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# Estimates of the Solutions of Two-Scale Difference Equations

ABSTRACT. The solutions of special classes of two-scale difference equations are estimated in both directions.

KEY WORDS. Two-scale difference equations, de Rham's function, one-sided approximations

## 1 Introduction

A two-scale difference equation is a functional equation of the form

$$\varphi\left(\frac{t}{2}\right) = \sum_{\nu=0}^{n} c_\nu \varphi(t-\nu) \quad (t \in \mathbb{R})\,, \tag{1.1}$$

where $c_\nu$ are given real constants with $c_0 c_n \neq 0$ and $n \in \mathbb{N}$, cf. [3]. As usual, we only seek non-trivial solutions of (1.1) which vanish for $t \leq 0$. Hence, by restriction to $t \leq 1$, equation (1.1) reduces to

$$\varphi\left(\frac{t}{2}\right) = a\varphi(t) \tag{1.2}$$

with $a = c_0$. By the further restriction to $t \geq 0$ and $a > 0$, this equation has the general solution

$$\varphi(t) = t^\alpha P\left(\frac{\ln t}{\ln 2}\right) \tag{1.3}$$

where $\alpha = -\frac{\ln a}{\ln 2}$ and where $P(\cdot)$ is an arbitrary 1-periodic function. Hence, for a continuous (real) solution $\varphi$ of (1.1) with $\varphi(1) \neq 0$ there always exist estimates of the form

$$\lambda_1 \varphi(1) t^\alpha \leq \varphi(t) \leq \lambda_2 \varphi(1) t^\alpha \quad (t \in [0,1]) \tag{1.4}$$

with $\lambda_1 \leq 1 \leq \lambda_2$ in the case $\varphi(1) > 0$. The optimal constants are

$$\lambda_1 = \min \frac{P(\cdot)}{P(0)}, \quad \lambda_2 = \max \frac{P(\cdot)}{P(0)},$$

however since, usually speaking, the function $P$ is unknown, we are going to determine estimates for these optimal constants. In order to prove the estimates (1.4) for concrete $\lambda_1$, $\lambda_2$ it suffices to prove them for $t \in \left(\frac{1}{2}, 1\right]$ only because, if (1.4) is satisfied for a fixed $t$ then, in view of the periodicity of $P(\cdot)$, it is automatically satisfied for $2^{-k}t$ with arbitrary $k \in \mathbb{N}$, cf. (1.3).

In the following we shall determine $\lambda_1$, $\lambda_2$ in (1.3) for some classes of equations (1.1) up to $n = 3$. The special case

$$\varphi\left(\frac{t}{2}\right) = a\varphi(t) + (1-a)\varphi(t-1) \tag{1.5}$$

of (1.1) with $n = 1$ and $a \in (0, 1)$ was already treated in [1] in connection with investigations concerning the Hölder continuity of $\varphi$, but here we shall give a new prove which also works in more complicated cases. In the case (1.5) one of the constants $\lambda_1$, $\lambda_2$ in (1.4) is always equal to 1 but, in general, they are both different from 1.

## 2 Estimates for de Rham's function

We begin with the equation (1.5). After the normalization $\varphi(1) = 1$ it has the continuous solution $\varphi(t) = 0$ for $t \leq 0$, $\varphi(t) = 1$ for $t \geq 1$ and $\varphi(t) = \varphi_a(t)$ for $t \in [0, 1]$, where $\varphi_a$ is de Rham's function, which is the simultaneous solution of (1.2) and

$$\varphi\left(\frac{t+1}{2}\right) = a + (1-a)\varphi(t), \tag{2.1}$$

both equations for $t \in [0, 1]$. In [1] it was proved:

**Proposition 2.1** *In the case $a \in \left(0, \frac{1}{2}\right]$ de Rham's function satisfies the estimates*

$$C(a)t^\alpha \leq \varphi_a(t) \leq t^\alpha \ (t \in [0, 1]) \tag{2.2}$$

*and in the case $a \in \left[\frac{1}{2}, 1\right)$ it satisfies*

$$t^\alpha \leq \varphi_a(t) = C(a)t^\alpha \ (t \in [0, 1]) \tag{2.3}$$

*where $\alpha = -\frac{\ln a}{\ln 2}$,*

$$C(a) = \left(1 - q^{\frac{1}{1-\alpha}}\right)^{\alpha-1} \tag{2.4}$$

*and $q = \frac{1}{a} - 1$.*

**Proof:** The case $a = \frac{1}{2}$ is elementary and can be excluded now, since $\alpha = C\left(\frac{1}{2}\right) = 1$ as well as $\varphi_{\frac{1}{2}}(t) = t$. We first deal with the case $a \in \left(0, \frac{1}{2}\right)$ where $\alpha > 1$. We assume that the inequalities (2.2) are satisfied for a fixed $t$ and show that they are also satisfied for $\frac{t+1}{2}$ instead of $t$. From (2.1) and (2.2) we obtain

$$a + (1-a)C(a)t^\alpha \le \varphi_a\left(\frac{t+1}{2}\right) \le a + (1-a)t^\alpha. \tag{2.5}$$

In view of $\frac{1}{2^\alpha} = a$ we have to show that

$$aC(a)(t+1)^\alpha \le a + (1-a)C(a)t^\alpha, \quad a + (1-a)t^\alpha \le a(t+1)^\alpha$$

for $t \in [0,1]$, i.e. that

$$C(a) \le \frac{1}{(t+1)^\alpha - qt^\alpha} \leqq 1\,, \tag{2.6}$$

since $(t+1)^\alpha \ge \frac{1}{\alpha}t^\alpha > qt^\alpha$. For this reason we introduce the auxiliary function $f(s) = \left(s^{\frac{1}{\alpha}} + 1\right)^\alpha - qs$ with $s = t^\alpha \in [0,1]$. We have $f''(s) = \frac{1-\alpha}{\alpha}\left(s^{\frac{1}{\alpha}} + 1\right)^{\alpha-2} s^{\frac{1}{\alpha}-2} \le 0$ and $f(0) = f(1) = 1$ so that $f$ is concave and the second inequality of (2.6) is proved. The maximum of $f$ is attained at $s_0 = \left(q^{\frac{1}{\alpha-1}} - 1\right)^{-\alpha}$, i.e. the minimum of the quotient in (2.6) at $t_0 = \left(q^{\frac{1}{\alpha-1}} - 1\right)^{-1}$. A short calculation shows that this minimum is equal to (2.4) so that the first inequality of (2.6) is proved, too.

In the case $a \in \left(\frac{1}{2}, 1\right)$ where $\alpha \in (0,1)$, we assume that (2.3) is satisfied for a fixed $t$, conclude from (2.1) and (2.3) that

$$a + (1-a)t^\alpha \le \varphi_a\left(\frac{t+1}{2}\right) \le a + (1-a)C(a)t^\alpha\,, \tag{2.7}$$

and have to show that

$$a(t+1)^\alpha \le a + (1-a)t^\alpha, \quad a + (1-a)C(a)t^\alpha \le aC(a)(t+1)^\alpha$$

for $t \in [0,1]$, i.e.

$$1 \le \frac{1}{(t+1)^\alpha - qt^\alpha} \le C(a)\,. \tag{2.8}$$

In this case the auxiliary function $f$ is convex in view of $\alpha \in (0,1)$, and the quotient in (2.8) attains its minimum at the same $t_0$ as before, i.e. also (2.8) is proved.

After these preliminaries we can prove the inequalities (2.2) and (2.3) by means of an inductive process. In the case $a \in \left(0, \frac{1}{2}\right)$ we see from (2.6) that $C(a) \le 1$, and in the case $a \in \left(\frac{1}{2}, 1\right)$ we see from (2.8) that $1 \le C(a)$. Hence according to $\varphi(1) = 1$, the inequalities

(2.2) and (2.3) are satisfied for $t = 1$, and in view of $\alpha > 0$ they are also satisfied for $t = 0$. Now, we assume that the inequalities in question are satisfied for $t = \frac{\nu}{2^k}$, $\nu = 0, 1, \ldots, 2^k$ and a fixed $k \in \mathbb{N}_0$. According to a remark in the introduction, they are also satisfied for $t = \frac{\nu}{2^{k+1}}$, $\nu = 0, 1, \ldots, 2^k$, but in view of (2.5)-(2.8), they are moreover satisfied for $t = \frac{\nu}{2^{k+1}}$, $\nu = 2^k + 1, 2^k + 2, \ldots, 2^{k+1}$. Hence by induction, the inequalities (2.2) and (2.3) are satisfied for all (rational) dyadic points, and by continuity for all points of $[0, 1]$, so that the proposition is proved. ■

**Improvement** As already mentioned (but not proved) in [1], the constant $C(a)$ in Proposition 2.1 is not optimal for $a \neq \frac{1}{2}$. It can be improved by splitting the interval $[0, 1]$ into several subintervals. In the simplest case of two subintervals we have to use the equation (2.1) only for $t \in \left[0, \frac{1}{2}\right]$ and additionally the consequence of (2.1)

$$\varphi\left(\frac{t+3}{4}\right) = 2a - a^2 + (1-a)^2\varphi(t)$$

for $t \in [0, 1]$. In order to prove $Ct^\alpha \leq \varphi(t)$ for $a \in \left(0, \frac{1}{2}\right)$ inductively, we now have to fulfil the former inequality $aC(t+1)^\alpha \leq a + (1-a)Ct^\alpha$ only for $t \in \left[0, \frac{1}{2}\right]$, and additionally the inequality

$$a^2C(t+3)^\alpha \leq 2a - a^2 + (1-a)^2Ct^\alpha$$

for $t \in [0, 1]$. After determining maximal $C_\nu$ ($\nu = 1, 2$) with $C_\nu \leq g_\nu(t)$ where

$$g_1(t) = \frac{1}{(t+1)^\alpha - qt^\alpha} \qquad \left(t \in \left[0, \frac{1}{2}\right]\right),$$
$$g_2(t) = \frac{2q+1}{(t+3)^\alpha - q^2t^\alpha} \qquad (t \in [0, 1]),$$

we have to choose $C = \min(C_1, C_2)$. For $t \in [0, 1]$ the minimum of $g_1$ is attained as before at $t_0 = \left(q^{\frac{1}{\alpha-1}} - 1\right)^{-1}$. For $a \in \left(\frac{1}{4}, \frac{1}{2}\right)$ we have $t_0 \in \left(\frac{1}{3}, \frac{1}{2}\right)$, and we get no improvement. However, for $a \in \left(0, \frac{1}{4}\right)$ we have $t_0 > \frac{1}{2}$, so that for $t \in \left[0, \frac{1}{2}\right]$ the function $g_1(t)$ attains its minimum at $t_1 = \frac{1}{2}$, and it follows

$$C_1 = \frac{1}{a(3^\alpha - q)}.$$

The function $g_2$ attains its minimum at $t_2 = 3\left(q^{\frac{2}{\alpha-1}} - 1\right)^{-1}$ and this is equal to

$$C_2 = \frac{2q+1}{3^\alpha q^2}\left(q^{\frac{2}{\alpha-1}} - 1\right)^{\alpha-1}.$$

It turns out that $C_1 > C_2$ in the case $a < 0.10322$, and $C_1 < C_2$ in the case $a > 0.10322$, i.e. $C = C_2$ in the first and $C = C_1$ in the second case. However, the improvement in comparison with $C(a)$ is not essential and likewise not optimal.

**Example 2.1** The special case

$$\varphi\left(\frac{t}{2}\right) = a\varphi(t) + \varphi(t-1) + (1-a)\varphi(t-2) \tag{2.9}$$

of (1.1) with $n = 2$ and $a \in (0,1)$ can be reduced to the foregoing one, since (2.9) has a continuous solution with

$$\varphi(t) = \varphi_a(t), \quad \varphi(t+1) = \varphi_{1-a}(1-t)$$

for $t \in [0,1]$ and $\varphi(t) = 0$ for $t \notin (0,2)$, cf. [1] and in particular the relation $\varphi_{1-a}(1-t) = 1 - \varphi_a(t)$. Hence, besides of the estimates (2.2) resp. (2.3) we get the

**Corollary 2.1** *In the case $a \in \left(0, \frac{1}{2}\right]$ the foregoing solution of (2.9) satisfies the estimates*

$$(1-t)^\beta \leq \varphi(t+1) \leq C(1-a)(1-t)^\beta \qquad (t \in [0,1])\,, \tag{2.10}$$

*and in the case $a \in \left[\frac{1}{2}, 1\right)$ it satisfies*

$$C(1-a)(1-t)^\beta \leq \varphi(t+1) \leq (1-t)^\beta \qquad (t \in [0,1])\,, \tag{2.11}$$

*where $\beta = -\frac{\ln(1-a)}{\ln 2}$ and $C(\cdot)$ as in (2.4).*

**Example 2.2** The special case

$$\varphi\left(\frac{t}{2}\right) = a\varphi(t) + a\varphi(t-1) + (1-a)\varphi(t-2) + (1-a)\varphi(t-3) \tag{2.12}$$

of (1.1) with $n = 3$, $a \in (0,1)$, can also be reduced to Proposition 2.1, since it has a continuous solution with

$$\varphi(t) = a\varphi_a(t),\ \varphi(t+1) = a + (1-2a)\varphi_a(t),\ \varphi(t+2) = (1-a)\varphi_{1-a}(1-t)$$

for $t \in [0,1]$, and $\varphi(t) = 0$ for $t \notin (0,3)$.

**Example 2.3** Moreover, the special case

$$\varphi\left(\frac{t}{2}\right) = a\varphi(t) + (1-a)\varphi(t-1) + a\varphi(t-2) + (1-a)\varphi(t-3) \tag{2.13}$$

of (1.1) with $n = 3$, $a \in (0,1)$, can be reduced to Proposition 2.1, since it has a continuous solution $\Phi$ with the representation

$$\Phi(t) = \varphi(t) - \varphi(t-2)\,,$$

where $\varphi$ is the solution of (1.5) studied before. This means in particular that

$$\Phi(t) = \varphi_a(t), \quad \Phi(t+1) = 1, \quad \Phi(t+2) = \varphi_{1-a}(1-t)$$

for $t \in [0,1]$ and $\Phi(t) = 0$ for $t \notin (0,3)$. In the last two cases the corresponding estimates are obvious in view of Proposition 2.1.

**Example 2.4** Finally, we can estimate the solution

$$\varphi(t) = \int_0^t \varphi_a(t-\tau)\varphi_b(\tau)d\tau$$

of the equation

$$\varphi\left(\frac{t}{2}\right) = \frac{ab}{2}\varphi(t) + \left(\frac{a+b}{2} - ab\right)\varphi(t-1) + \frac{(1-a)(1-b)}{2}\varphi(t-2)\,, \tag{2.14}$$

$a, b \in (0,1)$, by means of Proposition 2.1. Namely, after writing the estimates (2.2) resp. (2.3) in the form

$$\lambda_1 t^\alpha \leq \varphi_a(t) \leq \lambda_2 t^\alpha\,, \quad \lambda_3 t^\beta \leq \varphi_b(t) \leq \lambda_4 t^\beta$$

with $\alpha$ as before and $\beta = -\frac{\ln b}{\ln 2}$, we immediately obtain

$$\lambda_1\lambda_3 \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} t^{\alpha+\beta+1} \leq \varphi(t) \leq \lambda_2\lambda_4 \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} t^{\alpha+\beta+1}$$

for $t \in [0,1]$. The method can easily be transferred to more complicated cases, but we do not deal with that.

## 3 The four-coefficient equation

Usually, two-scale difference equations (1.1) are considered under the so-called sum rules

$$\Sigma c_{2\nu} = 1\,, \quad \Sigma c_{2\nu+1} = 1\,, \tag{3.1}$$

cf. [3]. These are satisfied in the cases (2.9) and (2.12), but not in the cases (1.5), (2.13) and (2.14). The general form of (1.1) with $n = 3$ and (3.1) is a generalization of (2.12) and reads

$$\varphi\left(\frac{t}{2}\right) = a\varphi(t) + (1-b)\varphi(t-1) + (1-a)\varphi(t-2) + b\varphi(t-3)\,, \tag{3.2}$$

this equation was well investigated in [2]. We only look for compactly supported solutions, i.e. for solutions with $\varphi(t) = 0$ for $t \notin (0,3)$, so that it suffices to restrict ourselves to $t \in [0,3]$ in what follows. We assume that

$$0 < a\,, \quad b < 1\,. \tag{3.3}$$

Then there exists a continuous solution with

$$\varphi(1) = a\,, \quad \varphi(2) = b\,, \tag{3.4}$$

$$\varphi(t) + \varphi(t+1) + \varphi(t+2) = a + b \qquad (t \in [0,1])\,. \tag{3.5}$$

The equations (3.2) and (3.5) imply

$$\varphi\left(\frac{t+1}{2}\right) = a(a+b) + (1-a-b)\varphi(t) - a\varphi(t+2) \qquad (t \in [0,1])\,. \tag{3.6}$$

We begin with two special cases. In the case $b = \frac{1}{2} - a$ and $a \in \left(0, \frac{1}{2}\right)$ the solution of (3.2) is even differentiable, cf. [2], and satisfies the additional relation $\varphi(t) - \varphi(t+2) = a + \frac{1}{2}(t-1)$ for $t \in [0,1]$, so that (3.6) with $a + b = \frac{1}{2}$ simplifies to

$$\varphi\left(\frac{t+1}{2}\right) = a\left(a + \frac{t}{2}\right) + \left(\frac{1}{2} - a\right)\varphi(t) \qquad (t \in [0,1])\,. \tag{3.7}$$

**Proposition 3.1** *The solution of* (3.2) *with* $b = \frac{1}{2} - a$, $a \in \left(0, \frac{1}{2}\right)$ *and* $\varphi(1) = a$ *satisfies the estimates* (1.4) *with* $\alpha = -\frac{\ln a}{\ln 2}$ *and*

$$\lambda_1 = \min f(t)\,, \quad \lambda_2 = \max f(t) \qquad (t \in [0,1]) \tag{3.8}$$

*where*

$$f(t) = \frac{2a+t}{2a(t+1)^\alpha - (1-2a)t^\alpha}\,. \tag{3.9}$$

**Proof:** Following the philosophy of the inductive proof of Proposition 2.1, and using (3.7) instead of (2.1), we only have to show that

$$\begin{aligned} a\lambda_1(t+1)^\alpha &\le a + \frac{t}{2} + \left(\frac{1}{2} - a\right)\lambda_1 t^\alpha\,, \\ a + \frac{t}{2} + \left(\frac{1}{2} - a\right)\lambda_2 t^\alpha &\le a\lambda_2(t+1)^\alpha \end{aligned}$$

for $t \in [0,1]$. According to $2a(t+1)^\alpha - (1-2a)t^\alpha > (1+2a)t^\alpha$, the denominator of (3.9) is positive, so that the inequalities in question are equivalent to $\lambda_1 \le f(t) \le \lambda_2$. In view of (3.8) these inequalities are evident, and the proposition is proved. ∎

| $a$ | $t_1$ | $t_2$ | $\lambda_1$ | $\lambda_2$ |
|---|---|---|---|---|
| .01 | .938013 | .153740 | .975664 | 3.359434 |
| .1 | .791639 | .163395 | .972564 | 1.105538 |
| .2 | .672925 | .114190 | .991617 | 1.007689 |
| .3 | .057474 | .568021 | .998655 | 1.006186 |
| .4 | .010468 | .473003 | .999829 | 1.009885 |
| .45 | .000558 | .428706 | .999997 | 1.006748 |

Table 1: Some extremal points $t_\nu$ and the bounds $\lambda_\nu = f(t_\nu)$, $\nu = 1, 2$

Numerical results are contained in Table 1. The case $a = \frac{1}{4}$ with $\alpha = 2$ is elementary with $f(t) = \lambda_1 = \lambda_2 = 1$ and therefore $\varphi(t) = \frac{1}{4}t^2$ for $t \in [0,1]$. Elementary is also the limit case $a = \frac{1}{2}$ with $\alpha = 1$ and $\varphi(t) = \frac{1}{2}t$, where (3.2) reduces to (2.9) with $a = \frac{1}{2}$.

Second, we consider the symmetric case $b = a$ in (3.2), so that $\varphi(t) = \varphi(3-t)$, and (3.6) can be written in the form

$$\varphi\left(\frac{t+1}{2}\right) = 2a^2 + (1-2a)\varphi(t) - a\varphi(1-t) \qquad (t \in [0,1])\,. \tag{3.10}$$

**Proposition 3.2** *The solution of* (3.2) *with* $b = a$, $a \in \left(0, \frac{1}{2}\right)$ *and* $\varphi(1) = a$ *satisfies the estimates* (1.4) *with* $\alpha = -\frac{\ln a}{\ln 2}$, $\lambda_1 = 1 - \varepsilon$, $\lambda_2 = 1 + \varepsilon$ *and*

$$\varepsilon = \max|f(t)| \qquad (t \in [0,1]) \tag{3.11}$$

*where*

$$f(t) = \frac{2a + (1-2a)t^\alpha - a(t+1)^\alpha - a(1-t)^\alpha}{a(t+1)^\alpha - a(1-t)^\alpha - (1-2a)t^\alpha}\,. \tag{3.12}$$

**Proof:** Once more, we follow the philosophy of the foregoing proofs, however, since (3.10) contains a negative coefficient, we have to show that

$$\begin{aligned} a\lambda_1(t+1)^\alpha &\le 2a + (1-2a)\lambda_1 t^\alpha - a\lambda_2(1-t)^\alpha\,, \\ 2a + (1-2a)\lambda_2 t^\alpha - a\lambda_1(1-t)^\alpha &\le a\lambda_2(t+1)^\alpha\,. \end{aligned}$$

For $t = 0$ these inequalities imply that $\lambda_1 + \lambda_2 \le 2 \le \lambda_1 + \lambda_2$ must be satisfied, i.e. $\lambda_1 = 1-\varepsilon$ and $\lambda_2 = 1 + \varepsilon$ with a nonnegative $\varepsilon$. Hence, the foregoing inequalities turn over into

$$-\varepsilon N(t) \le 2a + (1-2a)t^\alpha - a(t+1)^\alpha - a(1-t)^\alpha \le \varepsilon N(t) \tag{3.13}$$

with $N(t) = a(t+1)^\alpha - a(1-t)^\alpha - (1-2a)t^\alpha$. Substituting $x = \frac{1}{t} \ge 1$, we obtain

$$t^{-\alpha}N(t) = a(x+1)^\alpha - a(x-1)^\alpha - (1-2a)\,.$$

According to $\alpha > 1$ and $2^\alpha = \frac{1}{a}$, the derivative of this function with respect to $x$ reads $a\alpha\left((x+1)^{\alpha-1} - (x-1)^{\alpha-1}\right) > 0$, and the initial value for $x = 1$ is equal to $N(1) = 2a$. Hence $N(t) \ge 0$ for $t \in [0,1]$, and the inequalities (3.13) are equivalent to $-\varepsilon \le f(t) \le \varepsilon$. In view of (3.11) these inequalities are evident, and the proposition is proved. ■

| $a$ | $t_1$ | $f(t_1)$ |
|---|---|---|
| .1 | .506816 | -.030511 |
| .2 | .505478 | -.013624 |

| $a$ | $t_2$ | $f(t_2)$ |
|---|---|---|
| .3 | .490557 | .012852 |
| .4 | .457641 | .024888 |

Table 2: Some extremal points $t_\nu$ and the extrema $f(t_\nu)$, $\nu = 1, 2$

Numerical results are contained in Table 2, where $f \le 0$ for $0 < a \le \frac{1}{4}$ and $f \ge 0$ for $a \in \left[\frac{1}{4}, \frac{1}{2}\right)$, so that $\varepsilon = -f_{\min}$ in the first and $\varepsilon = f_{\max}$ in the second case. Again the cases $a = \frac{1}{4}$ and $a = \frac{1}{2}$, the last as a limit case, are elementary.

For $a \in \left(\frac{1}{2}, 1\right)$ the denominator $N(t)$ of (3.12) changes its sign so that the foregoing method fails. However, the method works again if (3.10) is replaced by

$$\varphi\left(\frac{t+1}{2}\right) = 2a^2(1-a) + (1-a)\varphi(t) + a(2a-1)\varphi(|2t-1|)\,,$$

but we are not concerned with this case.

Finally, we come back to the general four-coefficient equation (3.2), where besides of (1.4) with $\varphi(1) = a$ we also ask for estimates of the form

$$b\lambda_3(1-t)^\beta \le \varphi(t+2) \le b\lambda_4(1-t)^\beta \qquad (t \in [0,1]) \tag{3.14}$$

with $\beta = -\frac{\ln b}{\ln 2}$ and $\lambda_3 \le 1 \le \lambda_4$, cf. (3.4). The following fact can easily be seen: If $\varphi(t)$ is a solution of (3.2) with (3.4), then $\varphi(3-t)$ satisfies the same equations, only with interchanged coefficients. In particular, the inequalities (1.4) and (3.14) interchange, if we interchange $a$ and $b$, and (3.6) interchanges with

$$\varphi\left(2+\frac{t}{2}\right) = b(a+b) + (1-a-b)\varphi(2+t) - b\varphi(t) \qquad (t \in [0,1])\,. \tag{3.15}$$

In the following we use the notations

$$\left.\begin{aligned} f(t,a,b) &= a+b+(1-a-b)t^\alpha - a(t+1)^\alpha - b(1-t)^\beta\,,\\ g(t,a,b) &= a(t+1)^\alpha - a(1-t)^\beta - (1-a-b)t^\alpha\,. \end{aligned}\right\} \tag{3.16}$$

**Proposition 3.3** *Under the conditions* (3.3) *and* $a+b \le 1$ *the solution of* (3.2) *with* (3.4) *satisfies the estimates*

$$\varphi(t) \le a\lambda_2 t^\alpha\,, \quad b\lambda_3(1-t)^\beta \le \varphi(t+2) \qquad (t \in [0,1]) \tag{3.17}$$

*with* $\lambda_2 = 1 + b\varepsilon$, $\lambda_3 = 1 - a\varepsilon$ *and* $\varepsilon \ge 0$, *so long as both inequalities*

$$f(t,a,b) \le bg(t,a,b)\varepsilon\,, \quad -f(t,b,a) \le ag(t,b,a)\varepsilon \qquad (t \in [0,1]) \tag{3.18}$$

*are satisfied. It satisfies the estimates*

$$a\lambda_1 t^\alpha \le \varphi(t)\,, \quad \varphi(t+2) \le b\lambda_4(1-t)^\beta \qquad (t \in [0,1]) \tag{3.19}$$

*with* $\lambda_1 = 1 - b\varepsilon$, $\lambda_4 = 1 + a\varepsilon$ *and* $\varepsilon \ge 0$, *so long as both inequalities*

$$f(t,b,a) \le ag(t,b,a)\varepsilon\,, \quad -f(t,a,b) \le bg(t,a,b)\varepsilon \qquad (t \in [0,1]) \tag{3.20}$$

*are satisfied.*

**Proof:** The inequalities (3.17) are satisfied for $t = 0$ and $t = 1$. In order to prove them for all $t \in [0, 1]$, we proceed analogously as before using (3.6) and (3.15), i.e. we have to show that both inequalities

$$a + b + (1 - a - b)\lambda_2 t^\alpha - b\lambda_3(1 - t)^\beta \le a\lambda_2(t + 1)^\alpha$$

and

$$b\lambda_3(2 - t)^\beta \le a + b + (1 - a - b)\lambda_3(1 - t)^\beta - a\lambda_2 t^\alpha$$

are satisfied. Replacing $t$ by $1 - t$ in the last inequality, it turns over into

$$b\lambda_3(t + 1)^\beta \le a + b + (1 - a - b)\lambda_3 t^\beta - a\lambda_2(1 - t)^\alpha\,.$$

For $t = 0$ we obtain from the first and the last inequality

$$a\lambda_2 + b\lambda_3 \le a + b \le a\lambda_2 + b\lambda_3$$

and therefore $a\lambda_2 + b\lambda_3 = a + b$. Choosing $\lambda_2 = 1 + b\varepsilon$ and $\lambda_3 = 1 - a\varepsilon$ with a non-negative $\varepsilon$ and considering the notations (3.16), we see that the inequalities in question are equivalent with (3.18). The inequalities (3.20) can be derived analogously or by interchanging $a$ and $b$. Hence, the proposition is proved. ■

There are two interesting special cases which shall be formulated for the conditions (3.18) only, but which can also be transferred to the conditions (3.20).

**Corollary 3.1** *The conditions* (3.18) *of Proposition* 3.3 *simplify to*

$$f(t, a, b) \le 0 \le f(t, b, a) \qquad (t \in [0, 1]) \tag{3.21}$$

*in the case* $\varepsilon = 0$, *and to* $\varepsilon = \max(\varepsilon_1, \varepsilon_2)$ *with*

$$\left.\begin{aligned} \varepsilon_1 &= \max \frac{f(t, a, b)}{bg(t, a, b)} \qquad (t \in [0, 1])\,, \\ \varepsilon_2 &= \max -\frac{f(t, b, a)}{ag(t, b, a)} \qquad (t \in [0, 1]) \end{aligned}\right\} \tag{3.22}$$

*in the case that the denominators in* (3.22) *are non-negative.*

**Example 3.1** As an example to the case $\varepsilon = 0$ we shall show that the conditions (3.21) with $a + b = 1$, i.e. the inequalities

$$1 - \left(\frac{t+1}{2}\right)^\alpha - \left(\frac{1-t}{2}\right)^\beta \le 0 \le 1 - \left(\frac{t+1}{2}\right)^\beta - \left(\frac{1-t}{2}\right)^\alpha \qquad (t \in [0, 1])\,, \tag{3.23}$$

are satisfied in the case $a \in \left(0, \frac{1}{2}\right)$ and therefore $\alpha > 1$, $b \in \left(\frac{1}{2}, 1\right)$, $\beta \in (0, 1)$. (Again, the case $a = \frac{1}{2}$ is elementary). Substituting $t + 1 = 2x$ so that $1 - t = 2(1 - x)$, we have to show that the auxiliary function $h(x) = (1 - (1 - x)^\beta)^{\frac{1}{\alpha}} - x$ satisfies $h(x) \geq 0$ for $x \in \left[0, \frac{1}{2}\right]$, and $h(x) \leq 0$ for $x \in \left[\frac{1}{2}, 1\right]$. From

$$h''(x) = \frac{\beta}{\alpha}(1 - (1 - x)^\beta)^{\frac{1}{\alpha}-2}(1 - x)^{\beta-2}\left(\left(\frac{\beta}{\alpha} - 1\right)(1 - x)^\beta + 1 - \beta\right)$$

we see that $h''(x)$ has exactly one zero in $(0, 1)$, and that $h''(x) < 0$ before as well as $h''(x) > 0$ afterwards. This means that $h(x)$ has exactly one turning point in $[0, 1]$, and $h(x)$ is concave before as well as convex afterwards. Together with $h(0) = h\left(\frac{1}{2}\right) = h(1) = 0$ this proves the assertion, and Corollary 3.1 implies the estimates (3.17) with $\lambda_2 = \lambda_3 = 1$ in the case $a \in \left(0, \frac{1}{2}\right)$.

Of course, these estimates are already known from Example 2.3 (where the normalization $\varphi(1) = 1$ was used instead of $\varphi(1) = a$ here), but the conditions (3.23) are only sufficient, and therefore no consequences of the results in Example 2.3.

Moreover, the conditions (3.21) and therefore also the estimates (3.17) with $\lambda_2 = \lambda_3 = 1$ are satisfied in a subdomain of $\left[\frac{1}{2}, 1\right] \times \left[\frac{1}{2}, 1\right]$, which above and at the right is bounded by $a + b = 1$, at the left by the curve $a\alpha = b\beta$, and below by the envelope of the curves $f(t, a, b) = 0$ with the parameter $t \in [0, 0.61]$.

| $t$ | | .6 | .55 | .5 | .45 | .4 | .3 | .2 | .1 | .05 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $a$ | .25 | .2557 | .2839 | .3103 | .3350 | .3583 | .4006 | .4382 | .4714 | .4864 | .5 |
| $b$ | .5 | .5015 | .5078 | .5121 | .5148 | .5163 | .5165 | .5138 | .5083 | .5045 | .5 |

Table 3: Some points of the envelope $f(t, a, b) = f_t(t, a, b) = 0$

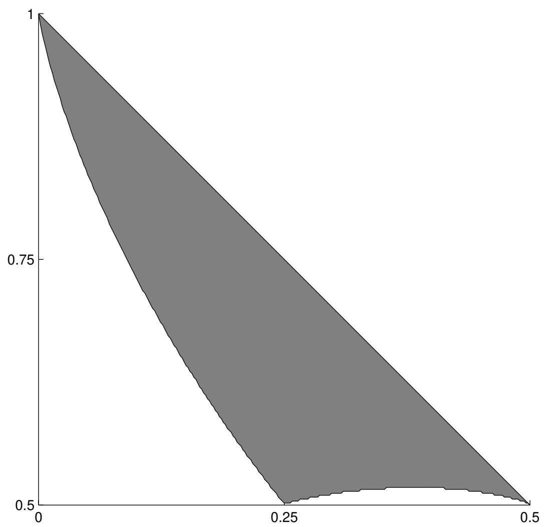


Figure 1: The level surface $\varepsilon = 0$

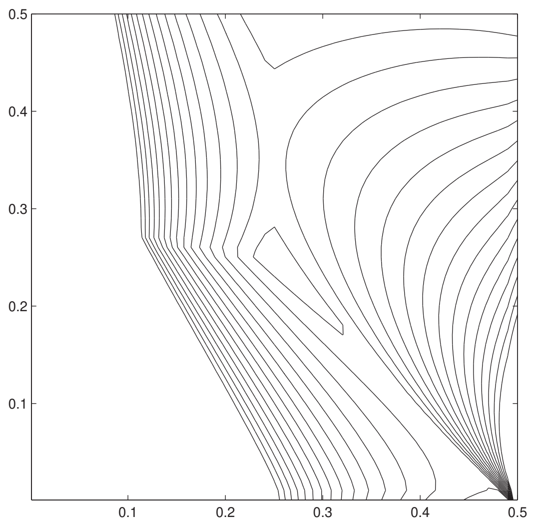


Figure 2: Some level lines for $\varepsilon > 0$

Numerically, this envelope is given by Table 3, where the corner points belong to all parameters $t$, and the complete subdomain is the black domain in Figure 1. The curves in the subdomain $\left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$ of the $(a, b)$-plane, which are shown in Figure 2, are the level lines of $\varepsilon$ from Corollary 3.1 for $\frac{1}{15}\left(\frac{1}{15}\right)1$ around the three points with $\varepsilon = 0$ in $\left(\frac{1}{4}, \frac{1}{4}\right)$ and $\left(\frac{1}{4}, \frac{1}{2}\right)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$, the last two are already known from Figure 1. Both Figures were worked out by K. Frischmuth [4].

In the case $\varepsilon > 0$ the estimates of the general Proposition 3.3 are worse than those following from the Propositions 3.1 and 3.2 in the special cases considered there, because there we have used more informations about the solutions.

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