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Nontrivial Positive Solutions for Systems with the p-Laplacian in $\mathbb{R}^N$ Involving Critical Exponents

ABSTRACT. In this paper we give some results for the existence of positive solutions of p-Laplacian systems in $\mathbb{R}^N$ involving critical exponents. These solutions can be obtained by variational methods, more precisely by the mountain pass lemma and the concentration compactness lemma.

1 Introduction

In this paper we are interested in the existence of nontrivial positive solutions for Systems of q and p Laplacian involving critical exponents in $\mathbb{R}^N$. Our study follows several works for the case of a simple equation which take their origin in the paper of Brézis and Nirenberg [2] concerning a semilinear equation of the following form

$$(EC) \qquad \begin{cases} -\Delta u = u^{2^*-1} + \lambda u & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain and 2^* is the Sobolev exponent. We recall that for $1 \le p < +\infty$, $p^* = \frac{Np}{N-p}$.

They give existence of positive solutions when $N \ge 3$ and $\lambda \in (0, \lambda_1)$, where λ_1 is the first eigenvalue for the Laplacian operator.

Recently, several authors were interested by this type of problems in the case of the p-Laplacian (i.e. $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$). See [10], [5], [6], [8] and the references therein.

For $1 < p < N$, Drabeck and Huang in [6] consider the equation

$$(EC1) \qquad -\Delta_p u = \lambda g(x)|u|^{p-2}u + f(x)|u|^{p^*-2}u, \quad x \in \mathbb{R}^N.$$

They introduce $\lambda_1 > 0$, principal eigenvalue for the problem

$$\begin{cases} -\Delta_p u = \lambda g(x)|u|^{p-2}u & x \in \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} g(x)|u|^p dx > 0, \end{cases}$$

and they prove that Equation $(EC1)$ admits at least one weak positive solution if $\lambda \in (0, \lambda_1)$. This solution lies in $\mathcal{D}^{1,p}(I\!R^N)$, i.e. the closure of $\mathcal{C}_0^\infty(I\!R^N)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,p}(I\!R^N)} = \|\nabla u\|_{L^p(I\!R^N)}.$$

In this paper we extend the results above ([6], [2]) to the system

$$(SC) \qquad \begin{cases} -\Delta_p u = a(x)|u|^{p^*-2}u + \lambda b(x)|u|^{\alpha-1}u|v|^{\beta+1}, \\ -\Delta_q v = c(x)|v|^{q^*-2}v + \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v \qquad \text{in } I\!R^N, \\ \lim_{|x|\to+\infty} u(x) = \lim_{|x|\to+\infty} v(x) = 0, \\ u > 0,\ v > 0, \end{cases}$$

with

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1, \quad \alpha+\beta+2 < N, \tag{1.1}$$

$$p > 1, \quad q > 1, \quad \max\{p, q\} < N, \quad \alpha \geq 0, \quad \beta \geq 0. \tag{1.2}$$

We prove the existence of a positive solution for any $\lambda \in (0, \lambda_1)$. Here λ_1 is the principal eigenvalue associated to the problem

$$(VP) \qquad \begin{cases} -\Delta_p u = \lambda b(x)|u|^{\alpha-1}u|v|^{\beta+1}, \\ -\Delta_q v = \lambda b(x)|u|^{\alpha+1}|v|^{\beta-1}v \qquad \text{in } I\!R^N, \\ u > 0,\ v > 0, \\ \lim_{|x|\to+\infty} u(x) = \lim_{|x|\to+\infty} v(x) = 0. \end{cases}$$

J. Fleckinger, R.F. Manàsevich, N.M. Stavrakakis and F. de Thélin [7] prove that under hypotheses (1.1), (1.2) and

(H1) b is a $\mathcal{C}^{0,\gamma}(I\!R^N)$ function with $\gamma \in (0,1)$ which belongs to $L^{\frac{N}{\alpha+\beta+2}}(I\!R^N) \cap L^\infty(I\!R^N)$,

there exists a positive eigenvalue denoted by λ_1.
Results concerning the subcritical case have been obtained independently by [1] and [11]. But, from our knowledge, there are no results concerning the critical case for systems in $I\!R^N$. In the case of a bounded domain and $p = q$ see [4].

Now we come back to the critical system (SC). Here we shall prove the existence of positive solutions by a variational method, more precisely by the Montain Pass Theorem [3]. One of the difficulties in the critical case is that the Palais Smale conditions will be satisfied only for $c>0$ satisfying

$$c<\min\left\{\frac{\alpha+1}{N}\mathcal{S}_p^{\frac{N}{p}}\|a\|_{L^\infty(I\!R^N)}^{\frac{p-N}{p}},\ \frac{\beta+1}{N}\mathcal{S}_q^{\frac{N}{q}}\|c\|_{L^\infty(I\!R^N)}^{\frac{q-N}{q}}\right\}.$$

We recall that $(u,v)\in\mathcal{D}^{1,p}(I\!R^N)\times\mathcal{D}^{1,q}(I\!R^N)$ is a weak solution if it satisfies (SC) in weak sense. And the Sobolev constant $\mathcal{S}_p$ is given by

$$\mathcal{S}_p=\inf\left\{\frac{\|\nabla u\|_{L^p(I\!R^N)}^p}{\|u\|_{L^{p^*}(I\!R^N)}^p}\quad u\in W^{1,p}(I\!R^N)\backslash\{0\}\right\}.$$

Now, we introduce the hypotheses below what we need in our proofs:

(H2) a and c are nonnegative functions in $L^\infty(I\!R^N)$.

(H3) There exist $R>0$ and $b_0>0$ such that

$$b(x)\geq b_0>0\qquad\text{in } B_R.$$

(H4) If $\frac{\alpha+1}{N}\mathcal{S}_p^{\frac{N}{p}}\|a\|_{L^\infty(I\!R^N)}^{\frac{p-N}{p}}\leq\frac{\beta+1}{N}\mathcal{S}_q^{\frac{N}{q}}\|c\|_{L^\infty(I\!R^N)}^{\frac{q-N}{q}}$ we assume
- $\frac{(N-p)(\alpha+1)}{p}>\frac{pq}{p-1}$,
- $a(x)=a(0)+O(|x|^{k_1})$ near 0, where $k_1>\frac{\delta p}{p-1}$ and $\delta=N-\frac{N-p}{p}(\alpha+1)+\frac{N-p}{p^2}(\alpha+1)$,
- $a(0)=\|a\|_{L^\infty(I\!R^N)}$ and $a(x)>0$ in B_{2R}.

If $\frac{\alpha+1}{N}\mathcal{S}_p^{\frac{N}{p}}\|a\|_{L^\infty(I\!R^N)}^{\frac{p-N}{p}}\geq\frac{\beta+1}{N}\mathcal{S}_q^{\frac{N}{q}}\|c\|_{L^\infty(I\!R^N)}^{\frac{q-N}{q}}$ we assume
- $\frac{(N-q)(\beta+1)}{q}>\frac{pq}{q-1}$,
- $c(x)=c(0)+O(|x|^{k_2})$ near 0, where $k_2>\frac{\gamma q}{q-1}$ and $\gamma=N-\frac{N-q}{q}(\beta+1)+\frac{N-q}{q^2}(\beta+1)$,
- $c(0)=\|c\|_{L^\infty(I\!R^N)}$ and $c(x)>0$ in B_{2R}.

Example: If $a(x)=a(0)$ on B_R, $a(0)>\left\{\frac{\alpha+1}{\beta+1}\frac{\mathcal{S}_p^{\frac{N}{p}}}{\mathcal{S}_q^{\frac{N}{q}}}\|c\|_\infty^{\frac{N-q}{q}}\right\}^{\frac{p}{N-p}}$ and $N-p>\frac{pp'q}{\alpha+1}$, then the hypothesis (**H**4) is satisfied.

Theorem 1.1 *If the hypotheses* (**H**1) - (**H**4), (1.1) *and* (1.2) *are satisfied, then System* (SC) *admits a weak solution* $(u,v)\neq(0,0)$.

Remark 1.2 In fact, we prove that $u\neq 0$ and $v\neq 0$ (see Corollary 3.1).

2 Preliminaries

Lemma 2.1 *Assume that hypotheses* (**H**1), (1.1) *and* (1.2) *are satisfied; then for all* $\lambda \in (0, \lambda_1)$, *there exists* $\alpha_0 > 0$ *(depending of* λ*) such that* $\forall (u,v) \in \mathcal{D}^{1,p}(I\!R^N) \times \mathcal{D}^{1,q}(I\!R^N)$ *we have*

$$\begin{aligned} &\frac{\alpha+1}{p}\|u\|^p_{\mathcal{D}^{1,p}(I\!R^N)} + \frac{\beta+1}{q}\|v\|^q_{\mathcal{D}^{1,q}(I\!R^N)} - \lambda \int_{I\!R^N} b(x)|u|^{\alpha+1}|v|^{\beta+1}dx \\ &\geq \alpha_0 \left(\|u\|^p_{\mathcal{D}^{1,p}(I\!R^N)} + \|v\|^q_{\mathcal{D}^{1,q}(I\!R^N)} \right). \end{aligned} \tag{2.3}$$

■

Now, we introduce the functional

$$\begin{aligned} J_\lambda(u,v) = &\frac{\alpha+1}{p}\|u\|^p_{\mathcal{D}^{1,p}(I\!R^N)} + \frac{\beta+1}{q}\|v\|^q_{\mathcal{D}^{1,q}(I\!R^N)} - \frac{\alpha+1}{p^*}\int_{I\!R^N} a(x)|u|^{p^*}dx \\ &- \frac{\beta+1}{q^*}\int_{I\!R^N} c(x)|v|^{q^*}dx - \lambda \int_{I\!R^N} b(x)|u|^{\alpha+1}|v|^{\beta+1}dx. \end{aligned} \tag{2.4}$$

Classical arguments show that J_λ is well defined.
In the following lemma we recall Lions's concentration-compactness Lemma.

Lemma 2.2 [6], [9], [8].
Let $\{u_n\}_n$ *converge weakly to* u *in* $\mathcal{D}^{1,p}(I\!R^N)$ *such that* $\{|u_n|^{p^*}\}_n$ *and* $\{|\nabla u_n|^p\}_n$ *converge weakly to the nonnegative measure* ν, μ *on* $I\!R^N$ *respectively. Then*

1. $\nu = |u|^{p^*} + \sum_{j\in J} \nu_j \delta_{x_j}$ *with* $\nu_j > 0 \quad \forall j \in J$,
2. $\mu \geq |\nabla u|^p + \sum_{j\in J} \mu_j \delta_{x_j}$ *with* $\mu_j > 0 \quad \forall j \in J$,
3. $\mathcal{S}_p \nu_j^{\frac{p}{p^*}} \leq \mu_j, \quad \forall j \in J$,

where $x_j \in I\!R^N$, δ_x *is the Dirac measure,* J *is a countable set and* $\mathcal{S}_p$ *is the Sobolev constant.*

Proposition 2.3 *Let* $\{u_n\}_n$ *converge weakly to* u_0 *in* $\mathcal{D}^{1,p}(I\!R^N)$. *Then for all* $h \in L^\infty(I\!R^N)$, *there exists a subsequence of* $\{u_n\}_n$ *denoted again by* $\{u_n\}_n$ *such that for all* $\phi \in \mathcal{C}_0^\infty(I\!R^N)$

$$\int_{I\!R^N} h|u_n|^{p^*-2}u_n\phi dx \to \int_{I\!R^N} h|u_0|^{p^*-2}u_0\phi dx,$$

and

$$\int_{I\!R^N} h|u_n|^{p-2}u_n\phi dx \to \int_{I\!R^N} h|u_0|^{p-2}u_0\phi dx.$$

See [6] for the proof of Proposition 2.3.

Proposition 2.4 *Let $\{(u_n, v_n)\}_n$ be a bounded sequence converging weakly to (u, v) in $\mathcal{D}^{1,p}(I\!R^N) \times \mathcal{D}^{1,q}(I\!R^N)$ and satisfying*

$$J_\lambda(u_n, v_n) \to C, \tag{2.5}$$

and

$$J'_\lambda(u_n, v_n) \to 0. \tag{2.6}$$

Then there exists a subsequence still denoted by $\{(u_n, v_n)\}_n$ such that

$$|\nabla u_n|^{p-2}\nabla u_n \to |\nabla u|^{p-2}\nabla u \qquad \textit{weakly in } \left(L^p(I\!R^N)\right)', \tag{2.7}$$

$$|\nabla v_n|^{q-2}\nabla v_n \to |\nabla v|^{q-2}\nabla v \qquad \textit{weakly in } \left(L^q(I\!R^N)\right)'. \tag{2.8}$$

Proof: It is clear from Sobolev embedding that $\{(u_n, v_n)\}_n$ converges strongly in $L^p_{loc}(I\!R^N) \times L^q_{loc}(I\!R^N)$ and so there exists a subsequence still denoted by $\{(u_n, v_n)\}_n$ such that

$$(u_n, v_n) \to (u, v) \qquad \text{a.e. in } \; I\!R^N.$$

Observe, that all hypotheses of Lemma 2.2 are satisfied, and thus, there exist two countable sets J and $\bar{J}$ such that $(u_n)_n$ and $(v_n)_n$ satisfy the properties 1, 2 and 3 (Lemma 2.2) in J and $\bar{J}$ respectively.

First, we show that J and $\bar{J}$ are finite.

From (2.4) we have

$$\begin{aligned} J'_\lambda(u,v)(w,z) &= (\alpha+1)\int_{I\!R^N} |\nabla u|^{p-2}\nabla u.\nabla w dx + (\beta+1)\int_{I\!R^N} |\nabla v|^{q-2}\nabla v.\nabla z dx \\ &- (\alpha+1)\int_{I\!R^N} a(x)|u|^{p^*-2}uw dx - (\beta+1)\int_{I\!R^N} c(x)|v|^{q^*-2}vz dx \\ &- \lambda\int_{I\!R^N} \left[(\alpha+1)b(x)|u|^{\alpha-1}uw|v|^{\beta+1} + (\beta+1)b(x)|u|^{\alpha+1}|v|^{\beta-1}vz\right] dx, \end{aligned} \tag{2.9}$$

for all $(u, v),\ (w, z) \in \mathcal{D}^{1,p}(I\!R^N) \times \mathcal{D}^{1,q}(I\!R^N)$.

Now, for $x_j \in \{x_i,\ i \in J\}$ we introduce a function $\phi_j \in \mathcal{C}_0^\infty(I\!R^N)$ satisfying $\phi_j \equiv 1$ in $B(x_j, \epsilon)$, $\phi_j \equiv 0$ in $I\!R^N / B(x_j, 2\epsilon)$ and $|\nabla \phi_j| \le \dfrac{c}{\epsilon}$.

Substituting $w = \phi_j u_n$, $z \equiv 0$ in (2.9) and using (2.6) we obtain

$$\begin{aligned} &\lim_{n\to+\infty}\int_{I\!R^N} |\nabla u_n|^{p-2}u_n(\nabla u_n.\nabla\phi_j)dx \\ &\quad = \lim_{n\to\infty}\left[\int_{I\!R^N} a(x)|u_n|^{p^*}\phi_j dx + \lambda\int_{I\!R^N} b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\phi_j dx - \int_{I\!R^N} |\nabla u_n|^p\phi_j dx\right]. \end{aligned} \tag{2.10}$$

Classical arguments (see [1]) show that

$$\lim_{n\to+\infty}\int_{I\!R^N} b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}\phi_j dx = \int_{I\!R^N} b(x)|u|^{\alpha+1}|v|^{\beta+1}\phi_j dx. \tag{2.11}$$

And from Lemma 2.2, (2.11) and (2.10), we obtain

$$\begin{aligned}&\lim_{n\to+\infty}\int_{I\!R^N}|\nabla u_n|^{p-2}u_n(\nabla u_n.\nabla\phi_j)dx\\ &\quad=\int_{I\!R^N}a(x)\phi_j d\nu+\lambda\int_{I\!R^N}b(x)|u|^{\alpha+1}|v|^{\beta+1}\phi_j dx-\int_{I\!R^N}\phi_j d\mu.\end{aligned} \tag{2.12}$$

From [6] we have

$$\left|\lim_{n\to+\infty}\int_{I\!R^N}|\nabla u_n|^{p-2}u_n(\nabla u_n.\nabla\phi_j)dx\right|\to 0, \tag{2.13}$$

as $\epsilon\to 0$.
Letting ϵ go to 0 in (2.12) and using (2.13), we obtain

$$a(x_j)\nu_j=\mu_j. \tag{2.14}$$

We deduce from this last equality

$$a(x_j)>0 \quad \forall j\in J, \tag{2.15}$$

$$\nu_j\geq\left(\frac{\mathcal{S}_p}{a(x_j)}\right)^{\frac{N}{p}}, \tag{2.16}$$

$$\text{J is finite.} \tag{2.17}$$

Indeed, if J is infinite we deduce from (2.16) and Lemma 2.2 that $\int_{I\!R^N}|u_n|^{p^*}dx\to+\infty$; a contradiction.
Now, we let $J=\{1,...,m\}$ and $\Omega_{\epsilon_0}=\{x\in I\!R^N:\ dist(x_j,x)>\epsilon_0,\ \forall j=1,...,m\}$.
Arguing as in [8] we prove that (see the details in [1])

$$\int_{\Omega_{\epsilon_0}}(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u)(\nabla u_n-\nabla u)dx\to 0 \quad \forall\epsilon_0>0. \tag{2.18}$$

Classical arguments show that

$$\frac{\partial u_n}{\partial x_i}\to\frac{\partial u}{\partial x_i} \quad a.e.\ in\ \Omega_{\epsilon_0} \quad 1\leq i\leq N.$$

Since ϵ_0 is arbitrary, we conclude by a diagonal process that there exists a subsequence denoted again by (u_n) such that

$$\frac{\partial u_n}{\partial x_i}\to\frac{\partial u}{\partial x_i} \quad a.e.\ in\ I\!R^N \quad 1\leq i\leq N.$$

Since $\{|\nabla u_n|^{p-2}\dfrac{\partial u_n}{\partial x_i}\}$ is bounded in $(L^p(I\!R^N))'$, we conclude that

$$|\nabla u_n|^{p-2}\frac{\partial u_n}{\partial x_i} \to |\nabla u|^{p-2}\frac{\partial u}{\partial x_i} \quad \textit{weakly in} \;\; (L^p(I\!R^N))', \quad 1 \le i \le N.$$

Hence (2.7).
Similarly we obtain (2.8). ■

In the following lemma we prove Palais Smale conditions.

Lemma 2.5 *Suppose that the hypotheses* (**H**1), (**H**2), (1.1), (1.2) *hold and* $\lambda \in (0, \lambda_1)$. *We define*

$$\mathcal{S}_0 = \min\left\{\frac{\alpha+1}{N}\mathcal{S}_p^{\frac{N}{p}}\,\|a\|_{L^\infty(I\!R^N)}^{\frac{p-N}{p}}, \frac{\beta+1}{N}\mathcal{S}_q^{\frac{N}{q}}\,\|c\|_{L^\infty(I\!R^N)}^{\frac{q-N}{q}}\right\}. \tag{2.19}$$

Then the function J_λ satisfies the Palais Smale conditions at C for all $C < \mathcal{S}_0$.

Proof: Let $\{(u_n, v_n)\}_n \subset \mathcal{D}^{1,p}(I\!R^N) \times \mathcal{D}^{1,q}(I\!R^N)$ be so that (2.5) and (2.6) are satisfied.
First, we will prove that $\{(u_n,\ v_n)\}_n$ is bounded.
Suppose the contrary, so $\|(u_n,\ v_n)\|_{\mathcal{D}^{1,p}(I\!R^N)\times\mathcal{D}^{1,q}(I\!R^N)} \to +\infty$.
We deduce from (2.4) and (2.9)

$$\begin{aligned} &J_\lambda(u_n, v_n) - J'_\lambda(u_n, v_n)(\frac{u_n}{p}, \frac{v_n}{q}) \\ &\quad = \frac{\alpha+1}{N}\int_{I\!R^N} a(x)|u_n|^{p^*}dx + \frac{\beta+1}{N}\int_{I\!R^N} c(x)|v_n|^{q^*}dx \\ &\quad = C + o(1)\|(u_n,\ v_n)\|_{\mathcal{D}^{1,p}(I\!R^N)\times\mathcal{D}^{1,q}(I\!R^N)}. \end{aligned} \tag{2.20}$$

Thus

$$\frac{\alpha+1}{p}\|u_n\|^p_{\mathcal{D}^{1,p}(I\!R^N)} + \frac{\beta+1}{q}\|v_n\|^q_{\mathcal{D}^{1,q}(I\!R^N)} - \lambda\int_{I\!R^N} b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}dx \le c\|(u_n, v_n)\|, \tag{2.21}$$

and with Lemma 2.1, it follows

$$\alpha_0(\|u_n\|^p_{\mathcal{D}^{1,p}(I\!R^N)} + \|v_n\|^q_{\mathcal{D}^{1,q}(I\!R^N)}) \le c\|(u_n, v_n)\|_{\mathcal{D}^{1,p}(I\!R^N)\times\mathcal{D}^{1,q}(I\!R^N)}, \tag{2.22}$$

which implies that $\{(u_n,\ v_n)\}_n$ is a bounded sequence.
Now, we prove that we can extract a converging subsequence.

It is clear that there exists a subsequence denoted again by $\{(u_n, v_n)\}_n$ which converges weakly to (u, v) in $\mathcal{D}^{1,p}(I\!R^N) \times \mathcal{D}^{1,q}(I\!R^N)$, and satisfies

$$\begin{aligned} |u_n|^{p^*} &\to \nu, \qquad & |\nabla u_n|^p &\to \mu, \\ |v_n|^{q^*} &\to \bar{\nu}, \qquad & |\nabla v_n|^q &\to \bar{\mu}, \end{aligned}$$

weakly.
From Propositions 2.3, 2.4 we have

$$J'_\lambda(u, v) = 0. \tag{2.23}$$

It is easy to show that the hypotheses of Lemma 2.2 are satisfied and from the proof of Proposition 2.4 that

$$a(x_j)\nu_j = \mu_j \quad \forall j \in J, \qquad b(x_j)\bar{\nu}_j = \bar{\mu}_j \quad \forall j \in \bar{J}, \tag{2.24}$$

and

$$\nu_j \geq [\frac{\mathcal{S}_p}{a(x_j)}]^{\frac{N}{p}} \quad \forall j \in J, \qquad \bar{\nu}_j \geq [\frac{\mathcal{S}_q}{c(x_j)}]^{\frac{N}{q}} \quad \forall j \in \bar{J}. \tag{2.25}$$

J and $\bar{J}$ are finit sets.
We will prove that J and $\bar{J}$ are empty sets.
Suppose that $J \cup \bar{J} \neq \emptyset$.
Since $\{(u_n, v_n)\}_n$ converges weakly to (u, v) and

$$\int_{I\!R^N} b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}dx \to \int_{I\!R^N} b(x)|u|^{\alpha+1}|v|^{\beta+1}dx,$$

we deduce that for any $\sigma > 0$ and n sufficiently large

$$\begin{aligned} C + \sigma &> J_\lambda(u_n, v_n) - J'_\lambda(u_n, v_n)(\frac{u_n}{p^*}, \frac{v_n}{q^*}) \\ &= \frac{\alpha+1}{N}\|u_n\|^p_{\mathcal{D}^{1,p}(I\!R^N)} + \frac{\beta+1}{N}\|v_n\|^q_{\mathcal{D}^{1,q}(I\!R^N)} \\ &\quad - \lambda(\frac{\alpha+\beta+2}{N}) \int_{I\!R^N} b(x)|u_n|^{\alpha+1}|v_n|^{\beta+1}dx. \end{aligned} \tag{2.26}$$

Taking $n \to +\infty$, we obtain

$$\begin{aligned} C + \sigma \geq{}& \frac{\alpha+1}{N}\|u\|^p_{\mathcal{D}^{1,p}(I\!R^N)} + \frac{\beta+1}{N}\|v\|^q_{\mathcal{D}^{1,q}(I\!R^N)} \\ &- \lambda(\frac{\alpha+\beta+2}{N}) \int_{I\!R^N} b(x)|u|^{\alpha+1}|v|^{\beta+1}dx \\ &+ \frac{\alpha+1}{N}\sum_{j\in J}\mu_j + \frac{\beta+1}{N}\sum_{j\in \bar{J}}\bar{\mu}_j. \end{aligned} \tag{2.27}$$

Taking into account (2.23), (2.24) and (2.25) we obtain

$$\begin{aligned}
C+\sigma &\geq \frac{\alpha+1}{N}\int_{I\!R^N} a(x)|u|^{p^*}dx + \frac{\beta+1}{N}\int_{I\!R^N} c(x)|v|^{q^*}dx \\
&+ \frac{\alpha+1}{N}\sum_{j\in J} a(x_j)\nu_j + \frac{\beta+1}{N}\sum_{j\in \bar{J}} c(x_j)\bar{\nu}_j \\
&> \frac{\alpha+1}{N}\int_{I\!R^N} a(x)|u|^{p^*}dx + \frac{\beta+1}{N}\int_{I\!R^N} c(x)|v|^{q^*}dx \\
&+ \min\left\{\frac{\alpha+1}{N}\mathcal{S}_p^{\frac{N}{p}}\|a\|_{L^\infty(I\!R^N)}^{\frac{p-N}{p}}, \frac{\beta+1}{N}\mathcal{S}_q^{\frac{N}{q}}\|c\|_{L^\infty(I\!R^N)}^{\frac{q-N}{q}}\right\}.
\end{aligned}$$

Since $C < \mathcal{S}_0$ and σ is arbitrary this implies

$$\frac{\alpha+1}{N}\int_{I\!R^N} a(x)|u|^{p^*}dx + \frac{\beta+1}{N}\int_{I\!R^N} c(x)|v|^{q^*}dx < 0, \tag{2.28}$$

which contradicts (**H**2).
Therefore J and $\bar{J}$ are empty, hence

$$\int_{I\!R^N} |u_n|^{p^*}dx \to \int_{I\!R^N} |u|^{p^*}dx \quad \text{and} \quad \int_{I\!R^N} |v_n|^{q^*}dx \to \int_{I\!R^N} |v|^{q^*}dx.$$

With the weak convergence of $\{(u_n, v_n)\}_n$ to (u, v) in $\mathcal{D}^{1,p}(I\!R^N)\times\mathcal{D}^{1,q}(I\!R^N)$, we have

$$(u_n, v_n) \to (u, v) \qquad \text{strongly in } L^{p^*}(I\!R^N)\times L^{q^*}(I\!R^N). \tag{2.29}$$

■

Lemma 2.6 *Suppose that the hypotheses* (**H**1), (**H**2), (1.1) *and* (1.2) *are satisfied, then there exist two constants* $\delta > 0, \rho > 0$ *such that*

$$J_\lambda(u, v) \geq \rho, \qquad \textit{for all } \|(u,v)\|_{\mathcal{D}^{1,p}(I\!R^N)\times\mathcal{D}^{1,q}(I\!R^N)} = \delta \quad \textit{and} \quad \lambda\in(0,\lambda_1). \tag{2.30}$$

Proof: We deduce from (2.3) and the Sobolev inequality that

$$J_\lambda(u,v) \geq \alpha_0(\|u\|^p_{\mathcal{D}^{1,p}(I\!R^N)} + \|v\|^q_{\mathcal{D}^{1,q}(I\!R^N)}) - c\|u\|^{p^*}_{\mathcal{D}^{1,p}(I\!R^N)} - c\|v\|^{q^*}_{\mathcal{D}^{1,q}(I\!R^N)}.$$

So, for δ sufficiently small we have

$$J_\lambda(u, v) \geq \rho.$$

■

Let $R > \epsilon > 0$, $0 < r \leq R$ and ϕ_r be a nonnegative function in $\mathcal{C}_0^\infty(I\!R^N)$ satisfying $0 \leq \phi_r \leq 1$, $\phi_r \equiv 1$ in $B(0, r)$ and $\phi \equiv 0$ in $I\!R^N/B(0, 2r)$.
We define

$$u_\epsilon(x) = \frac{\phi_R(x)}{(\epsilon + |x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}}, \qquad w_\epsilon(x) = \frac{u_\epsilon}{\|u_\epsilon\|_{L^{p^*}(I\!R^N)}}, \qquad z_\epsilon(x) = \phi_\epsilon(x).$$

Lemma 2.7 *We have*

1) $\|u_\epsilon\|^p_{L^{p^*}(I\!R^N)} = c\epsilon^{\frac{p-N}{p}} + O(1).$

For ϵ sufficiently small

2) $\dfrac{1}{\|u_\epsilon\|^{p^*}_{L^{p^*}(I\!R^N)}} \displaystyle\int_{I\!R^N} \frac{a(0)-a(x)}{(\epsilon+|x|^{\frac{p}{p-1}})^{\frac{N-p}{p}}}dx = O(\epsilon^{k_1(\frac{p-1}{p})}).$

3) $\displaystyle\int_{I\!R^N} w_\epsilon(x)^{\alpha+1}\phi_\epsilon(x)^{\beta+1}dx \geq c\epsilon^\delta,$
where $\delta = N - (\frac{N-p}{p})(\alpha+1) + (\frac{N-p}{p^2})(\alpha+1).$

Proof: For the proof of 1) and 2) see [2].
3) We have

$$\begin{aligned}
\int_{I\!R^N} u_\epsilon^{\alpha+1}\phi_\epsilon^{\beta+1}dx \geq \int_{B_\epsilon} u_\epsilon^{\alpha+1}dx &= \int_0^\epsilon \frac{s^{N-1}}{\left(\epsilon+s^{\frac{p}{p-1}}\right)^{\frac{(N-p)(\alpha+1)}{p}}}ds \\
&= \epsilon^{(\frac{p-1}{p})(N)-(\frac{N-p}{p})(\alpha+1)} \int_0^{\epsilon^{\frac{1}{p}}} \frac{r^{N-1}}{\left(1+r^{\frac{p}{p-1}}\right)^{\frac{(N-p)(\alpha+1)}{p}}}dr \qquad (2.31)\\
&\geq c\epsilon^{(\frac{p-1}{p})(N)-(\frac{N-p}{p})(\alpha+1)} \int_0^{\epsilon^{\frac{1}{p}}} r^{N-1}dr \\
&\geq c\epsilon^{N-(\frac{N-p}{p})(\alpha+1)}.
\end{aligned}$$

On the other hand, for ϵ sufficiently small

$$\frac{1}{\|u_\epsilon\|^{\alpha+1}_{L^{p^*}(I\!R^N)}} \geq c\epsilon^{(\frac{N-p}{p^2})(\alpha+1)}. \qquad (2.32)$$

Therefore

$$\int_{I\!R^N} w_\epsilon^{\alpha+1}\phi_\epsilon^{\beta+1}dx \geq c\epsilon^{N-(\frac{N-p}{p})(\alpha+1)+(\frac{N-p}{p^2})(\alpha+1)}. \qquad (2.33)$$

Lemma 2.8 *Suppose that the hypotheses* (**H**1) - (**H**4), (1.1) *and* (1.2) *are satisfied, then there exist $\epsilon > 0$ and $t_1 > 0$ such that for all $\lambda \in (0,\lambda_1)$* $J_\lambda(t_1^{\frac{1}{p}}w_\epsilon, t_1^{\frac{1}{q}}\phi_\epsilon) < 0$ *and*

$$0 < \sup_{t>0} J_\lambda(t^{\frac{1}{p}}w_\epsilon, t^{\frac{1}{q}}\phi_\epsilon) < \frac{\alpha+1}{N}\mathcal{S}_p^{\frac{N}{p}}\|a\|_\infty^{\frac{p-N}{p}}. \qquad (2.34)$$

Proof: We have

$$\begin{aligned} &J_\lambda(t^{\frac{1}{p}}w_\epsilon, t^{\frac{1}{q}}\phi_\epsilon)\\ &= t\left(\frac{\alpha+1}{p}\|w_\epsilon\|^p_{\mathcal{D}^{1,p}(I\!\!R^N)} + \frac{\beta+1}{q}\|\phi_\epsilon\|^q_{\mathcal{D}^{1,q}(I\!\!R^N)} - \lambda\int_{I\!\!R^N} b(x)|w_\epsilon|^{\alpha+1}|\phi_\epsilon|^{\beta+1}dx\right)\\ &- \frac{\alpha+1}{p^*}t^\tau\int_{I\!\!R^N} a(x)|w_\epsilon|^{q^*}dx - \frac{\beta+1}{q^*}t^{\bar\tau}\int_{I\!\!R^N} c(x)|\phi_\epsilon|^{p^*}dx. \end{aligned} \tag{2.35}$$

Since $t \to -J_\lambda(t^{\frac{1}{p}}w_\epsilon, t^{\frac{1}{q}}\phi_\epsilon)$ is a convex differentiable function it admits a minimum at $t_\epsilon \in [0,+\infty)$ satisfying

$$\frac{dJ_\lambda}{dt}(t_\epsilon^{\frac{1}{p}}w_\epsilon, t_\epsilon^{\frac{1}{q}}\phi_\epsilon) = 0.$$

Therefore

$$\begin{aligned} &\frac{\alpha+1}{p}t_\epsilon^{\tau-1}\int_{I\!\!R^N} a(x)|w_\epsilon|^{p^*}dx + \frac{\beta+1}{q}t_\epsilon^{\bar\tau-1}\int_{I\!\!R^N} c(x)|\phi_\epsilon|^{q^*}dx\\ &\le \frac{\alpha+1}{p}\|w_\epsilon\|^p_{L^{p^*}(I\!\!R^N)} + \frac{\beta+1}{q}\|\phi_\epsilon\|^q_{L^{q^*}(I\!\!R^N)} - \lambda\int_{I\!\!R^N} b(x)|w_\epsilon|^{\alpha+1}|\phi_\epsilon|^{\beta+1}dx. \end{aligned} \tag{2.36}$$

First, we show that t_ϵ is bounded from above and below.

• Indeed, it follows from (2.36), (**H**1), (**H**2) and (**H**4)

$$\begin{aligned} &\frac{\alpha+1}{p}t_\epsilon^{\tau-1}a(0)\int_{I\!\!R^N}|w_\epsilon|^{p^*}dx - \frac{\alpha+1}{p}t_\epsilon^{\tau-1}\int_{I\!\!R^N}(a(0)-a(x))|w_\epsilon|^{p^*}dx\\ &\le \frac{\alpha+1}{p}\|w_\epsilon\|^p_{\mathcal{D}^{1,p}(I\!\!R^N)} + \frac{\beta+1}{q}\|\phi_\epsilon\|^q_{\mathcal{D}^{1,q}(I\!\!R^N)}. \end{aligned}$$

From 2) in Lemma 2.7, the fact that $\|w_\epsilon\|_{\mathcal{D}^{1,p}(I\!\!R^N)} = \mathcal{S}_p$, and $\int_{I\!\!R^N}|w_\epsilon|^{p^*} = 1$, we have

$$\frac{\alpha+1}{p}[a(0) - O(\epsilon^{k_1(\frac{p-1}{p})})]t_\epsilon^{\tau-1} \le \frac{\alpha+1}{p}\mathcal{S}_p + \frac{\beta+1}{q}\|\phi_\epsilon\|^q_{\mathcal{D}^{1,q}(I\!\!R^N)}.$$

By taking ϵ small enough such that $O(\epsilon^{k_1(\frac{p-1}{p})}) \le \dfrac{a(0)}{2}$ and remarking that $\|\phi_\epsilon\|^q_{\mathcal{D}^{1,q}(I\!\!R^N)} \le c\epsilon^{N-q}$ we obtain

$$\frac{\alpha+1}{2p}a(0)t_\epsilon^{\tau-1} \le \frac{\alpha+1}{p}\mathcal{S}_p + c. \tag{2.37}$$

This proves that t_ϵ is bounded from above.

• On the other hand, observe that

$$\begin{aligned} J_\lambda(t^{\frac{1}{p}}w_\epsilon, t^{\frac{1}{q}}\phi_\epsilon) &\ge \alpha_0 t\left(\|w_\epsilon\|^p_{\mathcal{D}^{1,p}(I\!\!R^N)} + \|\phi_\epsilon\|^q_{\mathcal{D}^{1,q}(I\!\!R^N)}\right)\\ &- \frac{\alpha+1}{p^*}t^\tau\int_{I\!\!R^N} a(x)|w_\epsilon|^{p^*}dx - \frac{\beta+1}{q^*}t^{\bar\tau}\int_{I\!\!R^N} c(x)|\phi_\epsilon|^{q^*}dx\\ &\ge \alpha_0 t\mathcal{S}_p^p - \frac{\alpha+1}{p^*}\|a\|_{L^\infty(I\!\!R^N)}t^\tau - \frac{\beta+1}{q^*}\|c\|_{L^\infty(I\!\!R^N)}t^{\bar\tau} = \Theta(t). \end{aligned} \tag{2.38}$$

It is clear that

$$\frac{dJ_\lambda}{dt}(0) \geq \Theta'(0) = \alpha_0 \mathcal{S}_p^p = k > 0,$$

and by continuity there exists $\eta > 0$ such that $J'_\lambda(t^{\frac{1}{p}} w_\epsilon, t^{\frac{1}{q}} \phi_\epsilon) \geq \Theta'(t) > \frac{k}{2}$ for any $|t| < \eta$. Hence $t_\epsilon \geq \eta$.

On the other hand, we deduce from (2.38) that for $\epsilon > 0$ sufficiently small $J_\lambda(t^{\frac{1}{p}} w_\epsilon, t^{\frac{1}{q}} \phi_\epsilon) > 0$ which implies that $J_\lambda(t_\epsilon^{\frac{1}{p}} w_\epsilon, t_\epsilon^{\frac{1}{q}} \phi_\epsilon) > 0$. Now, we write $J_\lambda(t_\epsilon^{\frac{1}{p}} w_\epsilon, t_\epsilon^{\frac{1}{q}} \phi_\epsilon)$ as below

$$J_\lambda(t_\epsilon^{\frac{1}{p}} w_\epsilon, t_\epsilon^{\frac{1}{q}} \phi_\epsilon) = \sup_{t \geq 0} J_\lambda(t^{\frac{1}{p}} w_\epsilon, t^{\frac{1}{q}} \phi_\epsilon) = E_\epsilon - F_\epsilon,$$

where

$$\begin{aligned} E_\epsilon = {} & t_\epsilon \frac{\alpha+1}{p} \|w_\epsilon\|^p_{\mathcal{D}^{1,p}(I\!\!R^N)} - \frac{\alpha+1}{p} a(0) t_\epsilon^\tau \int_{I\!\!R^N} w_\epsilon^{p^*} dx \\ & + t_\epsilon \frac{\beta+1}{q} \|\phi_\epsilon\|^q_{\mathcal{D}^{1,p}(I\!\!R^N)} - \frac{\beta+1}{q^*} t_\epsilon^{\bar{\tau}} \int_{I\!\!R^N} c(x) \phi_\epsilon^{q^*} dx, \end{aligned}$$

$$F_\epsilon = \lambda t_\epsilon \int_{I\!\!R^N} b(x) |w_\epsilon|^{\alpha+1} |\phi_\epsilon|^{\beta+1} dx - \frac{\alpha+1}{p^*} t_\epsilon^\tau \int_{I\!\!R^N} (a(0) - a(x)) w_\epsilon^{p^*} dx.$$

We consider the function

$$m(t) = Mt - \frac{L}{\tau} t^\tau,$$

m attains its maximum at $t_1 = \left(\frac{M}{L}\right)^{\frac{N-p}{p}}$. Since t_ϵ is bounded from above, since $\|\phi_\epsilon\|^q_{\mathcal{D}^{1,q}(I\!\!R^N)}$ $= O(\epsilon^{N-q})$ and c is nonnegative, we deduce that

$$\begin{aligned} E_\epsilon & \leq \frac{\alpha+1}{p} \left(\frac{\|w_\epsilon\|^N_{\mathcal{D}^{1,p}(I\!\!R^N)}}{a(0)^{\frac{N-p}{p}}} - \frac{1}{\tau} \frac{\|w_\epsilon\|^N_{\mathcal{D}^{1,p}(I\!\!R^N)}}{a(0)^{\frac{N-p}{p}}} \right) + O(\epsilon^{N-q}) \\ & = \frac{\alpha+1}{N} \|a\|_\infty^{\frac{p-N}{p}} \mathcal{S}^{\frac{N}{p}} + O(\epsilon^{N-q}). \end{aligned} \tag{2.39}$$

(Remark that $\|w_\epsilon\|^N_{\mathcal{D}^{1,p}(I\!\!R^N)} = \mathcal{S}_p^{\frac{N}{p}}$.)

Since t_ϵ is bounded from below and above, it follows from Lemma 2.7 that

$$\begin{aligned} \lambda t_\epsilon \int_{I\!\!R^N} b(x) |w_\epsilon|^{\alpha+1} |\phi_\epsilon|^{\beta+1} & \geq \lambda t_\epsilon \int_{I\!\!R^N} b_0 |w_\epsilon|^{\alpha+1} |\phi_\epsilon|^{\beta+1} dx \\ & \geq c\epsilon^\delta. \end{aligned}$$

and

$$\frac{\alpha+1}{p} t_\epsilon^\tau \int_{I\!\!R^N} (a(0) - a(x)) w_\epsilon^{p^*} dx = O(\epsilon^{k_1(\frac{p-1}{p})}) \tag{2.40}$$

Observing that $\delta < k_1(\frac{p-1}{p})$, this implies

$$F_\epsilon \geq c\epsilon^\delta.$$

Therefore

$$J_\lambda(t_\epsilon^{\frac{1}{p}} w_\epsilon, t_\epsilon^{\frac{1}{p}} \phi_\epsilon) \leq \frac{\alpha+1}{N} \mathcal{S}^{\frac{N}{p}} \|a\|_{L^\infty(I\!R^N)}^{\frac{p-N}{p}} - c\epsilon^\delta + O(\epsilon^{N-q}).$$

On the other hand, it follows from (**H**4) that $\delta < N - q$. Thus, we obtain

$$J_\lambda(t_\epsilon^{\frac{1}{p}} w_\epsilon, t_\epsilon^{\frac{1}{p}} \phi_\epsilon) \leq \frac{\alpha+1}{N} \mathcal{S}^{\frac{N}{p}} \|a\|_{L^\infty(I\!R^N)}^{\frac{p-N}{p}} - c\epsilon^\delta,$$

hence (2.34).

By taking t sufficiently large in (2.35), we have $J_\lambda(t^{\frac{1}{p}} w_\epsilon,\ t^{\frac{1}{q}} \phi_\epsilon) < 0$, and thus there exists a t_1 such that $J_\lambda(t_1^{\frac{1}{p}} w_\epsilon,\ t_1^{\frac{1}{q}} \phi_\epsilon) < 0$, which concludes the proof of Lemma 2.8.

■

3 Proof of Theorem 1.1

We let $X = \mathcal{D}^{1,p}(I\!R^N) \times \mathcal{D}^{1,q}(I\!R^N)$, $u_0 = (t_1^{\frac{1}{p}} w_\epsilon, t_1^{\frac{1}{q}} z_\epsilon)$ and $C = \inf_{h\in\Gamma} \sup_{t\in[0,1]} J(h(t))$, where

$$\Gamma = \{h \in \mathcal{C}([0,1], X) :\ h(0) = 0,\ h(1) = u_0\}.$$

It is clear from Lemma 2.8 that $C < \dfrac{\alpha+1}{N} \mathcal{S}_p^{\frac{N}{p}} \|a\|_\infty^{\frac{p-N}{p}}$, furthermore (**H**4) and Lemma 2.5 imply that J_λ satisfies Palais Smale conditions at C. Thus, by Lemmas 2.8, 2.6 and the Mountain Pass Lemma we deduce that (SC) admits a solution $(u, v) \neq (0, 0)$. Moreover this solution is nonnegative because $J_\lambda(h(t)) = J_\lambda(|h(t)|)$.

■

Corollary 3.1 *The solution (u, v) obtained in Theorem 1.1 satisfies $u \neq 0$ and $v \neq 0$.*

Proof: It is clear from Theorem 1.1 that there exist other solutions than $(0, 0)$. Suppose (for example) that $u \neq 0$ and $v = 0$. In this case System (SC) is reduced to the simple equation

$$-\Delta_p u = a(x)|u|^{p^*-2}u. \tag{3.41}$$

Multiplying equation (3.41) by u, and integrating over $I\!R^N$, we have

$$\|u\|_{\mathcal{D}^{1,p}(I\!R^N)}^p = \int_{I\!R^N} a(x)|u|^{p^*} dx \leq \|a\|_{L^\infty(I\!R^N)} \int_{I\!R^N} |u|^{p^*} dx. \tag{3.42}$$

From the definition of $\mathcal{S}_p$

$$\mathcal{S}_p^{\frac{p^*}{p}} \int_{I\!R^N} |u|^{p^*} dx \leq \|u\|^{p^*}_{\mathcal{D}^{1,p}(I\!R^N)}. \tag{3.43}$$

Combining (3.42) and (3.43), we have

$$\|u\|^p_{\mathcal{D}^{1,p}(I\!R^N)} \leq \|a\|_{L^\infty(I\!R^N)} \int_{I\!R^N} |u|^{p^*} dx \leq \mathcal{S}_p^{-\frac{p^*}{p}} \|a\|_{L^\infty(I\!R^N)} \|u\|^{p^*}_{\mathcal{D}^{1,p}(I\!R^N)}. \tag{3.44}$$

Therefore

$$\|u\|^p_{\mathcal{D}^{1,p}(I\!R^N)} \geq \|a\|^{\frac{p-N}{p}}_{L^\infty(I\!R^N)} \mathcal{S}_p^{\frac{N}{p}}. \tag{3.45}$$

Using (3.42), we have

$$J_\lambda(u,0) = \frac{\alpha+1}{p} \|u\|^p_{\mathcal{D}^{1,p}(I\!R^N)} - \frac{\alpha+1}{p^*} \int_{I\!R^N} a(x)|u|^{p^*} dx = \frac{\alpha+1}{N} \|u\|^p_{\mathcal{D}^{1,p}(I\!R^N)}. \tag{3.46}$$

So, we deduce from (3.45) and (3.46) that

$$J_\lambda(u,0) \geq \frac{\alpha+1}{N} \|a\|^{\frac{p-N}{p}}_{L^\infty(I\!R^N)} \mathcal{S}_p^{\frac{N}{p}}. \tag{3.47}$$

But from Theorem 1.1 $C = J_\lambda(u,0) < \dfrac{\alpha+1}{N} \|a\|^{\frac{p-N}{p}}_{L^\infty(I\!R^N)} \mathcal{S}_p^{\frac{N}{p}}$, which contradicts (3.47).

■

Corollary 3.2 *Suppose that the functions a, b, c are in $L^\infty(I\!R^N)$, $q \geq p$ and (1.1), (1.2) are satisfied. Then for each solution $(u,v) \in \mathcal{D}^{1,p}(I\!R^N) \times \mathcal{D}^{1,q}(I\!R^N)$ we have the following assertions:*

- $\lim\limits_{|x| \to +\infty} u(x) = \lim\limits_{|x| \to +\infty} v(x) = 0.$
- *Let $x \in I\!R^N$ and $R > 0$ be such that*

$$\begin{aligned} &\max\left(\|a\|_\infty, \|c\|_\infty, \|b\|_\infty\right) \max\left\{2^p \mathcal{S}_p \tau^{p-1}, 2^{2q-p} \mathcal{S}_q |B_1|^{\frac{q-p}{N}} R^{q-p} \tau^{q-1}\right\} \times \\ &\times \left(\|u\|^{p(\tau-1)}_{L^{p^*}(B_{2R}(x))} + \|v\|^{q(\tau-1)}_{L^{q\tau}(B_{2R}(x))}\right) < 1. \end{aligned} \tag{3.48}$$

Then each solution (u,v) of (SC) belongs to $\mathcal{D}^{1,p}(I\!R^N) \times \mathcal{D}^{1,q}(I\!R^N) \cap \left(\mathcal{C}^{1,\alpha}(B_{\frac{R}{2}}(x)\right)^2$.

The proof of this corollary is a consequence of Theorem 4.2.1 in [1] and Tolksdorfs Theorem [12].

■

Corollary 3.3 *Under the hypotheses of Corollary 3.2 the solution (u, v) of (SC) is positive.*

Observe that (3.48) is satisfied for all $x \in \mathbb{R}^N$ and R sufficiently small. So by applying the strong Maximum Principle of Vazquez [13] in the ball $B_{\frac{R}{2}}(x)$, we deduce that $u(x)$ and $v(x)$ are positive; since x is arbitrary this implies that u and v are positive everywhere on $\mathbb{R}^N$.

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