Rostock. Math. Kolloq. **56**, 55–61 (2002)

Subject Classification (AMS)
11B68,11C08,39A10

Lothar Berg

On Polynomials Related with Generalized Bernoulli Numbers

ABSTRACT. The generalized Bernoulli numbers are connected with a sequence of polynomials, which are defined recursively and which can be represented by means of a generalized Rodrigues formula. A relation between the coefficients of a polynomial and corresponding moments is derived. A generating function with an unusual shape is constructed.

KEY WORDS. Bernoulli numbers, recursions, Rodrigues formula, moments, generating function

The generalized Bernoulli numbers $B_m^{(z)}$ are defined by

$$\left(\frac{w}{e^w - 1}\right)^z = \sum_{m=0}^{\infty} \frac{1}{m!} B_m^{(z)} w^m \ (z \in \mathbb{C})\,, \tag{1}$$

cf. [1], [2], they contain the ordinary Bernoulli numbers B_m as special case with $z = 1$. The generalized Bernoulli numbers satisfy the difference equation

$$B_m^{(z+1)} = \left(1 - \frac{m}{z}\right) B_m^{(z)} - m B_{m-1}^{(z)} \tag{2}$$

from which they can be calculated recursively for natural m, z subject to the initial values $B_0^{(z)} = 1$ and $B_m^{(1)} = B_m$. They are polynomials in z of degree m, the first of them are listed in [1, p. 459] for $0 \leq m \leq 12$.

The substitution

$$d_m(z) = (-1)^m \binom{z-1}{m} B_m^{(z)} \tag{3}$$

transfers (2) into the simpler recursion

$$d_m(z+1) - d_m(z) = z d_{m-1}(z)\,. \tag{4}$$

Under the initial values $d_0(z) = 1$, $d_m(1) = 0$ for $m \in \mathbb{N}$ the last has polynomial solutions of the form

$$d_m(z) = \sum_{\mu=1}^{m} c_{m\mu} \binom{z}{m+\mu} \tag{5}$$

from which, conversely, $B_m^{(z)}$ can be calculated by means of (3). Using the identity

$$z\binom{z}{n-1} = n\binom{z}{n} + (n-1)\binom{z}{n-1}$$

substitution of (5) into (4) yields the recursion

$$c_{m,\mu+1} = (m+\mu)(c_{m-1,\mu} + c_{m-1,\mu+1}) \quad (m \geq 2) \tag{6}$$

for $\mu = 0, 1, \ldots, m-1$ with $c_{11} = 1$ and $c_{m0} = c_{m,m+1} = 0$ ($m \in \mathbb{N}$). From (6) and the initial values we immediately obtain the values of $c_{m\mu}$ ($1 \leq \mu \leq m$) in form of a Pascal triangle

m	$c_{m\mu}$										
1						1					
2					2		3				
3				6		20		15			
4			24		130		210		105		
5		120		924		2380		2520		945	
6	720		7308		26432		44100		34650		10395

and, moreover,

$$c_{mm} = (2m-1)!!,\ c_{m,m-1} = \frac{2}{3}(m-1)(2m-1)!!,\ c_{m1} = m!\,. \tag{7}$$

In order to learn more about the coefficients $c_{m\mu}$ we introduce the polynomials

$$f_n(z) = \sum_{\nu=0}^{n} c_{n+1,\nu+1} z^\nu \quad (n \in \mathbb{N}_0)\,. \tag{8}$$

The recursions (6) and the corresponding initial values imply

$$f_n(z) = ((n+2)z + n + 1) f_{n-1}(z) + z(z+1) f'_{n-1}(z) \quad (n \in \mathbb{N}) \tag{9}$$

and $f_0(z) = 1$. These equations yield immediately

$$f_n(0) = (n+1)!, \quad f_n(-1) = (-1)^n\,. \tag{10}$$

Equation (9) can also be written in the compact form

$$z^n f_n(z) = (z^{n+1}(z+1) f_{n-1}(z))' \quad (n \in \mathbb{N}) \tag{11}$$

from which, introducing the operator $D = \frac{d}{dz}[z^2(z+1)(.)]$, the generalized Rodrigues formula

$$f_n(z) = \frac{1}{z^n} D^n 1 \quad (n \in N_0) \tag{12}$$

can be derived. Using Rolle's theorem, formula (11) shows that f_n has exactly n simple zeros in (0,1), and that the zeros of f_{n-1} separate those of f_n.

For the next considerations the moments

$$J_{nk} = \int\limits_{-1}^{0} z^k f_n(z) dz \quad (n, k \in \mathbb{N}_0) \tag{13}$$

are a crucial remedy. Using (11), partial integration yields

$$J_{nk} = (n-k)(J_{n-1,k+1} + J_{n-1,k}) \quad (n \in \mathbb{N}, k \in \mathbb{N}_0) \tag{14}$$

with the special cases

$$J_{nn} = 0, \;\; J_{n+1,n} = J_{n,n+1}, \;\; J_{n+1,n-1} = 2J_{n,n-1} \quad (n \in \mathbb{N}) \tag{15}$$

whereas $J_{00} = 1$. Substituting (8) into (13) we see that

$$J_{nk} = (-1)^{n+k} \frac{k!}{(n+k+1)!} p_n(k) \tag{16}$$

with certain polynomials p_n of degree n, and by means of the DERIVE system it follows

$$\begin{aligned}
&p_0(k) = 1, \;\; p_1(k) = k-1, \;\; p_2(k) = (k-2)(k-3), \;\; p_3(k) = (k-2)(k-3)(k-11),\\
&p_4(k) = (k-2)(k-4)(k-5)(k-31), \;\; p_5(k) = (k-4)(k-5)(k^3 - 90k^2 + 1019k - 1770),\\
&p_6(k) = (k-4)(k-6)(k-7)(k^3 - 202k^2 + 4267k - 8490).
\end{aligned}$$

These special cases suggest:

Proposition 1 *For $n \geq 3$ the moments* (13) *vanish in the following cases:*

n odd and $k \in \{n-1, n\}$,
n even and $k \in \{n-2, n, n+1\}$.

Proof: In view of (15) we only have to show that $J_{n,n+1} = 0$ for positive even n. For this reason we introduce the abbreviations $\varphi(z) = z^2(z+1)$ and $\Phi_n(z) = z^n f_n(z)$ $(n \in \mathbb{N}_0)$, so that (11) turns over into

$$\Phi_n = (\varphi \Phi_{n-1})', \tag{17}$$

and (13) for $k = n + 1$ into

$$J_{n,n+1} = \int_{-1}^{0} z\Phi_n dz \,.$$

Using (17) and $\varphi = 0$ both for $z = 0$ and $z = -1$, partial integration yields

$$J_{n,n+1} = -\int_{-1}^{0} \varphi\Phi_{n-1} dz \tag{18}$$

and $k - 1$ further partial integrations yield

$$J_{n,n+1} = (-1)^k \int_{-1}^{0} \varphi\Phi_{k-1}\Phi_{n-k} dz \,.$$

For $k = n$ we obtain, in view of (18), $J_{n,n+1} = (-1)^{n+1} J_{n,n+1}$. This proves the assertion ■

From (15) and (16) we obtain the relations

$$p_{n+1}(n) = (n+1)p_n(n+1),\; p_{n+1}(n-1) = -(4n+2)p_n(n-1) \quad (n \in \mathbb{N})\,.$$

According to Proposition 1 the first of these are only interesting for n odd, and the last for n even.

Proposition 2 *The coefficients in* (8) *can be expressed by means of the polynomials in* (16):

$$c_{n+1,\nu+1} = (-1)^n \frac{1}{n!}\binom{n}{\nu} p_n(-\nu - 1)\,. \tag{19}$$

Proof: Substituting (8) into (13) we find

$$J_{nk} = \sum_{\nu=0}^{n} c_{n+1,\nu+1} \frac{(-1)^{k+\nu}}{k+\nu+1}\,,$$

and decomposition of (16) into partial fractions immediately yields (19) ■

According to (19) the relations (7) transfer into

$$p_n(-1) = (-1)^n n!(n+1)!,\; p_n(-n) = \frac{2}{3}(-1)^n n!(2n+1)!!\,,$$
$$p_n(-n-1) = (-1)^n n!(2n+1)!!\,.$$

Remark Here, we are interested in formula (19) only concerning the coefficients $c_{m\mu}$ from (5). But Proposition 2 is even valid for an arbitrary polynomial (8) with the moments (13) and polynomials $p_n(k)$ defined by (16).

Finally, we introduce the function

$$\psi = \ln\left(1+\frac{1}{z}\right) - \frac{1}{z}\,, \tag{20}$$

which is analytic for $z \notin \{-1, 0\}$. Both for $z < -1$ and for $z > 0$ the function ψ is real, whereas $\psi(z) = \ln\left|1+\frac{1}{z}\right| - \frac{1}{z} - i\pi$ for $-1 < z < 0$, cf. Figure 1. Differentiation of (20) yields $\psi' = \frac{1}{\varphi}$ where $\varphi = z^2(z+1)$ as before. Hence, for $z \notin \{-1, 0\}$ there exists the inverse function F with

$$F(\psi) = z\,. \tag{21}$$

We want to determine the generating function

$$G(z, w) = \sum_{n=0}^{\infty} \frac{1}{n!} f_n(z) w^n\,, \tag{22}$$

which will have an unusual shape.

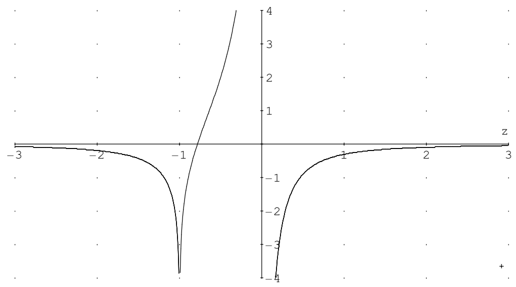

Figure 1: $\Re\left(\ln\left(1+\frac{1}{z}\right) - \frac{1}{z}\right)$

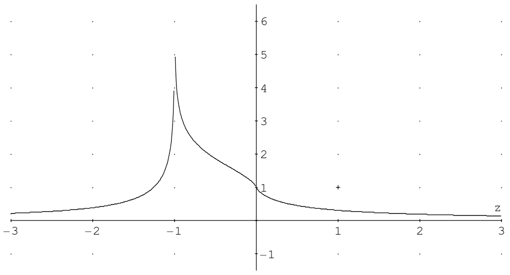

Figure 2: abs $\left[z \ln\left(1+\frac{1}{z}\right) - 1\right]$

Proposition 3 *With the function F defined by* (21) *and* (20) *the generating function* (22) *has the representation*

$$G(z, w) = \frac{1}{z^2(z+1)} F'\left(\frac{w-1}{z} + \ln\left(1+\frac{1}{z}\right)\right) \quad (z \notin \{-1, 0\})\,, \tag{23}$$

and (22) *converges for $|w| < |z\psi(z)|$, cf. Figure* 2.

Proof: We give two proofs of this proposition, the first will be a constructive one.

The recursion (9) can be transferred into the partial differential equation

$$z(z+1)G_z + (w(z+1) - 1)G_w + (3z+2)G = 0\,. \tag{24}$$

The corresponding characteristics satisfy the system

$$\frac{dz}{z(z+1)} = \frac{dw}{w(z+1) - 1} = -\frac{dG}{(3z+2)G}$$

with the integrals

$$c_1 = \frac{w-1}{z} + \ln\left(1+\frac{1}{z}\right), \quad c_2 = z^2(z+1)G\,.$$

Hence (24) has the general solution

$$G = \frac{1}{z^2(z+1)}\Phi\left(\frac{w}{z}+\psi(z)\right)$$

with (20) and an arbitrary differentiable function Φ. The initial condition $G(z,0)=1$ yields

$$\Phi(\psi(z)) = z^2(z+1)\,.$$

Differentiation of (21) yields $\Phi = F'$ which proves (23).

Second we want to check (23) directly. The function F is singular only in the point 0. Hence, the Taylor expansion (22) converges for $|w| < |z\psi(z)|$. Comparing it with (22) we find that

$$f_n = \frac{1}{\varphi z^n}F^{(n+1)}(\psi)$$

for $n \in \mathbb{N}_0$ and therefore $F^{(n)}(\psi) = \varphi z^{n-1} f_{n-1}$ for $n \in \mathbb{N}$. Differentiation with respect to z yields

$$\left(\varphi z^{n+1} f_{n-1}\right)' = \frac{d}{dz}F^{(n)}(\psi) = \frac{1}{\varphi}F^{(n+1)}(\psi) = z^n f_n\,,$$

i.e. equation (11) which, together with $f_0 = \frac{1}{\varphi}F'(\psi) = 1$, defines f_n uniquely ■

In the excluded cases $z \in \{-1, 0\}$, where (23) is not applicable, the formulas (10) imply the special values of (22)

$$G(0,w) = \frac{1}{(1-w)^2}\,, \quad G(-1,w) = e^{-w}\,.$$

References

[1] **Nörlund, N. E. :** *Vorlesungen über Differenzenrechnung.* Springer Berlin 1924

[2] **Temme, N. M. :** *Special Functions.* Wiley New York etc. 1996

received: June 29, 2001

Author:

Prof. i.R. Dr. Lothar Berg
Universität Rostock
Fachbereich Mathematik
18051 Rostock
Germany

e-mail: lothar.berg@mathematik.uni-rostock.de