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Some results on a nonunique fixed point

ABSTRACT. In the present paper, we obtain some nonunique fixed point theorems of single valued and multivalued maps in metric and generalized metric spaces, one of which generalizes the corresponding results of [1] and [2].

KEY WORDS AND PHRASES. Nonunique fixed point, T-orbitally continuous, T-orbitally complete, orbitally upper - semicontinuous.

1 Introduction

In [1], Pachpatte obtained some results on a nonunique fixed point in complete metric spaces and introduced an inequality as follows:

$$\begin{aligned} &\min\{[d(Tx,Ty)]^2, d(x,y)d(Tx,Ty), [d(y,Ty)]^2\} \\ &\quad - \min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \le rd(x,Tx)d(y,Ty) \end{aligned} \tag{I}$$

for any x, y in X, where r is in $(0, 1)$.

In [2], Liu generalized the above result for single valued maps and introduced the following:

$$\begin{aligned} &\min\{[d(Tx,Ty)]^2, d(x,y)d(Tx,Ty), d(x,y)d(y,TY), d(x,Tx)d(Tx,Ty), [d(y,Ty)]^2\} \\ &\quad - \min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \le r\cdot\max\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \end{aligned}$$

for any x, y in X, where r is in $(0, 1)$.

In the present paper, we obtain some results which generalize Theorem 1 of [1] and Theorem 1 of [2]. Furthermore, we give an example to show that our result indeed generalizes Theorem 1 of [1]. By the way, we show the example in [4] is false.

2 On a nonunique fixed point for single valued maps

Let $(X;d)$ be a metric space and T be a self map of X. T is called to be orbitally continuous if $\lim_i T^{n_i}x = u$ implies that $\lim_i TT^{n_i}x = Tu$ for each x in X. A metric space X is T-orbitally complete if every Cauchy sequence of the form $\{T^{n_i}x\}_{i\geq 1}$ converges in X for x in X. Throughout this paper R^+ denotes the set of nonnegative real numbers.

Theorem 2.1 *Let (X,d) be a T-orbitally complete metric space, T be an orbitally continuous self map of X. If T satisfies the following condition*

$$\begin{aligned}
&\min\{[d(Tx,Ty)]^2, d(x,y)d(Tx,Ty), d(x,y)d(y,Ty), d(x,Tx)d(Tx,Ty),\\
&\qquad d(x,Tx)d(y,Ty), d(y,Ty)d(Tx,Ty), [d(y,Ty)]^2\}\\
&\quad - \min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \qquad\qquad (2.1)\\
&\leq r\cdot\max\{d(x,y)d(Tx,Ty), d(x,y)d(y,Ty), d(x,Tx)d(Tx,Ty), d(x,Tx)d(y,Ty),\\
&\quad d(x,Ty)d(y,Tx), d(y,Tx)d(Tx,Ty), d(y,Tx)d(y,Ty)\}
\end{aligned}$$

for any x,y in X, where r is in $(0,1)$, then T has a fixed point and for each x in X the sequence $\{T^n x\}_{n\geq 1}$ converges to a fixed point of T.

Proof: Let x be in X. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for $n \geq 0$, where $x_0 = x$. If $x_n = x_{n+1}$ for some $n \geq 0$, then the assertion follows immediately. Therefore we assume that $x_n \neq x_{n+1}$ for each $n \geq 0$. Put $d_n = d(x_n, x_{n+1})$ for $n \geq 0$. By (2.1) we obtain

$$\begin{aligned}
&\min\{d_{n+1}^2, d_nd_{n+1}, d_nd_{n+1}, d_nd_{n+1}, d_nd_{n+1}, d_{n+1}^2, d_{n+1}^2\}\\
&\quad - \min\{d_nd_{n+1}, d(x_n,x_{n+2})d(x_{n+1},x_{n+1})\}\\
&\leq r\cdot\max\{d_nd_{n+1}, d_nd_{n+1}, d_nd_{n+1}, d_nd_{n+1}, d(x_n,x_{n+2})d(x_{n+1},x_{n+1}),\\
&\qquad d(x_{n+1},x_{n+1})d_{n+1}, d(x_{n+1},x_{n+1})d_{n+1}\}
\end{aligned}$$

i.e.,

$$d_{n+1}^2 = \min\{d_{n+1}^2, d_2d_{n+1}\} \leq rd_2d_{n+1}$$

which implies that $d_{n+1} \leq rd_n$. It is easy to see that $\{x_n\}_{n\geq 1}$ is a Cauchy sequence. Since X is orbitally complete there is some u in X such that $u = \lim_n T^n x$. By the T-orbitally continuity of T, $Tu = \lim_n TT^n x = u$. This completes the proof.

Remark 2.1 Theorem 2.1 extends Theorem 1 of [1] and Theorem 1 of [2]. The following example shows that Theorem 2.1 is a proper generalization of Theorem 1 of [1].

Example 2.1 Let $X = \{0,1,2,3,4\}$, $d(x,y) = d(y,x)$ for all x in X and $d(x,y) = 0$ if and only if $x = y$, $d(0,1) = 1$, $d(0,2) = 2.5$, $d(0,3) = 1$, $d(0,4) = 1$, $d(1,2) = 1.5$, $d(1,3) = 2$,

$d(1,4) = 1$, $d(2,3) = 2$, $d(2,4) = 1.5$, $d(3,4) = 1$. Obviously, (X, d) is a complete metric space. Now let $T : X \to X$, $T0 = 1$, $T1 = 0$, $T2 = 3$, $T3 = 2$, $T4 = 4$. It is easy to verify that the conditions of Theorem 2.1 are satisfied for $r = 0.3$. But Theorem 1 of [1] is not applicable, because T doesn't satisfy (I) for $x = 0$, $y = 2$ and all r in $(0, 1)$.

Remark 2.2 In 1990, Ciric [4] gave an example to show that the corresponding results of Dhage [5], Mishra [6] and Pathak [7] are false. Unfortunately the example is false. In fact, through strictly examining the proofs of Dhage, Pathak and Mishra's results we assert that the results of [5], [6] and [7] are true.

Mishra [6], Daghe [5], Pathak [7] assume that T satisfies respectively the following conditions (A), (B) and (C):

(A) $\min\{d(Tx,Ty), d(x,Tx), d(y,Ty), d(Tx,T^2x), d(y,T^2x)\}$
$- \min\{d(x,ty), d(y,Tx), d(x,T^2x), d(Ty,T^2x)\} \le q\, d(x,y)$
for all x, y in X, where $0 \le q < 1$;

(B) $\min\{d(Tx,Ty), d(x,Tx), d(y,Ty)\} + a \cdot \min\{d(x,Ty), d(y,Tx)\} \le qd(x,y) + pd(x,Tx)$
for all x, y in X, where $0 < p + q < 1$, a is a real number;

(C) $\min\{d(Tx,Ty), d(y,Ty)\} + a \cdot \min\{d(x,Ty), d(y,Tx)\} \le qd(x,y) + pd(x,Tx) + rd(x,Ty)$
for all x, y in X, where a, p, q and r are real numbers such that $0 \le r < 1$, $0 < p + q + 2r < 1$.

The example of Ciric [4] is as follows:

Let $M = \{0, 1, 3\}$ with the usual metric $d(x,y) = |x - y|$. Define the mapping T by $T0 = 1$, $T1 = 3$, $T3 = 0$.

Ciric [4] claimed that T satisfies each of conditions (A), (B) and C. We find that T doesn't satisfy any one of (A), (B) and (C), because if T satisfies (A), taking $x = 0$, $y = 1$, we have from (A)

$$\min\{d(1,3), d(0,1), d(1,3), d(1,3)\} - \min\{d(0,3), d(1,1), d(0,3), d(3,3)\} \le qd(0,1)$$

i.e., $1 \le q$. This contradicts the condition $0 \le q < 1$; if T satisfies (B), similarly we have $1 \le q + p < 1$, which is a contradiction, too; if T satisfies (C), we have $2 \le p + q + 3r$. Since $0 < p + q + 2r < 1$, it follows that $2 \le p + q + 3r < r + 1 < 2$, which is impossible.

Theorem 2.2 *Let (X, d) be a T-orbitally complete metric space, T be an orbitally continuous self map of X. If T satisfies the following condition*

$$\begin{aligned} & a_1[d(Tx,Ty)]^2 + a_2 d(x,y)d(Tx,Ty) + a_3 d(x,y)d(y,Ty) + a_4 d(x,Tx)d(Tx.Ty) \\ & \quad + a_5 d(x,Tx)d(y,Ty) + a_6 d(y,Ty)d(Tx,Ty) + a_7[d(y,Ty)]^2 \\ & \quad - \min\{d(x,Tx)d(y,Ty), d(x,Ty)d(y,Tx)\} \qquad (2.2) \\ & \leq r \cdot \max\{d(x,y)d(Tx,Ty), d(x,y)d(y,Ty), d(x,Tx)d(Tx,Ty), d(x,Tx)d(y,Ty), \\ & \qquad d(x,Ty)d(y,Tx), d(y,Tx)d(Tx,Ty), d(y,Tx)d(y,Ty)\} \end{aligned}$$

for all x, y in X, where $\sum_{i=1}^{7} a_i < 1$ and a_i is in R^+ for $i = 1, 2, \ldots, 7$, then T has a fixed point and the sequence $\{T^n x\}_{n\geq 0}$ converges to a fixed point of T for x in X.

Proof: Note that (2.2) implies (2.1). Theorem 2.2 follows immediately from Theorem 2.1.

3 On a nonunique fixed point for multivalued maps

We recall that (X, d) is a generalized metric space if X is a set and $d : X \times X \to R^+ \bigcup\{\infty\}$ satisfies all the properties of being a metric for X besides that d may have "infinite values". An orbit of F at the point x in X is a sequence $\{x_n : x_n \in Fx_{n-1}\}$, where $x_0 = x$. A multivalued map F on X is orbitally upper − semicontinuous if $x_n \to u \in X$ implies $u \in Fu$, whenever $\{x_n\}$ is an orbit of F at each x in X. A space X is F − orbitally complete if every orbit of F at all x in X which is a Cauchy sequence, converges in X. Let A and B be nonempty subsets of X. Denote

$$\begin{aligned} D(A,B) &= \inf\{d(a,b) : a \in A, b \in B\}, \\ CL(X) &= \{A : A \subset X, A \text{ is closed}\}, \\ N(\varepsilon, A) &= \{x \in X : d(x,a) < \varepsilon \text{ for some } a \text{ in } A\},\ \varepsilon > 0, \\ H(A,B) &= \begin{cases} \inf\{\varepsilon > 0 : A \subseteq N(\varepsilon,B) \text{ and } B \subseteq N(\varepsilon,A)\}, & \text{if the infimum exists}, \\ \infty, & \text{otherwise.} \end{cases} \end{aligned}$$

Ciric [3] introduced the following inequality:

$$\min\{H(Fx,Fy), D(x,Fx), D(y,Fy)\} - \min\{D(x,Fy), D(y,Fx)\} \leq qd(x,y)$$

for all x, y in M and some $q < 1$. Motivated by it, we obtain the following results.

Theorem 3.1 *Let (X, d) be a generalized metric space, $F : X \to CL(X)$ be orbitally upper–semicontinuous. If X is F-orbitally complete and F satisfies the following condition*

$$\begin{aligned}
&\min\{[H(Fx,Fy)]^2, d(x,y)H(Fx,Fy), d(x,y)D(y,Fy), D(x,Fx)H(Fx,Fy),\\
&\qquad D(x,Fx)D(y,Fy), D(y,Fy)H(Fx,Fy), [D(y,Fy)]^2\}\\
&\quad - \min\{D(x,Fx)D(y,Fy), D(x,Fy)D(y,Fx)\} \qquad (3.1)\\
&\le r\cdot\max\{d(x,y)d(Fx,Fy), d(x,y)D(y,Fy), D(x,Fx)D(Fx,Fy),\\
&\quad D(x,Fx)D(y,Fy), D(x,Fy)D(y,Fx), D(y,Fx)H(Fx,Fy), D(y,Fx)D(y,Fy)\}
\end{aligned}$$

for all x, y in X, where r is in $(0,1)$, then F has a fixed point.

Proof: Let $a > 0$ be a real number less than $1/2$. We define a single valued map $T : X \to X$ by letting $Tx = y \in Fx$ that satisfies

$$d(x,y) \le r^{-a} D(x,Fx)\,. \qquad (3.2)$$

Set $d_n = d(x_{n-1}, x_n), D_n = D(x_n, Fx_n)$ and $H_n = H(Fx_{n-1}, Fx_n)$ for $n \ge 0$. Now let's consider the following orbit of F at x in $X : x_0 = x, x_n = Tx_{n-1}$ for $n \ge 0$. We may assume that $x_{n-1} \ne x_n$ for any $n \ge 0$, otherwise the result is obtained at once. It follows from $x_n \in Fx_{n-1}$ that $D_n \le H_n, D(x_n, Fx_{n-1}) = 0$ and $D_{n-1} \le d_n$. By (3.1), we have

$$\begin{aligned}
&\min\{H_n^2, d_nH_n, d_nD_n, D_{n-1}H_n, D_{n-1}D_n, D_nH_n, D_n^2,\}\\
&\quad - \min\{D_{n-1}D_n, D(x_{n-1},Fx_n)D(x_n,Fx_{n-1})\}\\
&\le r\cdot\max\{d_nD(Fx_{n-1},Fx_n), d_nD_n, D_{n-1}D(Fx_{n-1},Fx_n), D_{n-1}D_n,\\
&\qquad D(x_{n-1},Fx_n)D(x_n,Fx_{n-1}), D(x_n,Fx_{n-1})H_n, D(x_n,Fx_{n-1})D_n\}
\end{aligned}$$

which implies that

$$\min\{D_n^2, D_{n-1}D_n\} = \min\{D_n^2, d_nD_n, D_{n-1}D_n\} \le r\cdot\max\{d_nD_n, D_{n-1}D_n\} = rd_nD_n\,,$$
$$\min\{r^{-2a}D_n^2, r^{-2a}D_{n-1}D_n\} \le r^{1-2a}d_nD_n \le r^{1-2a}d_nd_{n+1}\,,$$

on using (3.2)

$$\min\{d_{n+1}^2, d_nd_{n+1}\} \le r^{1-2a}d_nd_{n+1}\,.$$

Note that $0 < r^{1-2a} < 1$. If $d_n < d_{n+1}$, then

$$d_nd_{n+1} = \min\{d_{n+1}^2, d_nd_{n+1}\} \le r^{1-2a}d_nd_{n+1} < d_nd_{n+1}$$

a contradiction. Therefore $d_{n+1} < d_n$ and

$$d_{n+1}^2 = \min\{d_{n+1}^2, d_nd_{n+1}\} \le r^{1-2a}d_n^2$$

i.e., $d_{n+1} \leq bd_n$, where $b = r^{\frac{1}{2}-a}$. This implies $\{x_n\}_{n\geq 1}$ is a Cauchy sequence. Since X is F-orbitally complete, there exists some point u in X such that $\lim_n x_n = u$. Thus the orbitally upper–semicontinuity of F implies $u \in Fu$. This completes the proof.

Theorem 3.2 *Let (X, d) be a generalized space, $F : X \to CL(X)$ be orbitally upper–semicontinuous. If X is F-orbitally complete and F satisfies the following condition*

$$\begin{aligned} &a_1[H(Fx,Fy)]^2 + a_2 d(x,y)H(Fx,Fy) + a_3 d(x,y)D(y,Fy) + a_4 D(x,Fx)H(Fx,Fy) \\ &\quad + a_5 D(x,Fx)D(y,Fy) + a_6 D(y,Fy)H(Fx,Fy) + a_7[D(y,Fy)]^2 \\ &\quad - \min\{D(x,Fx)D(y,Fy), D(x,Fy)D(y,Fx)\} \\ &\leq r \cdot \max\{d(x,y)D(Fx,Fy), d(x,y)D(y,Fy), D(x,Fx)D(Fx,Fy), \\ &\qquad\qquad D(x,Fx)D(y,Fy), D(x,Fy)D(y,Fx), D(y,Fx)H(Fx,Fy), \\ &\qquad\qquad D(y,Fx)D(y,Fy)\} \end{aligned} \tag{3.3}$$

for all x, y in X, where $\sum_{i=1}^{7} a_i < 1$ and a_i is in R^+ for $i = 1, 2, \ldots, 7$, then F has a fixed point.

Proof: Since (3.3) implies (3.1), Theorem 3.2 follows immediately from Theorem 3.1.

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