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# Stationary points for set-valued mappings on two metric spaces

ABSTRACT. Stationary point theorems of set-valued mappings in complete and compact metric spaces are given. The corresponding results of Fisher [1] are generalized on set-valued mappings.

KEY WORDS AND PHRASES. Set-valued mappings, stationary points, complete metric spaces, compact metric spaces.

## 1 Preliminaries

In [1] and [2], Fisher and Popa have proved fixed point theorems for single valued mappings on two metric spaces. The purpose of this paper is to generalize these results for set-valued mappings. In this paper we show stationary point results of set-valued mappings in complete and compact metric spaces.

Let $(X, d)$ and $(Y, \rho)$ be complete metric spaces and $B(X)$ and $B(Y)$ be two families of all nonempty bounded subsets of $X$ and $Y$, respectively. The function $\delta(A, B)$ with $A$ and $B$ in $B(X)$ is defined as follows:

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\}\,.$$

Define $\delta(A) = \delta(A, A)$. Similarly, the function $\delta'(C, D)$ with $C$ and $D$ in $B(Y)$ is defined as follows:

$$\delta'(C, D) = \sup\{\rho(c, d) : c \in C, d \in D\}\,.$$

$\{A_n : n = 1, 2, \dots\}$ that is a sequence of sets in $B(X)$ converges to the set $A$ in $B(X)$ if

(i) each point $a$ in $A$ is the limit of some convergent sequence $\{a_n \in A_n : n = 1, 2, \dots\}$;

(ii) for arbitrary $\varepsilon > 0$, there exists an integer $N$ such that $A_n \subset A_\varepsilon$ for $n > N$, where $A_\varepsilon$ is the union of all open spheres with centers in $A$ and radius $\varepsilon$.

Let $T$ be a set-valued mapping of $X$ into $B(X)$, $z$ is a stationary point of $T$ if $Tz = \{z\}$. $T$ is continuous at $x$ in $X$ if whenever $\{x_n\}$ is a sequence of points in $X$ converging to $x$, the sequence $\{Tx_n\}$ in $B(X)$ converges to $Tx$ in $B(X)$. If $T$ is continuous at each point $x$ in $X$, then $T$ is a continuous mapping of $X$ into $B(X)$.

The following Lemmas 1.1 and 1.2 were proved in [3] and [4], respectively.

**Lemma 1.1** *If $\{A_n\}$ and $\{B_n\}$ are sequences of bounded subsets of a complete metric space $(X, d)$ which converge to the bounded subsets $A$ and $B$ respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

**Lemma 1.2** *Let $\{A_n\}$ be a sequence of nonempty subsets of $X$ and $x$ be a point of $X$ such that $\lim\limits_{n\to\infty} \delta(A_n, x) = 0$. Then the sequence $\{A_n\}$ converges to the set $\{x\}$.*

## 2 Stationary point results

Now we prove the following theorem for set-valued mappings.

**Theorem 2.1** *Let $(X, d)$ and $(Y, \rho)$ be complete metric spaces. If $T$ is a continuous mapping of $X$ into $B(Y)$ and $S$ is a continuous mapping of $Y$ into $B(X)$ satisfying the inequalities*

$$\delta(STx, STy) \le c \cdot \max\{\delta(x, y), \delta(x, STx), \delta(y, STy), \delta'(Tx, Ty)\}\,, \tag{1}$$

$$\delta'(TSx', TSy') \le c \cdot \max\{\delta'(x', y'), \delta'(x', TSx'), \delta'(y', TSy'), \delta(Sx', Sy')\} \tag{2}$$

for all $x, y$ in $X$ and $x', y'$ in $Y$, where $0 \le c < 1$, then $ST$ has a stationary point $z$ in $X$ and $TS$ has a stationary point $w$ in $Y$. Further $Tz = \{w\}$ and $Sw = \{z\}$.

**Proof:** From (1) and (2), it is easy to see that

$$\delta(STA, STB) \le c \cdot \max\{\delta(A, B), \delta(A, STA), \delta(B, STB), \delta'(TA, TB)\}\,, \tag{1'}$$

$$\delta'(TSA', TSB') \le c \cdot \max\{\delta'(A', B'), \delta'(A', TSA'), \delta'(B', TSB'), \delta(SA', SB')\} \tag{2'}$$

for all $A, B$ in $B(X)$ and $A', B'$ in $B(Y)$.

Let $x$ be an arbitrary point in $X$. Define sequences $\{x_n\}$ and $\{y_n\}$ in $B(X)$ and $B(Y)$ respectively by choose a point $x_n$ in $(ST)^n x = X_n$ and choose a point $y_n$ in $T(ST)^{n-1}x = Y_n$ for $n = 1, 2, \ldots$. From $(1')$ and $(2')$ we have

$$\begin{aligned}\delta(X_n, X_{n+1}) &= \delta(STX_{n-1}, STX_n) \\ &\le c \cdot \max\{\delta(X_{n-1}, X_n), \delta(X_{n-1}, X_n), \delta(X_n, X_{n+1}), \delta'(Y_n, Y_{n+1})\} \\ &\le c \cdot \max\{\delta(X_{n-1}, X_n), \delta'(Y_n, Y_{n+1})\}\,.\end{aligned}$$

Similarly $\delta'(Y_n, Y_{n+1}) \le c \cdot \max\{\delta'(Y_{n-1}, Y_n), \delta(X_{n-1}, X_n)\}$. Put $M = \max\{\delta(x, X_1), \delta'(Y_1, Y_2)\}$. From the above inequalities, we obtain immediatly

$$\delta(X_n, X_{n+1}) \le c^n \cdot M\,, \tag{3}$$

$$\delta'(Y_n, Y_{n+1}) \le c^n \cdot M \tag{4}$$

for $n = 1, 2, \ldots$. It follows from (2) that

$$\begin{aligned}\delta(X_n, X_{n+r}) &\le \delta(X_n, X_{n+1}) + \cdots + \delta(X_{n+r-1}, X_{n+r}) \\ &\le (c^n + \cdots + c^{n+r-1})M \\ &\le \frac{c^n}{1-c} M\,.\end{aligned}$$

Since $c < 1$, then $\delta(X_n, X_{n+r}) \to 0$ as $n \to \infty$. So $d(x_n, x_{n+r}) \le \delta(X_n, X_{n+r}) \to 0$ as $n \to \infty$. Thus $\{x_n\}$ is a Cauchy sequence. Completeness of $X$ implies that there exists $z$ in $X$ such that $x_n \to z$ as $n \to \infty$. Further

$$\delta(z, X_n) \le \delta(z, x_n) + \delta(x_n, X_n) \le \delta(z, x_n) + \delta(X_n, X_n) \le \delta(z, x_n) + 2\delta(X_n, X_{n+1})$$

which implies that $\delta(z, X_n) \to 0$ as $n \to \infty$. Similarly, there exists $w$ in $Y$ such that $y_n \to w$ and $\delta'(w, Y_n) \to 0$ as $n \to \infty$. Then

$$\delta'(w, Tx_n) \le \delta'(w, TX_n) = \delta'(w, Y_{n+1})\,.$$

By the continuity of $T$ and Lemma 1.1, we have $\delta'(w, Tz) \to 0$ as $n \to \infty$. From Lemma 1.2 it follows that $Tz = \{w\}$. Further

$$\delta(STz, x_n) \le \delta(STz, X_n) \le c \cdot \max\{\delta(z, X_{n-1}), \delta(z, STz), \delta(X_{n-1}, X_n), \delta'(Tz, TX_{n-1})\}\,.$$

Letting $n$ tend to infinity, we have $\delta(STz, z) \le c \cdot \max\{\delta(STz, z), 0\}$, which implies that $STz = \{z\} = Sw$. Similarly, we can show $w$ is a stationary point of $TS$. This completes the proof of the theorem.

**Theorem 2.2** *Let $(X, d)$ be a complete metric space, $S$ and $T$ be continuous mappings of $X$ into $B(X)$ and map bounded set into bounded set. If $S$ and $T$ satisfy the inequalities*

$$\delta(STx, STy) \le c \cdot \max\{\delta(x, y), \delta(x, STx), \delta(y, STy), \delta(x, STy), \delta(y, STx), \delta(Tx, Ty)\}, \tag{5}$$

$$\delta(TSx, TSy) \le c \cdot \max\{\delta(x, y), \delta(x, TSx), \delta(y, TSy), \delta(x, TSy), \delta(y, TSx), \delta(Sx, Sy)\} \tag{6}$$

*for all $x, y$ in $X$, where $0 \le c < 1$, then $ST$ has a stationary point $z$ and $TS$ has a stationary point $w$. Further $Tz = \{w\}$ and $Sw = \{z\}$. If $z = w$, then $z$ is the unique common stationary point of $S$ and $T$.*

**Proof:** Let $x$ be an arbitrary point in $X$, we define a sequence of sets $\{X_n\}$ by $T(ST)^{n-1}x = X_{2n-1}$, $(ST)^n x = X_{2n}$ for $n = 1, 2, \ldots$, and $X_0 = \{x\}$.

Now suppose that $\{\delta(X_n)\}$ is unbounded. Then the real-valued sequence $\{a_n\}$ is unbounded, where $a_{2n-1} = \delta(X_{2n-1}, X_3)$, $a_{2n} = \delta(X_{2n}, X_2)$ for $n = 1, 2, \ldots$, and so there exists an integer $k$ such that

$$a_k > \frac{c}{1-c} \max\{\delta(x, X_2), \delta(X_1, X_3)\}\,, \tag{7}$$

$$a_k > \max\{a_1, \ldots, a_{k-1}\}\,. \tag{8}$$

Suppose that $k$ is even, put $k = 2n$. From (7) and (8) we have

$$\begin{aligned} c \cdot \delta(X_{2r}, x) &\le c \cdot [\delta(X_{2r}, X_2) + \delta(X_2, x)] < \delta(X_{2n}, X_2)\,, \\ c \cdot \delta(X_{2r-1}, X_1) &\le c \cdot [\delta(X_{2r-1}, X_3) + \delta(X_3, X_1)] < \delta(X_{2n}, X_2)\,. \end{aligned}$$

That is

$$\delta(X_{2n}, X_2) > c \cdot \max\{\delta(X_{2r}, x), \delta(X_{2r-1}, X_1) : 1 \le r \le n\}\,. \tag{9}$$

We prove that the following (10) is true for $m \ge 1$:

$$\delta(X_{2n}, X_2) \le c^m \cdot \max\{\delta(X_{2r}, X_{2s}), \delta(X_{2r'-1}, X_{2s'-1}) : 1 \le r, s \le n, 2 \le r', s' \le n\}\,. \tag{10}$$

From (5) we have

$$\begin{aligned} \delta(X_{2n}, X_2) &= \delta(STX_{2n-2}, STx) \\ &\le c \cdot \max\{\delta(X_{2n-2}, x), \delta(X_{2n-2}, X_{2n}), \delta(x, X_2), \delta(x, X_{2n}), \\ &\qquad \delta(X_{2n-2}, X_2), \delta(X_{2n-1}, X_1)\}\,. \end{aligned}$$

It follows from (8) and (9) that $\delta(X_{2n}, X_2) \le c \cdot \delta(X_{2n-2}, X_{2n})$.

Now suppose that (10) is true for some $m$. From (5), (6), (8) and (9) we have

$$\begin{aligned} \delta(X_{2n}, X_2) &\le c^m \cdot \max\{\delta(X_{2r}, X_{2s}), \delta(X_{2r'-1}, X_{2s'-1}) : 1 \le r, s \le n, 2 \le r', s' \le n\} \\ &\le c^{m+1} \cdot \max\{\delta(X_{2r-2}, X_{2s-2}), \delta(X_{2r-2}, X_{2r}), \delta(X_{2s-2}, X_{2s}), \delta(X_{2r-2}, X_{2s})\,, \\ &\qquad \delta(X_{2s-2}, X_{2r}), \delta(X_{2r-1}, X_{2s-1}), \delta(X_{2r'-3}, X_{2s'-3})\,, \\ &\qquad \delta(X_{2r'-3}, X_{2r'-1}), \delta(X_{2s'-3}, X_{2s'-1}) : 1 \le r, s \le n, 2 \le r', s' \le n\} \\ &\le c^{m+1} \cdot \max\{\delta(X_{2r}, X_{2s}), \delta(X_{2r'-1}, X_{2s'-1}) : 1 \le r, s \le n, 2 \le r', s' \le n\}\,. \end{aligned}$$

So (10) is true for all $m \ge 1$. Letting $m$ tend to infinity, from (8) and (9) we have $0 < \delta(X_{2n}, X_2) \le 0$, which is impossible. Similarly, when $k$ is odd, $2n-1$, say, we also have $0 < \delta(X_{2n-1}, X_3) \le 0$, which is also impossible. Hence $\{\delta(X_n)\}$ is bounded.

Let $M = \sup\{\delta(X_r, X_s) : r, s = 0, 1, 2, \dots\} < \infty$. For arbitrary $\varepsilon > 0$, choose a positive integer $N$ such that $c^N \cdot M < \varepsilon$. Thus for $m, n$ greater than $2N$ with $m$ and $n$ both even or both odd, from (5) and (6) we have

$$\begin{aligned}
\delta(X_m, X_n) &\le c \cdot \max\{\delta(X_{m-2}, X_{n-2}), \delta(X_{m-2}, X_m), \delta(X_{n-2}, X_n), \delta(X_{m-2}, X_n), \\
&\qquad \delta(X_{n-2}, X_m), \delta(X_{m-1}, X_{n-1})\} \\
&\le c \cdot \max\{\delta(X_r, X_s), \delta(X_r, X_{r'}), \delta(X_s, X_{s'}) : m-2 \le r, r' \le m, n-2 \le s, s' \le n\} \\
&\le c^N \cdot \max\{\delta(X_r, X_s), \delta(X_r, X_{r'}), \delta(X_s, X_{s'}) : m-2N \le r, r' \le m, \\
&\qquad\qquad n-2N \le s, s' \le n\} \\
&\le c^N \cdot M < \varepsilon\,.
\end{aligned}$$

So $\delta(X_{2n})$ and $\delta(X_{2n+1}) \to 0$ as $n \to \infty$. Take a point $x_n$ in $X_n$ for $n \ge 1$. Since $d(x_{2n}, x_{2n+2p}) \le \delta(X_{2n}, X_{2n+2p}) \to 0$ as $n \to \infty$, hence $\{x_{2n}\}$ is a Cauchy sequence. Completeness of $X$ implies that $\{x_{2n}\}$ has a limit $z$ in $X$. Further

$$\delta(z, X_{2n}) \le \delta(z, x_{2n}) + \delta(x_{2n}, X_{2n}) \le \delta(z, x_{2n}) + \delta(X_{2n})\,.$$

That is $\delta(z, X_{2n}) \to 0$ as $n \to \infty$. Similarly $\{x_{2n+1}\}$ converges to some point $w$ in $X$ and $\delta(w, X_{2n+1}) \to 0$ as $n \to \infty$. Since $\delta(w, TX_{2n}) = \delta(w, X_{2n+1})$, by the continuity of $T$ and Lemma 1.1, we have $\delta(w, Tz) \to 0$ as $n \to \infty$. From Lemma 1.2 it follows that $Tz = \{w\}$. Further

$$\begin{aligned}
\delta(STz, x_{2n}) &\le \delta(STz, X_{2n}) \\
&\le c \cdot \max\{\delta(z, X_{2n-2}), \delta(z, STz), \delta(X_{2n-2}, X_{2n}), \delta(z, X_{2n}), \\
&\qquad \delta(X_{2n-2}, STz), \delta(Tz, X_{2n-1})\}
\end{aligned}$$

which implies that $\delta(STz, z) \le c \cdot \max\{\delta(z, STz), 0\}$ as $n \to \infty$. Since $c < 1$, $\delta(STz, z) = 0$. Therefore $STz = \{z\} = Sw$ and $TSw = Tz = \{w\}$.

Now suppose that $z = w$ and that $z'$ is a second common stationary point of $S$ and $T$. Using (5)

$$\begin{aligned}
\delta(z, z') = \delta(STz, STz') &\le c \cdot \max\{\delta(z, z'), \delta(z, STz), \delta(z', STz'), \delta(z', STz), \\
&\qquad \delta(z, STz'), \delta(Tz, Tz')\} \\
&\le c \cdot \delta(z, z')\,.
\end{aligned}$$

So $z = z'$ and this completes the proof of the theorem.

**Remark 2.1** If we use single valued mappings in place of set-valued mappings in Theorems 2.1 and 2.2, Theorems 2 and 3 of Fisher [1] can be attained.

**Remark 2.2** The following example 2.1 demonstrates that the continuity of $S$ and $T$ in Theorems 2.1 and 2.2 is necessary.

**Example 2.1** Let $X = \{0\} \cup \left\{\frac{1}{n} : n \geq 1\right\} = Y$ with the usual metric. Define mappings $S$, $T$ by $T0 = \{1\}$, $T\frac{1}{n} = \left\{\frac{1}{2n}\right\}$ for $n \geq 1$ and $S = T$. It is easy to prove that all the conditions of Theorems 2.1 and 2.2 are satisfied except that the mappings $S$ and $T$ are continuous. But $ST$ and $TS$ have no stationary points.

Now we give the following theorem for the compact metric spaces.

**Theorem 2.3** *Let $(X, d)$ and $(Y, \rho)$ be compact metric spaces. If $T$ is a continuous mapping of $X$ into $B(Y)$ and $S$ is a continuous mapping of $Y$ into $B(X)$ satisfying the following inequaltities*

$$\delta(STx, STy) < \max\{\delta(x, y), \delta(x, STx), \delta(y, STy), \delta'(Tx, Ty)\}\,, \tag{11}$$

$$\delta'(TSx', TSy') < \max\{\delta'(x', y'), \delta'(x', TSx'), \delta'(y', TSy'), \delta(Sx', Sy')\} \tag{12}$$

*for all distinct $x, y$ in $X$ and distinct $x', y'$ in $Y$, then $ST$ has a stationary point $z$ and $TS$ has a stationary point $w$. Further $Tz = \{w\}$ and $Sw = \{z\}$.*

**Proof:** Let us suppose that the right-hand sides of inequalities (11) and (12) are positive for all distinct $x, y$ in $X$ and distinct $x', y'$ in $Y$. Define the real valued function $f(x, y)$ in $X \times X$ as follows:

$$f(x, y) = \delta(STx, STy)/\max\{\delta(x, y), \delta(x, STx), \delta(y, STy), \delta'(Tx, Ty)\}\,.$$

Since $S$ and $T$ are continuous, $f$ is continuous and achieves the maximum value $s$ on the compact metric space $X \times X$. Inequality (11) implies $s < 1$, that is

$$\delta(STx, STy) \leq s \cdot \max\{\delta(x, y), \delta(x, STx), \delta(y, STy), \delta'(Tx, Ty)\} \tag{13}$$

for all distinct $x, y$ in $X$. It is obvious that inequality (13) is also true for $x = y$. Similarly, there exists $t < 1$ such that

$$\delta'(TSx', TSy') \leq t \cdot \max\{\delta'(x', y'), \delta'(x', TSx'), \delta'(y', TSy'), \delta(Sx', Sy')\} \tag{14}$$

for all $x', y'$ in $Y$. So Theorem 2.3 follows immediately from Theorem 2.1.

Now suppose there exist $z, z'$ in $X$ such that

$$\max\{\delta(z, z'), \delta(z, STz), \delta(z', STz'), \delta'(Tz, Tz')\} = 0$$

which implies $\{z\} = \{z'\} = STz$ and $Tz = Tz'$, a singleton, $\{w\}$, say. Therefore we have $STz = Sw = \{z\}$, $TSw = Tz = \{w\}$. If there exist $w, w'$ in $Y$ such that

$$\max\{\delta'(w, w'), \delta'(w, TSw), \delta'(w', TSw'), \delta(Sw, Sw')\} = 0\,,$$

similarly, we also have $STz = Sw = \{z\}$, $TSw = Tz = \{w\}$. This completes the proof of the theorem.

**Remark 2.3** Theorem 4 of Fisher [1] is a particular case of our Theorem 2.3 if set-valued mappings are replaced by single valued mappings in Theorem 2.3.

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