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Characterizations of common fixed points in 2-metric spaces

ABSTRACT. In this paper we obtain a few necessary and sufficient conditions for the existence of a common fixed point of a pair of mappings in 2-metric spaces. Our results generalize, improve and unify a number of fixed point theorems given by Cho [1], Constantin [2], Khan and Fisher [11], Kubiak [14], Rhoades [25], Singh, Tiwari and Gupta [31] and others.

KEY WORDS AND PHRASES. 2-metric spaces, common fixed points, compatible mappings.

1 Introduction

Gähler [4] introduced the concept of 2-metric space. A 2-metric space is a set X with a function $d : X \times X \times X \to [0, \infty)$ satisfying the following conditions:

(G1) for two distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$,

(G2) $d(x, y, z) = 0$ if at least two of x, y, z are equal,

(G3) $d(x, y, z) = d(x, z, y) = d(y, z, x)$,

(G4) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$ for all $x, y, z, u \in X$.

It has been shown by Gähler [4] that a 2-metric d is a continuous function of any one of its three arguments but it need not be continuous in two arguments. If it is continuous in two arguments, then it is continuous in all three arguments. A 2-metric d which is continuous in all of its arguments will be called continuous.

Iséki [7], for the first time, developed a fixed point theorem in 2-metric spaces. Since then a quite number of authors ([1]-[3], [5]-[36]) have extended and generalized the result of Iséki and various other results involving contractive and expansive type mappings. Especially, Murthy, Chang, Cho and Sharma [17] introduced the concepts of compatible mappings and compatible mappings of type (A) in 2-metric spaces, derived some relations between these mappings and proved common fixed point theorems for compatible mappings of type (A) in 2-metric spaces.

On the other hand, Cho [1], Constantin [2], Khan and Fisher [11] and Kubiak [14] established some necessary and sufficient conditions which guarantee the existence of a common fixed point for a pair of continuous mappings in 2-metric spaces.

In this paper we establish criteria for the existence of a common fixed point of a pair of mappings in 2-metric spaces. Our results generalize, improve and unify the corresponding results of Cho [1], Constantin [2], Khan and Fisher [11], Kubiak [14], Rhoades [25], Singh, Tiwari and Gupta [31] and others.

2 Preliminaries

Throughout this paper, N and ω denote the sets of positive and nonnegative integers, respectively. Let $R^+ = [0, \infty)$ and

$$W = \{w : w : R^+ \to R^+ \text{ is continuous and satisfies } 0 < w(t) < t \text{ for } t > 0\}.$$

We consider the family Φ of all continuous functions $\varphi : (R^+)^5 \to R^+$ with the following properties:

(i) φ is non-decreasing in the 4^{th} and 5^{th} variables,

(ii) if $u, v \in R^+$ with $u \leq \max\{\varphi(v, v, u, u+v, 0), \varphi(v, u, v, u+v, 0), \varphi(v, v, u, 0, u+v), \varphi(v, u, v, 0, u+v)\}$, then $u \leq cv$ for some $c \in (0, 1)$,

(iii) if $u \in R^+$ with $u \leq \max\{\varphi(u, 0, 0, u, u), \varphi(0, u, 0, u, u), \varphi(0, 0, u, u, u)\}$, then $u = 0$.

Let f be a mapping of a 2-metric space (X, d) into itself and $B \subset X$. Define $d(x, B, a) = \inf_{b \in B} d(x, b, a)$ for $x, a \in X$ and $F(f) = \{t : t = ft \in X\}$.

Definition 2.1 *A sequence $\{x_n\}_{n \in N}$ in a 2-metric space (X, d) is said to be convergent to a point $x \in X$ if $\lim_{n \to \infty} d(x_n, x, a) = 0$ for all $a \in X$. The point x is called the limit of the sequence $\{x_n\}_{n \in N}$ in X.*

Definition 2.2 *A sequence $\{x_n\}_{n \in N}$ in a 2-metric space (X, d) is said to be a Cauchy sequence if $\lim_{m,n \to \infty} d(x_m, x_n, a) = 0$ for all $a \in X$.*

Definition 2.3 *A 2-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.*

Note that, in a 2-metric space (X, d), a convergent sequence need not be a Cauchy sequence, but every convergent sequence is a Cauchy sequence when the 2-metric d is continuous on X ([19]).

Definition 2.4 *Let f and g be mappings from a 2-metric space (X, d) into itself. f and g are said to be compatible if*

$$\lim_{n\to\infty} d(fgx_n, gfx_n, a) = 0$$

for all $a \in X$, whenever $\{x_n\}_{n\in N} \subset X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$; f and g are said to be compatible of type (A) if

$$\lim_{n\to\infty} d(fgx_n, ggx_n, a) = \lim_{n\to\infty} d(gfx_n, ffx_n, a) = 0$$

for all $a \in X$, whenever $\{x_n\}_{n\in N} \subset X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Definition 2.5 *A mapping f from a 2-metric space (X, d) into itself is said to be continuous at $x \in X$ if for every sequence $\{x_n\}_{n\in N} \subset X$ such that $\lim_{n\to\infty} d(x_n, x, a) = 0$ for all $a \in X$, $\lim_{n\to\infty} d(fx_n, fx, a) = 0$. f is called continuous on X if it is so at all points of X.*

Lemma 2.1 **([17])** *Let f and g be compatible mappings from a 2-metric space (X, d) into itself. If $ft = gt$ for some $t \in X$, then $fgt = ggt = gft = fft$.*

Lemma 2.2 **([17])** *Let f and g be compatible mappings from a 2-metric space (X, d) into itself. If f is continuous at some $t \in X$ and if $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, then $\lim_{n\to\infty} gfx_n = ft$.*

3 Characterizations of common fixed points

Theorem 3.1 *Let (X, d) be a complete 2-metric space with d continuous on X and let h and t be two mappings of X into itself. Then the following conditions are equivalent:*

(1) *h and t have a common fixed point;*

(2) *there exist $r \in (0, 1)$, $f : X \to t(X)$ and $g : X \to h(X)$ such that*

  (a1) *the pairs f, h and g, t are compatible,*

  (a2) *one of f, g, h and t is continuous,*

  (a3) $d(fx, gy, a) \le r \max\{d(hx, ty, a), d(hx, fx, a), d(ty, gy, a), \frac{1}{2}[d(hx, gy, a) + d(ty, fx, a)]\}$ *for all $x, y, a \in X$;*

(3) *there exist* $w \in W, f : X \to t(X)$ *and* $g : X \to h(X)$ *satisfying* (a1), (a2) *and* (a4):

(a4) $$d(fx, gy, a) \le \max\big\{d(hx, ty, a), d(hx, fx, a), d(ty, gy, a), \frac{1}{2}[d(hx, gy, a) + d(ty, fx, a)]\big\} - w\big(\max\big\{d(hx, ty, a), d(hx, fx, a), d(ty, gy, a), \frac{1}{2}[d(hx, gy, a) + d(ty, fx, a)]\big\}\big)$$

for all $x, y, a \in X$;

(4) *there exist* $\varphi \in \Phi, f : X \to t(X)$ *and* $g : X \to h(X)$ *satisfying* (a1), (a2) *and* (a5):

(a5) $d(fx, gy, a) \le \varphi(d(hx, ty, a), d(hx, fx, a), d(ty, gy, a), d(hx, gy, a), d(ty, fx, a))$ *for all* $x, y, a \in X$.

Proof: (1) $\Rightarrow$ (2) and (4). Let z be a common fixed point of h and t. Define $f : X \to t(X)$ and $g : X \to h(X)$ by $fx = gx = z$ for all $x \in X$. Then (a1) and (a2) hold. For each $r \in (0, 1)$ and $\varphi \in \Phi$, (a3) and (a5) also hold.

(2) $\Rightarrow$ (3). Take $w(t) = (1 - r)t$. Then $w \in W$ and (a3) implies (a4).

(3) $\Rightarrow$ (1). Let x_0 be an arbitrary point in X. Since $f(X) \subset t(X)$ and $g(X) \subset h(X)$, there exist sequences $\{x_n\}_{n\in N}$ and $\{y_n\}_{n\in N}$ in X satisfying $y_{2n} = tx_{2n+1} = fx_{2n}$, $y_{2n+1} = hx_{2n+2} = gx_{2n+1}$ for $n \in \omega$. Define $d_n(a) = d(y_n, y_{n+1}, a)$ for $a \in X$ and $n \in \omega$. We claim that for any $i, j, k \in \omega$

$$d(y_i, y_j, y_k) = 0\,. \tag{3.1}$$

Suppose that $d_{2n}(y_{2n+2}) > 0$. Using (a4), we have

$$\begin{aligned} d(fx_{2n+2}, gx_{2n+1}, y_{2n}) \le{} & \max\big\{d(hx_{2n+2}, tx_{2n+1}, y_{2n}), d(hx_{2n+2}, fx_{2n+2}, y_{2n}), \\ & d(tx_{2n+1}, gx_{2n+1}, y_{2n}), \\ & \frac{1}{2}[d(hx_{2x+2}, gx_{2n+1}, y_{2n}) + d(tx_{2n+1}, fx_{2n+2}, y_{2n})]\big\} \\ & - w\big(\max\big\{d(hx_{2n+2}, tx_{2n+1}, y_{2n}), d(hx_{2n+2}, fx_{2n+2}, y_{2n}), \\ & d(tx_{2n+1}, gx_{2n+1}, y_{2n}), \\ & \frac{1}{2}[d(hx_{2n+2}, gx_{2n+1}, y_{2n}) + d(tx_{2n+1}, fx_{2n+2}, y_{2n})]\big\}\big) \end{aligned}$$

which implies that

$$\begin{aligned} d_{2n}(y_{2n+2}) &\le \max\{0, d_{2n}(y_{2n+2}), 0, 0\} - w(\max\{0, d_{2n}(y_{2n+2}), 0, 0\}) \\ &= d_{2n}(y_{2n+2}) - w(d_{2n}(y_{2n+2})) < d_{2n}(y_{2n+2}) \end{aligned}$$

which is a contradiction. Hence $d_{2n}(y_{2n+2}) = 0$. Similarly, we have $d_{2n+1}(y_{2n+3}) = 0$. Consequently, $d_n(y_{n+2}) = 0$ for all $n \in \omega$. Note that

$$\begin{aligned} d(y_n, y_{n+2}, a) &\le d_n(y_{n+2}) + d_n(a) + d_{n+1}(a) \\ &= d_n(a) + d_{n+1}(a)\,. \end{aligned} \tag{3.2}$$

By (a4) and (3.2) we have

$$\begin{aligned} d_{2n+1}(a) &= d(fx_{2n+2}, gx_{2n+1}, a) \\ &\le \max\big\{d(hx_{2n+2}, tx_{2n+1}, a), d(fx_{2n+2}, hx_{2n+2}, a), d(gx_{2n+1}, tx_{2n+1}, a), \\ &\qquad \frac{1}{2}[d(hx_{2n+2}, gx_{2n+1}, a) + d(tx_{2n+1}, fx_{2n+2}, a)]\big\} \\ &\qquad - w\big(\max\big\{d(hx_{2n+2}, tx_{2n+1}, a), d(fx_{2n+2}, hx_{2n+2}, a), d(gx_{2n+1}, tx_{2n+1}, a), \\ &\qquad \frac{1}{2}[d(hx_{2n+2}, gx_{2n+1}, a) + d(tx_{2n+1}, fx_{2n+2}, a)]\big\}\big) \\ &= \max\big\{d_{2n}(a), d_{2n+1}(a), d_{2n}(a), \frac{1}{2}[0 + d(y_{2n}, y_{2n+2}, a)]\big\} \\ &\qquad - w\big(\max\big\{d_{2n}(a), d_{2n+1}(a), d_{2n}(a), \frac{1}{2}[0 + d(y_{2n}, y_{2n+2}, a)]\big\}\big) \\ &= \max\big\{d_{2n}(a), d_{2n+1}(a), \frac{1}{2}[d_{2n}(a) + d_{2n+1}(a)]\big\} \\ &\qquad - w\big(\max\big\{d_{2n}(a), d_{2n+1}(a), \frac{1}{2}[d_{2n}(a) + d_{2n+1}(a)]\big\}\big) \\ &= \max\big\{d_{2n}(a), d_{2n+1}(a)\big\} - w\big(\max\{d_{2n}(a), d_{2n+1}(a)\}\big)\,. \end{aligned}$$

Suppose that $d_{2n+1}(a) > d_{2n}(a)$. Then $d_{2n+1}(a) \le d_{2n+1}(a) - w(d_{2n+1}(a)) < d_{2n+1}(a)$, which is a contradiction. Hence $d_{2n+1}(a) \le d_{2n}(a)$ and so $d_{2n+1}(a) \le d_{2n}(a) - w(d_{2n}(a)) < d_{2n}(a)$. Similarly, we have $d_{2n}(a) \le d_{2n-1}(a)$. That is, for all $n \in N$

$$d_{n+1}(a) \le d_n(a)\,. \tag{3.3}$$

Let n, m be in ω. If $n \ge m$, then $0 = d_m(y_m) \ge d_n(y_m)$; if $n < m$, then

$$\begin{aligned} d_n(y_m) &\le d_n(y_{m-1}) + d_{m-1}(y_n) + d_{m-1}(y_{n+1}) \\ &\le d_n(y_{m-1}) + d_n(y_n) + d_n(y_{n+1}) \\ &\le d_n(y_{m-1}) \le d_n(y_{m-2}) \le \cdots \le d_n(y_{n+1}) = 0\,. \end{aligned}$$

Thus, for any $n, m \in \omega$

$$d_n(y_m) = 0\,. \tag{3.4}$$

For all $i, j, k \in \omega$, we may, without loss of generality, assume that $i < j$. It follows from (3.4) that

$$\begin{aligned} d(y_i, y_j, y_k) &\le d_i(y_j) + d_i(y_k) + d(y_{i+1}, y_j, y_k) \\ &= d(y_{i+1}, y_j, y_k) \le d(y_{i+2}, y_j, y_k) \le \dots \\ &\le d(y_{j-1}, y_j, y_k) = d_{j-1}(y_k) = 0\,. \end{aligned}$$

Therefore (3.1) holds.

By virtue of (a4), (3.3) and (3.4) we have

$$\begin{aligned} d_{2n}(a) &= d(fx_{2n}, gx_{2n+1}, a) \\ &\le \max\big\{d(hx_{2n}, tx_{2n+1}, a), d(hx_{2n}, fx_{2n}, a), d(tx_{2n+1}, gx_{2n+1}, a), \\ &\qquad \frac{1}{2}[d(hx_{2n}, gx_{2n+1}, a) + d(tx_{2n+1}, fx_{2n}, a)]\big\} \\ &\quad - w\big(\max\big\{d(hx_{2n}, tx_{2n+1}, a), d(hx_{2n}, fx_{2n}, a), d(tx_{2n+1}, gx_{2n+1}, a), \\ &\qquad \frac{1}{2}[d(hx_{2n}, gx_{2n+1}, a) + d(tx_{2n+1}, fx_{2n}, a)]\big\}\big) \\ &= \max\big\{d_{2n-1}(a), 0, d_{2n}(a), \frac{1}{2}d(y_{2n-1}, y_{2n+1}, a)\big\} \\ &\quad - w\big(\max\big\{d_{2n-1}(a), 0, d_{2n}(a), \frac{1}{2}d(y_{2n-1}, y_{2n+1}, a)\big\}\big) \\ &= \max\big\{d_{2n-1}(a), d_{2n}(a), \frac{1}{2}[d_{2n-1}(a) + d_{2n}(a) + d_{2n+1}(y_{2n-1})]\big\} \\ &\quad - w\big(\max\big\{d_{2n-1}(a), d_{2n}(a), \frac{1}{2}[d_{2n-1}(a) + d_{2n}(a) + d_{2n+1}(y_{2n-1})]\big\}\big) \\ &= \max\{d_{2n-1}(a), d_{2n}(a)\} - w(\max\{d_{2n-1}(a), d_{2n}(a)\}) \\ &= d_{2n-1}(a) - w(d_{2n-1}(a))\,. \end{aligned}$$

Similarly, we have $d_{2n+1}(a) \le d_{2n}(a) - w(d_{2n}(a))$. It follows that

$$\sum_{i=0}^{n} w(d_i(a)) \le \sum_{i=0}^{n}[d_i(a) - d_{i+1}(a)] = d_0(a) - d_{n+1}(a) \le d_0(a)\,.$$

So the series of nonnegative terms $\sum\limits_{n=0}^{\infty} w(d_n(a))$ is convergent. This means that

$$\lim_{n\to\infty} w(d_n(a)) = 0\,. \tag{3.5}$$

(3.3) ensures that $\{d_n(a)\}_{n\in\omega}$ converges to some $r \ge 0$. In view of the continuity of w and (3.5) we have

$$w(r) = \lim_{n\to\infty} w(d_n(a)) = 0$$

which implies that $r = 0$. Hence

$$\lim_{n\to\infty} d_n(a) = 0\,. \tag{3.6}$$

In order to show that $\{y_n\}_{n\in\omega}$ is a Cauchy sequence, by (3.6), it is sufficient to show that $\{y_{2n}\}_{n\in\omega}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\in\omega}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and $a \in X$ such that for each even integer $2k$, there are even integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) > 2k$ and $d(y_{2m(k)}, y_{2n(k)}, a) \geq \varepsilon$.

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying the above inequality, so that

$$d(y_{2m(k)-2}, y_{2n(k)}, a) \leq \varepsilon\,, \quad d(y_{2m(k)}, y_{2n(k)}, a) > \varepsilon\,. \tag{3.7}$$

For each even integer $2k$, by (3.1) and (3.7) we have

$$\begin{aligned}
\varepsilon &< d(y_{2m(k)}, y_{2n(k)}, a)\\
&\leq d(y_{2m(k)-2}, y_{2n(k)}, a) + d(y_{2m(k)}, y_{2m(k)-2}, a) + d(y_{2m(k)}, y_{2n(k)}, y_{2m(k)-2})\\
&\leq \varepsilon + d(y_{2m(k)-2}, y_{2m(k)}, y_{2m(k)-1}) + d(y_{2m(k)-2}, y_{2m(k)-1}, a) + d(y_{2m(k)-1}, y_{2m(k)}, a)\\
&= \varepsilon + d_{2m(k)-2}(a) + d_{2m(k)-1}(a)
\end{aligned}$$

which implies that

$$\lim_{k\to\infty} d(y_{2m(k)}, y_{2n(k)}, a) = \varepsilon\,. \tag{3.8}$$

It follows from (3.7) that

$$\begin{aligned}
0 &< d(y_{2n(k)}, y_{2m(k)}, a) - d(y_{2n(k)}, y_{2m(k)-2}, a)\\
&\leq d(y_{2m(k)-2}, y_{2m(k)}, a)\\
&\leq d_{2m(k)-2}(a) + d_{2m(k)-1}(a)\,.
\end{aligned}$$

In view of (3.6) and (3.8) we immediately obtain

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-2}, a) = \varepsilon\,. \tag{3.9}$$

Note that

$$\begin{aligned}
\left|d(y_{2n(k)}, y_{2m(k)-1}, a) - d(y_{2n(k)}, y_{2m(k)}, a)\right| &\leq d_{2m(k)-1}(a) + d_{2m(k)-1}(y_{2n(k)})\,,\\
\left|d(y_{2n(k)+1}, y_{2m(k)}, a) - d(y_{2n(k)}, y_{2m(k)}, a)\right| &\leq d_{2n(k)}(a) + d_{2n(k)}(y_{2m(k)})\,,\\
\left|d(y_{2n(k)+1}, y_{2m(k)-1}, a) - d(y_{2n(k)}, y_{2m(k)-1}, a)\right| &\leq d_{2n(k)}(a) + d_{2n(k)}(y_{2m(k)-1})\,.
\end{aligned}$$

It is easy to see that

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1}, a) = \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)}, a) = \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1}, a) = \varepsilon\,. \tag{3.10}$$

It follows from (a4) that

$$\begin{aligned}
d(y_{2m(k)}, y_{2n(k)+1}, a) &= d(fx_{2m(k)}, gx_{2n(k)+1}, a)\\
&\le \max\big\{d(hx_{2m(k)}, tx_{2n(k)+1}, a), d(hx_{2m(k)}, fx_{2m(k)}, a),\\
&\qquad d(tx_{2n(k)+1}, gx_{2n(k)+1}, a),\\
&\qquad \tfrac{1}{2}[d(hx_{2m(k)}, gx_{2n(k)+1}, a) + d(tx_{2n(k)+1}, fx_{2m(k)}, a)]\big\}\\
&\quad - w\big(\max\big\{d(hx_{2m(k)}, tx_{2n(k)+1}, a), d(hx_{2m(k)}, fx_{2m(k)}, a),\\
&\qquad d(tx_{2n(k)+1}, gx_{2n(k)+1}, a),\\
&\qquad \tfrac{1}{2}[d(hx_{2m(k)}, gx_{2n(k)+1}, a) + d(tx_{2n(k)+1}, fx_{2m(k)}, a)]\big\}\big)\\
&= \max\big\{d(y_{2m(k)-1}, y_{2n(k)}, a), d_{2m(k)-1}(a), d_{2n(k)}(a),\\
&\qquad \tfrac{1}{2}[d(y_{2m(k)-1}, y_{2n(k)+1}, a) + d(y_{2n(k)}, y_{2m(k)}, a)]\big\}\\
&\quad - w\big(\max\big\{d(y_{2m(k)-1}, y_{2n(k)}, a), d_{2m(k)-1}(a), d_{2n(k)}(a),\\
&\qquad \tfrac{1}{2}[d(y_{2m(k)-1}, y_{2n(k)+1}, a) + d(y_{2n(k)}, y_{2m(k)}, a)]\big\}\big)\,.
\end{aligned}$$

Letting $k \to \infty$, by (3.10), (3.8), and (3.6) we have

$$\varepsilon \le \max\{\varepsilon, 0, 0, \varepsilon\} - w(\max\{\varepsilon, 0, 0, \varepsilon\}) = \varepsilon - w(\varepsilon) < \varepsilon$$

which is a contradiction. Therefore $\{y_{2n}\}_{n\in\omega}$ is a Cauchy sequence in X.

It follows from completeness of (X, d) that $\{y_n\}_{n\in\omega}$ converges to a point $u \in X$. Now, suppose that t is continuous. Since g and t are compatible and $\{gx_{2n+1}\}_{n\in\omega}$ and $\{tx_{2n+1}\}_{n\in\omega}$ converge to the point u, by Lemma 2.2 we get that gtx_{2n+1}, $tgx_{2n+1} \to tu$ as $n \to \infty$. In virtue of (a4) we have

$$\begin{aligned}
d(fx_{2n}, gtx_{2n+1}, a) &\le \max\big\{d(hx_{2n}, ttx_{2n+1}, a), d(hx_{2n}, fx_{2n}, a), d(ttx_{2n+1}, gtx_{2n+1}, a),\\
&\qquad \tfrac{1}{2}[d(hx_{2n}, gtx_{2n+1}, a) + d(ttx_{2n+1}, fx_{2n}, a)]\big\}\\
&\quad - w\big(\max\big\{d(hx_{2n}, ttx_{2n+1}, a), d(hx_{2n}, fx_{2n}, a),\\
&\qquad d(ttx_{2n+1}, gtx_{2n+1}, a),\\
&\qquad \tfrac{1}{2}[d(hx_{2n}, gtx_{2n+1}, a) + d(ttx_{2n+1}, fx_{2n}, a)]\big\}\big)\,.
\end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} d(u,tu,a) \le \max\big\{&d(u,tu,a),d(u,u,a),d(tu,tu,a),\frac{1}{2}[d(u,tu,a)+d(ttu,u,a)]\big\}\\ &- w\big(\max\big\{d(u,tu,a),d(u,u,a),d(tu,tu,a),\frac{1}{2}[d(u,tu,a)+d(ttu,u,a)]\big\}\big)\\ &= d(u,tu,a)-w(d(u,tu,a)) \end{aligned}$$

which implies that $w(d(u,tu,a)) \le 0$. This means that $u = tu$.

It follows from (a4) that

$$\begin{aligned} d(fx_{2n},gu,a) \le \max\big\{&d(hx_{2n},tu,a),d(hx_{2n},fx_{2n},a),d(tu,gu,a),\\ &\frac{1}{2}[d(hx_{2n},gu,a)+d(tu,fx_{2n},a)]\big\}\\ &- w\big(\max\big\{d(hx_{2n},tu,a),d(hx_{2n},fx_{2n},a),d(tu,gu,a),\\ &\frac{1}{2}[d(hx_{2n},gu,a)+d(tu,fx_{2n},a)]\big\}\big)\,. \end{aligned}$$

As $n \to \infty$, we have

$$\begin{aligned} d(u,gu,a) \le \max\big\{&d(u,u,a),d(u,u,a),d(u,gu,a),\frac{1}{2}[d(u,gu,a)+d(u,u,a)]\big\}\\ &- w\big(\max\big\{d(u,u,a),d(u,u,a),d(u,gu,a),\frac{1}{2}[d(u,gu,a)+d(u,u,a)]\big\}\big)\\ &= d(u,gu,a)-w(d(u,u,a)) \end{aligned}$$

which implies that $u = gu$. It follows from $g(X) \subset h(X)$ that there exists $v \in X$ with $u = gu = hv$. From (a4) we get

$$\begin{aligned} d(fv,u,a) &= d(fv,gu,a)\\ &\le \max\big\{d(hv,tu,a),d(hv,fv,a),d(tu,gu,a),\frac{1}{2}[d(hv,gu,a)+d(tu,fv,a)]\big\}\\ &\quad - w\big(\max\big\{d(hv,tu,a),d(hv,fv,a),d(tu,gu,a),\\ &\qquad \frac{1}{2}[d(hv,gu,a)+d(tu,fv,a)]\big\}\big)\\ &= d(u,fv,a)-w(d(u,fv,a))\,. \end{aligned}$$

Therefore, $u = fv$. Lemma 2.1 ensures that $fu = fhv = hfv = hu$. By (a4) we obtain again

$$\begin{aligned} d(fu,u,a) &= d(fv,gu,a)\\ &\le \max\big\{d(hu,tu,a),d(hu,fu,a),d(tu,gu,a),\frac{1}{2}[d(hu,gu,a)+d(tu,fu,a)]\big\}\\ &\quad - w\big(\max\big\{d(hu,tu,a),d(hu,fu,a),d(tu,gu,a),\\ &\qquad \frac{1}{2}[d(hu,gu,a)+d(tu,fu,a)]\big\}\big)\\ &= d(fu,u,a)-w(d(fu,u,a))\,. \end{aligned}$$

Hence $u = fu$. That is, u is a common fixed point of f, g, h and t. Similarly, we can complete the proof when f or g or h is continuous.

(4) $\Rightarrow$ (1). Let $\{x_n\}_{n\in\omega}$, $\{y_n\}_{n\in\omega}$, $d_n(a)$ be as in the proof of (3) $\Rightarrow$ (1). Analogous we conclude that for all $n, m \in \omega$

$$d_n(a) \le hd_{n-1}(a) \le \cdots \le h^n d_0(a) \tag{3.11}$$

and

$$d_n(y_m) = 0\,. \tag{3.12}$$

For any $n, m \in \omega$, by (3.11) and (3.12) we have

$$\begin{aligned} d(y_n, y_{n+m}, a) &\le d_n(a) + d(y_{n+1}, y_{n+m}, a) + d_n(y_{n+m}) \\ &= d_n(a) + d(y_{n+1}, y_{n+m}, a) \le \dots \\ &\le \sum_{k=n}^{n+m-1} d_k(a) \le \sum_{k=n}^{n+m-1} h^k d_0(a) \le \frac{h^n}{1-h} d_0(a) \end{aligned}$$

which implies that $\{y_n\}_{n\in\omega}$ is a Cauchy sequence. The remainder of the proof follows from the proof process of (3) $\Rightarrow$ (1).

This completes the proof.

From Theorem 3.1 we immediately have

Theorem 3.2 *Let (X, d) be a complete 2-metric space with d continuous on X and let h and t be two mappings of X into itself. Then (1) is equivalent to each of the following conditions:*

(5) *there exist $r \in (0, 1)$, $f : X \to t(X) \cap h(X)$ such that*

  (a6) *the pairs f, h and f, t are compatible,*

  (a7) *one of f, t and h is continuous,*

  (a8) $d(fx, fy, a) \le r \max\big\{d(hx, ty, a), d(hx, fx, a), d(ty, fy, a), \frac{1}{2}[d(hx, fy, a) + d(ty, fx, a)]\big\}$
  for all $x, y, a \in X$;

(6) *there exist $w \in W$, $f : X \to t(X) \cap h(X)$ satisfying* (a6), (a7) *and* (a9):

(a9) $$d(fx,fy,a) \le \max\big\{d(hx,ty,a), d(hx,fx,a), d(ty,fy,a), \frac{1}{2}[d(hx,fy,a)+d(ty,fx,a)]\big\} - w\big(\max\big\{d(hx,ty,a), d(hx,fx,a), d(ty,fy,a), \frac{1}{2}[d(hx,fy,a)+d(ty,fx,a)]\big\}\big)$$

for all $x, y, a \in X$;

(7) *there exist $\varphi \in \Phi$, $f : X \to t(X) \cap h(X)$ satisfying* (a6), (a7) *and* (a10):

(a10) $d(fx,fy,a) \le \varphi(d(hx,ty,a), d(hx,fx,a), d(ty,fy,a), d(hx,fy,a), d(ty,fx,a))$ *for all $x, y, a \in X$.*

Remark 3.1 Theorems 3.1 and 3.2 are still true even though the condition of the compatibility is replaced by the compatibility of type (A).

Remark 3.2 Theorems 3.1 and 3.2 generalize, improve and unify Theorem 4.8 of Cho [1], the Theorem of Constantin [2], Theorem 2 of Khan and Fisher [11], Theorem 1 of Kubiak [14], Theorem 4 of Rhoades [24], Theorem 1 of Singh, Tiwari and Gupta [30].

Theorem 3.3 *Let f be a mapping of a complete 2-metric space (X, d) into itself satisfying*

(a11) $$d(fx,fy,a) \le \max\big\{d(x,y,a), \frac{1}{2}[d(x,fx,a)+d(y,fy,a)], \frac{1}{2}[d(x,fy,a)+d(y,fx,a)], \frac{1}{2}[d(y,fy,a)+d(y,fx,a)], \frac{1}{2}[d(y,fy,a)+d(x,fy,a)]\big\}$$

for all $x, y, a \in X$.

Then the following conditions are equivalent:

(8) *$F(f)$ is nonempty and for each $x \in X$, the sequence of iterates $\{f^n x\}_{n\in\omega}$ converges to some point in $F(f)$;*

(9) *there exists a closed subset G of X such that*

(a12) $d(fx,p,a) \le d(x,p,a)$ *for $x, a \in X$, $p \in G$,*

(a13) $\liminf\limits_{n\to\infty} d(f^n x, G, a) = 0$ *for $a \in X$.*

Proof: (8) $\Rightarrow$ (9). Take $G = F(f)$ and $x \in X$. Then $\{f^n x\}_{n\in N}$ converges to some point $w \in G$. It follows that

$$\begin{aligned}\liminf_{n\to\infty} d(f^n x, G, a) &= \liminf_{n\to\infty} \big\{ \inf_{p\in G} d(f^n x, p, a)\big\} \le \liminf_{n\to\infty} d(f^n x, w, a)\\ &= \lim_{n\to\infty} d(f^n x, w, a) = 0\end{aligned}$$

which implies that $\liminf\limits_{n\to\infty} d(f^n, G, a) = 0$ for $a \in X$.

Let $x, a \in X$ and $p \in G$. In view of (a11) we have

$$\begin{aligned}d(fx, p, a) &= d(fx, fp, a)\\ &\le \max\Big\{d(x,p,a), \frac{1}{2}d(x,fx,a), \frac{1}{2}[d(x,p,a)+d(p,fx,a)],\\ &\qquad \frac{1}{2}d(p,fx,a), \frac{1}{2}d(x,p,a)\Big\}\\ &= \max\Big\{d(x,p,a), \frac{1}{2}(x,fx,a), \frac{1}{2}[d(x,p,a)+d(p,fx,a)]\Big\}.\end{aligned} \tag{3.13}$$

Using (3.13) we obtain

$$d(fx,p,x) \le \max\Big\{0, 0, \frac{1}{2}d(p,fx,x)\Big\} = \frac{1}{2}d(fx,p,x)$$

which implies that $d(fx,p,x) = 0$. Hence

$$\begin{aligned}d(x,fx,a) &\le d(x,fx,p) + d(x,p,a) + d(p,fx,a)\\ &= d(x,p,a) + d(p,fx,a).\end{aligned} \tag{3.14}$$

It follows from (3.13) and (3.14) that

$$d(fx,p,a) \le \max\Big\{d(x,p,a), \frac{1}{2}[d(x,p,a)+d(p,fx,a)]\Big\}$$

which implies that

$$d(fx,p,a) \le d(x,p,a). \tag{3.15}$$

Assume that $\{y_n\}_{n\in N} \subset G$ and $y_n \to y \in X$ as $n \to \infty$. It follows from (3.15) that

$$d(fy,y,a) \le d(fy,y,y_n) + d(fy,y_n,a) + d(y_n,y,a) \le 2d(y_n,y,a).$$

Letting $n \to \infty$, we easily conclude that $d(fy,y,a) = 0$ for all $a \in X$. Therefore $y = fy \in G$. That is, G is a closed.

(9) $\Rightarrow$ (8). Let x_0 and a be in X and $x_n = f^n x_0$ for all $n \in N$. It follows from (a12) that

$$d(x_n, G, a) = \inf_{p\in G}\big\{d(x_n,p,a)\big\} \le \inf_{p\in G}\big\{d(x_{n-1},p,a)\big\} = d(x_{n-1}, G, a).$$

That is, $\{d(x_n, G, a)\}_{n\in N}$ is non-increasing. Thus (a13) implies that $\lim_{n\to\infty} d(x_n, G, a) = \liminf_{n\to\infty} d(x_n, G, a) = 0$. Therefore, for any $\varepsilon > 0$, there exists $n_0(a) \in N$ such that for all $n \geq n_0(a)$

$$d(x_n, G, a) < \frac{1}{2}\varepsilon\,. \tag{3.16}$$

Now, suppose that $n > m \geq n_0(a)$ and $p \in G$. By (a12) we have for $k \geq n_0(a)$

$$d(x_n, x_m, p) = d(fx_{n-1}, p, x_m) \leq d(x_{n-1}, p, x_m) \leq \cdots \leq d(x_m, p, x_m) = 0 \tag{3.17}$$

and

$$d(x_k, p, a) = d(fx_{k-1}, p, a) \leq d(x_{k-1}, p, a) \leq \cdots \leq d(x_{n_0(a)}, p, a)\,. \tag{3.18}$$

In virtue of (3.17) and (3.18) we get

$$\begin{aligned} d(x_n, x_m, a) &\leq d(x_n, x_m, p) + d(x_n, p, a) + d(p, x_m, a) \\ &\leq 2d(x_{n_0(a)}, p, a) \end{aligned}$$

which implies that

$$d(x_n, x_m, a) \leq 2d(x_{n_0(a)}, G, a)\,.$$

By (3.16) and the above inequality we have for $n > m \geq n_0(a)$

$$d(x_n, x_m, a) < \varepsilon\,.$$

Hence $\{x_n\}_{n\in N}$ is a Cauchy sequence in X and has a limit $w \in X$ since (X, d) is complete. Letting $k \to \infty$ in (3.18), we have

$$d(w, p, a) \leq d(x_{n_0(a)}, p, a)$$

which implies that

$$d(w, G, a) = \inf_{p\in G} d(w, p, a) \leq d(x_{n_0(a)}, G, a)\,. \tag{3.19}$$

In view of (3.16) and (3.19) we obtain that $d(w, G, a) < \varepsilon$, which implies that $d(w, G, a) = 0$ for all $a \in X$. Since G is closed, so $w \in G$. It follows from (a12) that $d(fw, w, a) \leq d(w, w, a) = 0$ for all $a \in X$. This means that $w = fw \in F(f) \neq \phi$.

This completes the proof.

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