{C}^*})$ will be strongly complete.

Suppose that $h \notin \mathcal{C}^*$. Then, for each $j \in \mathbb{N}$ there is an $m_j \in w$ such that $j < \sum_{n=0}^{m_j} 2^{-n}h(n)$. Since $h \in \overline{F_1 \cap K}$, there exists $f \in F_1$ such that $|h(n) - f(n)| < 2^{-j}$ for $n = 0, 1, \dots, m_j$. So

$$\sum_{n=0}^{m_j} 2^{-n}|h(n) - f(n)| < 2^{-j}\sum_{n=0}^{m_j} 2^{-n} < 2^{-(j-1)}.$$

Since $d_{\mathcal{C}^*}(f_1, f) < 2^{-3}$, it follows that

$$\begin{aligned} j < \sum_{n=0}^{m_j} 2^{-n}h(n) &\leq 2^{-(j-1)} + \sum_{n=0}^{m_j} 2^{-n}f(n) \\ &< 2^{(j-1)} + 2^{-3} + \sum_{n=0}^{m_j} 2^{-n}f_1(n), \end{aligned}$$

which contradicts that $f_1 \in \mathcal{C}^*$. Therefore $h \in \mathcal{C}^*$.

Finally, we shall prove that $\mathcal{F}$ clusters to h with respect to $T((d_{\mathcal{C}^*})^s)$.

Fix $k \in \mathbb{N}$ and $F \in \mathcal{F}$. Since f_k and h are in $\mathcal{C}^*$, there is $n_0 \in \mathbb{N}$ with $n_0 > 3k$ such that

$$\sum_{n=n_0}^{\infty} 2^{-n}f_k(n) < 2^{-3k} \quad \text{and} \quad \sum_{n=n_0}^{\infty} 2^{-n}h(n) < 2^{-3k}.$$

On the other hand, since $h \in \overline{F \cap F_k \cap K}$, there is $f \in F \cap F_k$ such that $(u_P)^s(h, f) < 2^{-n_0}$, which implies that $\sum_{n=0}^{n_0-1} 2^{-n}\min\{1, |h(n) - f(n)|\} < 2^{n_0}$, i.e. $\sum_{n=0}^{n_0-1} 2^{-n}|h(n) - f(n)| < 2^{-n_0}$. Therefore,

$$\begin{aligned} (d_{\mathcal{C}^*})^s(h, f) &\leq \sum_{n=0}^{\infty} 2^{-n}|h(n) - f(n)| \\ &< 2^{-n_0} + \sum_{n=n_0}^{\infty} 2^{-n}h(n) + \sum_{n=n_0}^{\infty} 2^{-n}f(n). \end{aligned}$$

From $\sum_{n=n_0}^{\infty} 2^{-n}u(f_k(n), f(n)) \leq d_{\mathcal{C}^*}(f_k, f) < 2^{-3k}$ we deduce that $\sum_{n=n_0}^{\infty} 2^{-n}f(n) < 2^{-3k} + \sum_{n=n_0}^{\infty} 2^{-n}f_k(n) < 2^{-3k} + 2^{-3k}$, so

$$(d_{\mathcal{C}^*})^s(h, f) < 2^{-n_0} + 2^{-3k} + 2^{-3k} + 2^{-3k} < 4 \cdot 2^{-3k} \leq 2^{-k}.$$

We have shown that $\mathcal{F}$ clusters to h with respect to $T((d_{\mathcal{C}^*})^s)$. Consequently, the dual complexity space is strongly complete. ■

By using the quasi-isometry Ψ from $(\mathcal{C}^*, d_{\mathcal{C}^*})$ to $(\mathcal{C}, d_{\mathcal{C}})$ constructed above and the preceding theorem, we immediately deduce the following result.

Corollary 5 *The complexity space $(\mathcal{C}, d_{\mathcal{C}})$ is strongly complete.*

The notion of a Corson complete quasi-uniform space was introduced in [11]. A quasi-uniform space $(X,\mathcal{U})$ is said to be Corson complete if every weakly Cauchy filter on $(X,\mathcal{U})$ has a $T(\mathcal{U}^s)$-cluster point, where a filter $\mathcal{F}$ on $(X,\mathcal{U})$ is weakly Cauchy provided that for each $U \in \mathcal{U}$, $\cap_{F\in\mathcal{F}} U^{-1}(F) \neq \emptyset$ (see, for instance, [3]). Clearly, every Corson complete quasi-uniform space is strongly complete. The converse does not hold even for uniform spaces as a well-known example of J.R. Isbell shows (see [5]).

We conclude the paper by showing that the quasi-uniform space $(\mathcal{C}^*, U_{d_{\mathcal{C}^*}})$ is not Corson complete.

Example 3 For each $j,k \in \mathbb{N}$ define a function $f_k^j : \omega \to [0,+\infty)$ by $f_k^j(n) = j$ if $n < k$ and $f_k^j(n) = j + 2^{k-j}$ if $n \geq k$.

An easy computation shows that $\sum_{n=0}^{\infty} 2^{-n} f_k^j(n) = 2(j + 2^{-j})$, so $f_k^j \in \mathcal{C}^*$ for all $j,k \in \mathbb{N}$.

Now, for each $m \in \mathbb{N}$, define $F_m = \{f_k^j : j \geq 1 \text{ and } k \geq m\}$. Then $\{F_m : m \in \mathbb{N}\}$ is a base for a filter $\mathcal{F}$ on $\mathcal{C}^*$.

For each $j \in \mathbb{N}$ consider the constant function $g_j : \omega \to [0,+\infty)$ defined by $g_j(n) = j$ for all $n \in \omega$. Clearly each g_j is in $\mathcal{C}^*$ and, for each $j \in \mathbb{N}$, the sequence $(f_k^j)_{k\in\mathbb{N}}$ converges to g_j with respect to $T((u_P)^s)$.

Furthermore, for each $j,k \in \mathbb{N}$, we have

$$d_{\mathcal{C}^*}(g_j, f_k^j) = \sum_{n=k}^{\infty} 2^{-n}(j + 2^{k-j} - j) = 2^{-(j-1)},$$

which implies that $\mathcal{F}$ is a Corson filter on $(\mathcal{C}^*,\mathcal{U}_{d_{\mathcal{C}^*}})$.

Finally, suppose that $\mathcal{F}$ clusters to a function g with respect to $T((d_{\mathcal{C}^*})^s)$. Then there is a sequence $(f_{k_m}^{j_m})_{m\in\mathbb{N}}$ of distinct elements of $\mathcal{F}$ such that $f_{k_m}^{j_m} \in F_m$ for all $m \in \mathbb{N}$ and $(d_{\mathcal{C}^*})^s(g, f_{k_m}^{j_m}) \to 0$.

We have two cases:

Case 1. The sequence $(j_m)_{m\in\mathbb{N}}$ is bounded. Then there is an $i \in \mathbb{N}$ and a subsequence $(h_m)_{m\in\mathbb{N}}$ of $(f_{k_m}^{j_m})_{m\in\mathbb{N}}$ which is also a subsequence $(f_k^i)_{k\in\mathbb{N}}$. Hence, $(d_{\mathcal{C}^*})^s(g, h_m) \to 0$, so $(u_P)^s(g, h_m) \to 0$, and, by the triangle inequality, $(u_P)^s(g, g_i) = 0$, i.e. $g = g_i$, which is a contradiction because $d_{\mathcal{C}^*}(g_i, h_m) = 2^{-(i-1)}$ for all $m \in \mathbb{N}$.

Case 2. The sequence $(j_m)_{m\in\mathbb{N}}$ is not bounded. Let $m_0 \in \mathbb{N}$ such that $(d_{\mathcal{C}^*})^s(g, f_{k_m}^{j_m}) < 1$ for all $m \geq m_0$. Then, in particular, $|f_{k_m}^{j_m}(0) - g(0)| < 1$ for all $m \geq m_0$, which provides a new contradiction because $f_{k_m}^{j_m}(0) = j_m$, for all $m \in \mathbb{N}$.

We conclude that the Corson filter $\mathcal{F}$ has no cluster point in $(\mathcal{C}^*, (d_{\mathcal{C}^*})^s)$, and, thus, the quasi-uniform space $(\mathcal{C}^*,\mathcal{U}_{d_{\mathcal{C}^*}})$ is not Corson complete.

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